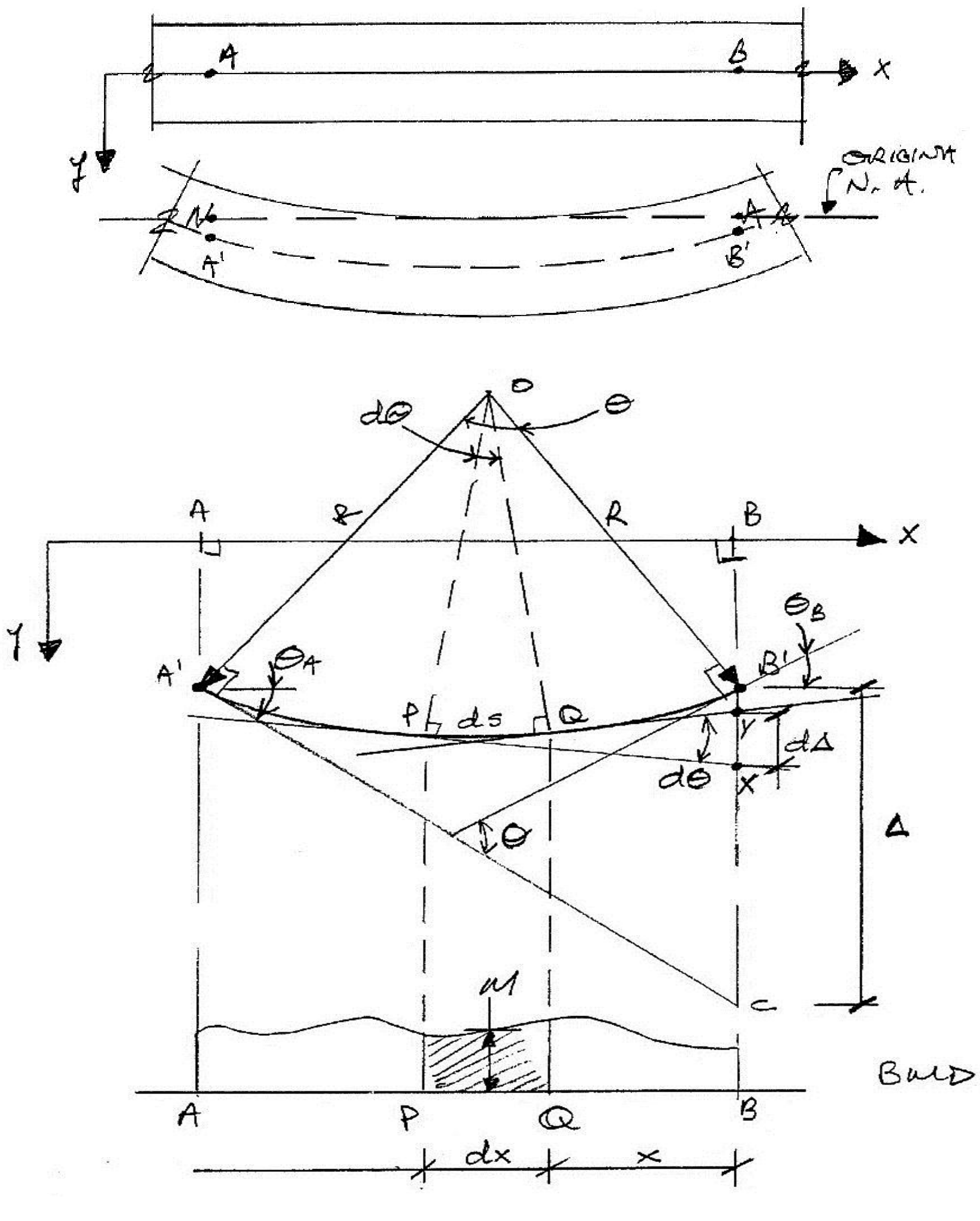
Moment-Area Method

The moment-area method; developed by Otto Mohr, is a powerful tool for finding the deflections of structures primarily subjected to bending and also provide a relatively easy way to derive many of the classical methods of structural analysis.

As illustrated below in figure, consider a portion of a beam in its undeformed and deformed state, where-

1 AB represent the unloaded position while A'B' is the deflected shape of the beam after loading.

2 θ is the angle subtended by the arc A'B' and is equal to change in slope from A' to B' (slope of the tangent at B' with respect the tangent at A').



5 dθ is the change in slope from P to Q.

6 M is the average bending moment over the portion dx between P and Q.

7 The distance Δ is known as the vertical intercept and is the distance from B' to the produced tangent to the curve at A' which crosses under B' at C. It is measured perpendicular to the undeformed neutral axis (i.e. the x-axis) and so is 'vertical'.

Moment- Area First Theorem (Mohr I): Noting that the angles are always measured in radians, we have:

$$ds = R \cdot d\theta$$
$$\therefore R = \frac{ds}{d\theta}$$

From the Euler-Bernoulli Theory of Bending, we know:

$$\frac{1}{R} = \frac{M}{EI}$$

Hence

$$d\theta = \frac{M}{EI} \cdot ds$$

But for small deflections, the chord and arc length are similar, i.e. ds ≈ dx , giving,

$$d\theta = \frac{M}{EI} \cdot dx$$

The total change in slope between A and B is thus:

$$\int_A^B d\theta = \int_A^B \frac{M}{EI}\,dx$$

The term M/EI is the curvature and the diagram of this term as it changes along a beam is the curvature diagram (or more simply the M/EI diagram). Thus we have:

$$d\theta_{BA} = \theta_B - \theta_A = \int_A^B \frac{M}{EI}\,dx$$

This is interpreted as: $\left[\text{Change in slope}\right]_{AB} = \left[\text{Area of } \frac{M}{EI} \text{ diagram}\right]_{AB}$

Thus the Mohr's First Theorem (Mohr I) can be stated as; *the change in slope over any length of a member subjected to bending is equal to the area of the curvature (M/EI) diagram over that length*. Usually the beam is prismatic and so E and I do not change over the length AB, whereas the bending moment M will change. Thus:

$$\theta_{AB} = \frac{1}{EI}\int_A^B M\,dx$$

$$[\text{Change in slope}]_{AB} = \frac{[\text{Area of } M \text{ diagram}]_{AB}}{EI}$$

Moment- Area Second Theorem (Mohr II): From the diagram, we have:

$$d\Delta = x \cdot d\theta$$

But

$$d\theta = \frac{M}{EI} \cdot dx$$

Thus

$$d\Delta = \frac{M}{EI} \cdot x \cdot dx$$

And so for the portion AB, we have:

$$\int_A^B d\Delta = \int_A^B \frac{M}{EI} \cdot x \cdot dx$$

$$\Delta_{BA} = \left[\int_A^B \frac{M}{EI} \cdot dx\right]\bar{x}$$

= First moment of M/EI diagram about B

This is easily interpreted as:

$$\begin{bmatrix}\text{Vertical}\\ \text{Intercept}\end{bmatrix}_{BA} = \begin{bmatrix}\text{Area of}\\ \dfrac{M}{EI}\ \text{diagram}\end{bmatrix}_{BA} \times \begin{bmatrix}\text{Distance from } B \text{ to centroid}\\ \text{of } \left(\dfrac{M}{EI}\right)_{BA} \text{ diagram}\end{bmatrix}$$

Thus the Mohr's Second Theorem (Mohr II) can be stated as; *for an originally straight beam, subject to bending moment, the vertical intercept between one terminal and the tangent to the curve of another terminal is the first moment of the curvature diagram about the terminal where the intercept is measured.*

There are few crucial things to note in Mohr's theorems

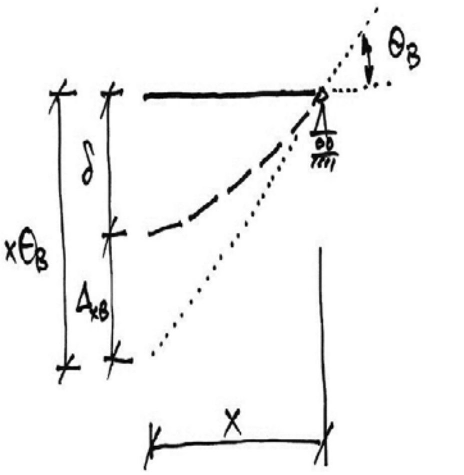
(i) $d\theta_{AB} \neq \theta_A \neq \theta_B$ — The change in slope between A and B($d\theta_{AB}$) is ***not*** the slope (θ_A or θ_B) of beam at section A or B (Mohr I)

(iii) $\Delta \neq \delta$ — Vertical intercept is (Δ) **not the** deflection (δ). It is the distance from the deformed position of the beam to the tangent of the deformed curve of the beam at another location (Mohr II)

(iii) $\Delta_{BA} \neq \Delta_{AB}$ — The moment of the curvature (M/EI) diagram must be taken about the point where the vertical intercept is required.

(iv) Mohr I and II — These theorems do not give the slopes and deflections directly, instead they give the change in slopes and deflections between tangents at two sections; $d\theta AB$ and Δ_{AB} are also re[presented by other symbols like $\theta A/B$, Q_{AB} and $\Delta_{A/B}$, $t_{A/B}$ etc.

Finding Deflections: To find the deflection at any location *x* from a support use the following relationships between slopes and vertical intercepts. Thus;

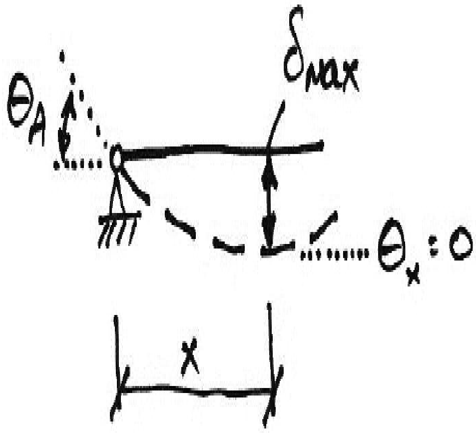


1. Find the slope at the support using Mohr II

2. For the location x, and from the diagram we have:

$$\delta_x = x \cdot \theta_B - \Delta_{xB}$$

Maximum Deflection: To find the maximum deflection we first need to find the location at which this occurs. We know from beam theory that:

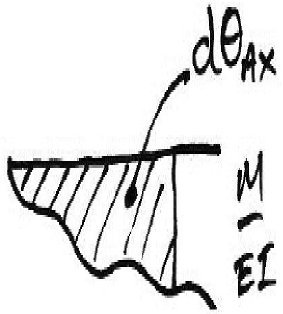


$$\delta = \frac{d\theta}{dx}$$

Hence, from calculus, the maximum deflection occurs at a slope, θ= 0. To find where the slope is zero:

1 Calculate a slope at some point, say support A, using Mohr II say;

2 Using Mohr I, determine at what distance from the point of known slope (A) the change in slope (MohrI), $d\theta_{AX}$ equals the known slope (θ_A)



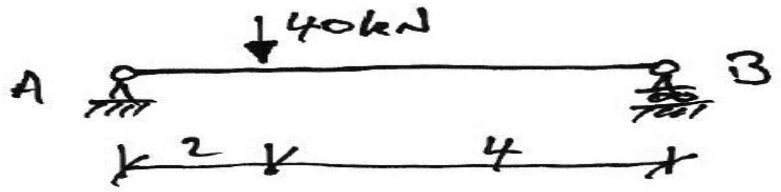
3 This is the point of maximum deflection since:

$$\theta_A - d\theta_{Ax} = \theta_A - \theta_A = 0$$

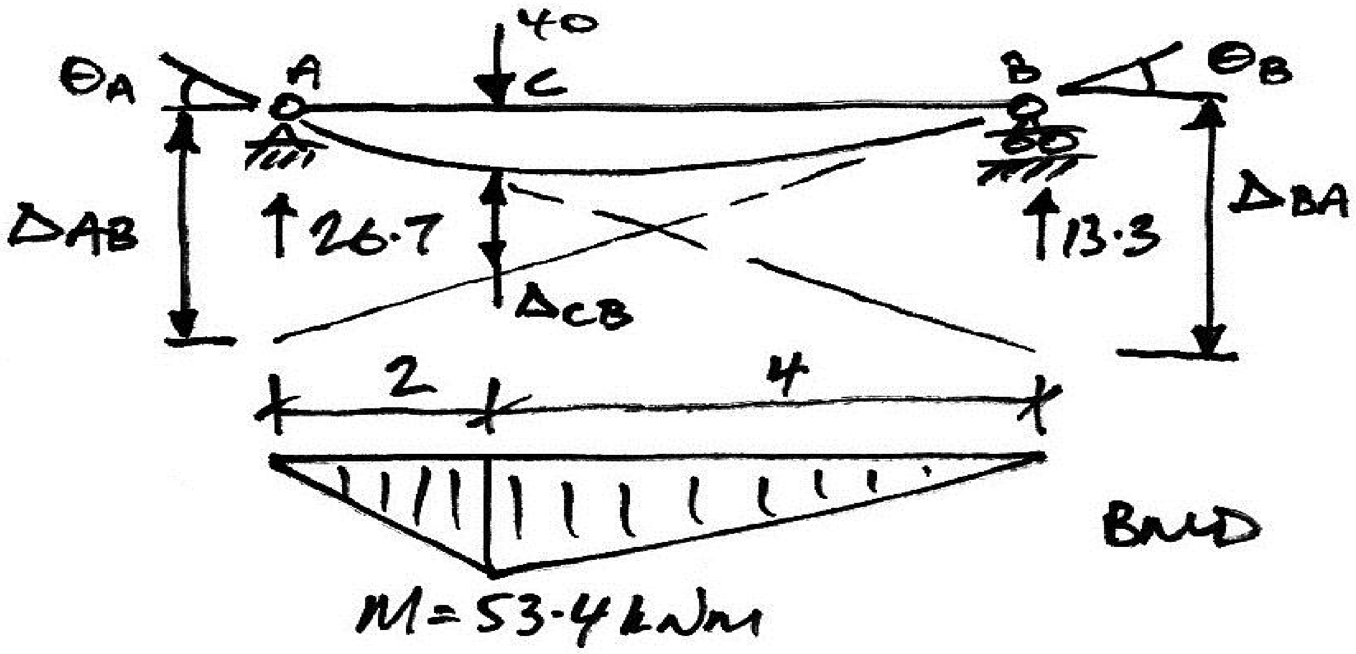
Example: For the beam of constant EI as shown in the figure:

(a) Determine θ_A, θ_B and δ at the centre.

(b) What is the maximum deflection and where is it located? Express answers in terms of EI.



The first step is to determine the BMD and draw the deflected shape diagram with slopes and tangents indicated:



Slopes at A and B

To calculate the slopes, we need to calculate the vertical intercepts and use the fact that the intercept is length times rotation (or slope). Thus, for the slope at B:

$$EI\Delta_{AB} = \left(\frac{2}{3}\cdot 2\right)\left(\frac{1}{2}\cdot 2\cdot 53.4\right) + \left(2+\frac{4}{3}\right)\left(\frac{1}{2}\cdot 4\cdot 53.4\right)$$

$$= 53.4\left(\frac{4}{3}+\frac{20}{3}\right)$$

$$= 427.2$$

$$\therefore \Delta_{AB} = \frac{427.2}{EI}$$

But, we also know that $\Delta_{AB} = 6\theta_B$

$$6\theta_B = \frac{427.2}{EI}$$
$$\therefore \theta_B = \frac{71.2}{EI}$$

Similarly for the slope at A:

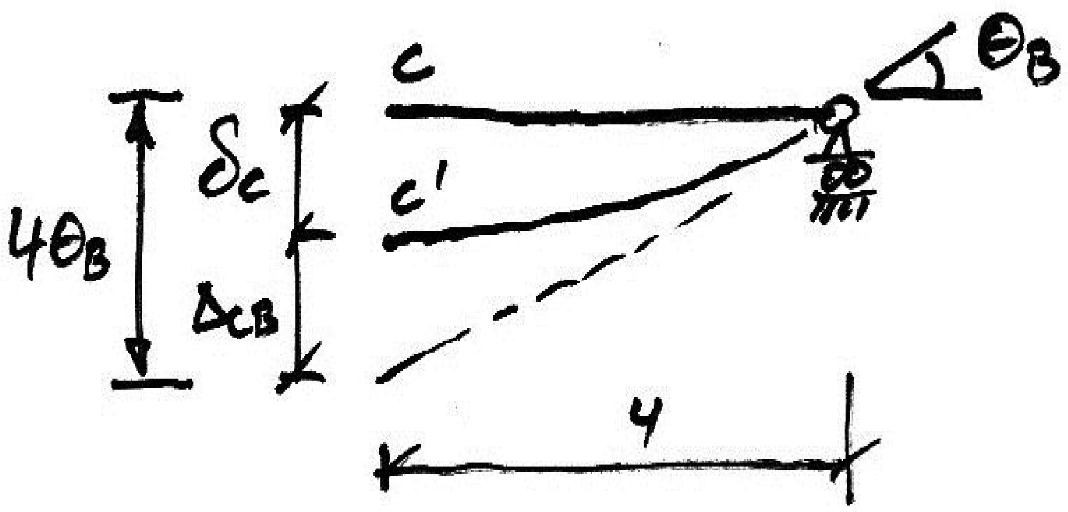
$$EI\Delta_{BA} = \left(\frac{2}{3}\cdot 4\right)\left(\frac{1}{2}\cdot 4\cdot 53.4\right) + \left(4 + \frac{1}{3}\cdot 2\right)\left(\frac{1}{2}\cdot 2\cdot 53.4\right)$$
$$= 53.4\left(\frac{16}{3} + \frac{14}{3}\right)$$
$$= 534$$
$$\therefore \Delta_{BA} = \frac{534}{EI}$$

But, we also know that $\Delta_{BA} = 6\theta_A$ and so:

$$6\theta_A = \frac{534}{EI}$$
$$\therefore \theta_A = \frac{89.2}{EI}$$

Deflection at *C*

To find the deflection at *C*, we use the vertical intercept and Δ_{CB} and $6\theta_B$



From the figure, we see:

$$\delta_C = 4\theta_B - \Delta_{CB}$$

And so from the BMD and slope at *B*:

$$EI\delta_C = 4(1.33 \cdot 53.4) - \left(\frac{1}{2} \cdot 4 \cdot 53.4\right)\left(\frac{4}{3}\right)$$
$$\therefore \delta_C = \frac{142.3}{EI}$$

Maximum Deflection

The first step in finding the maximum deflection is to locate it. We know two things:

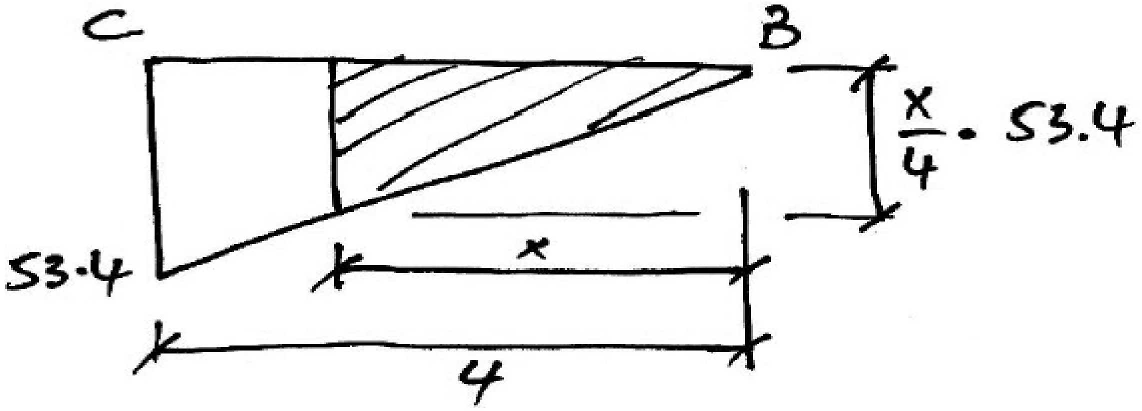
1. Maximum deflection occurs where there is zero slope;

2. Maximum deflection is always close to the centre of the span.

Based on these facts, we work with Mohr I to find the point of zero slope, which will be located between B and C, as follows:

$$\text{Change in rotation} = \theta_B - 0 = \theta_B$$

But since we know that the change in slope is also the area of the M EI diagram we need to find the point x where the area of the M EI diagram is equal to θ_B



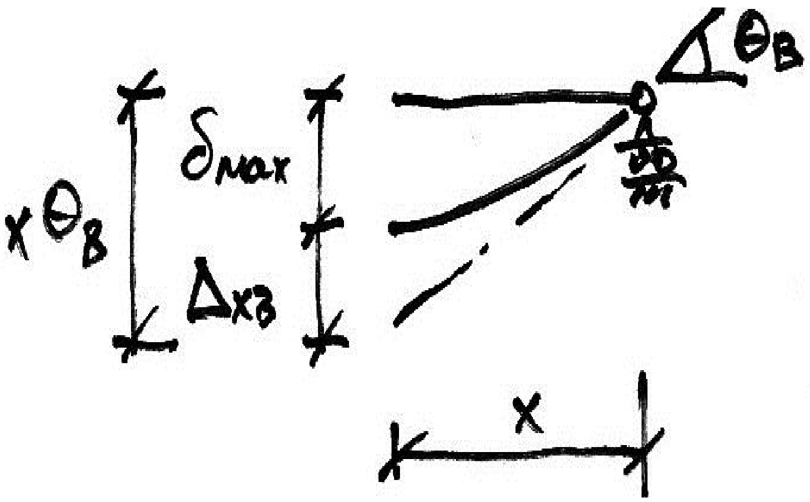
Thus:

$$EI(\theta_B - 0) = \left(53.4 \cdot \frac{x}{4}\right) \cdot \frac{1}{2} \cdot x$$
$$EI\theta_B = 53.4\frac{x^2}{8}$$

But we know that $\theta_B = \frac{71.2}{EI}$ hence :

$$EI\left(\frac{71.2}{EI}\right) = 53.4\frac{x^2}{8}$$
$$x^2 = 10.66$$
$$x = 3.265 \text{ m from } B \text{ or } 2.735 \text{ m from } A$$

So we can see that the maximum deflection is 265 mm shifted from the centre of the beam towards the load. Once we know where the maximum deflection is, we can calculate is based on the following diagram:



Thus:

$$\delta_{max} = x\theta_B - \Delta_{xB}$$

$$EI\delta_{max} = x(1.33 \cdot 53.4) - \left(53.4\frac{x^2}{8}\right)\left(\frac{x}{3}\right)$$
$$= M(4.342 - 1.450)$$
$$\delta_{max} = \frac{154.4}{EI}$$