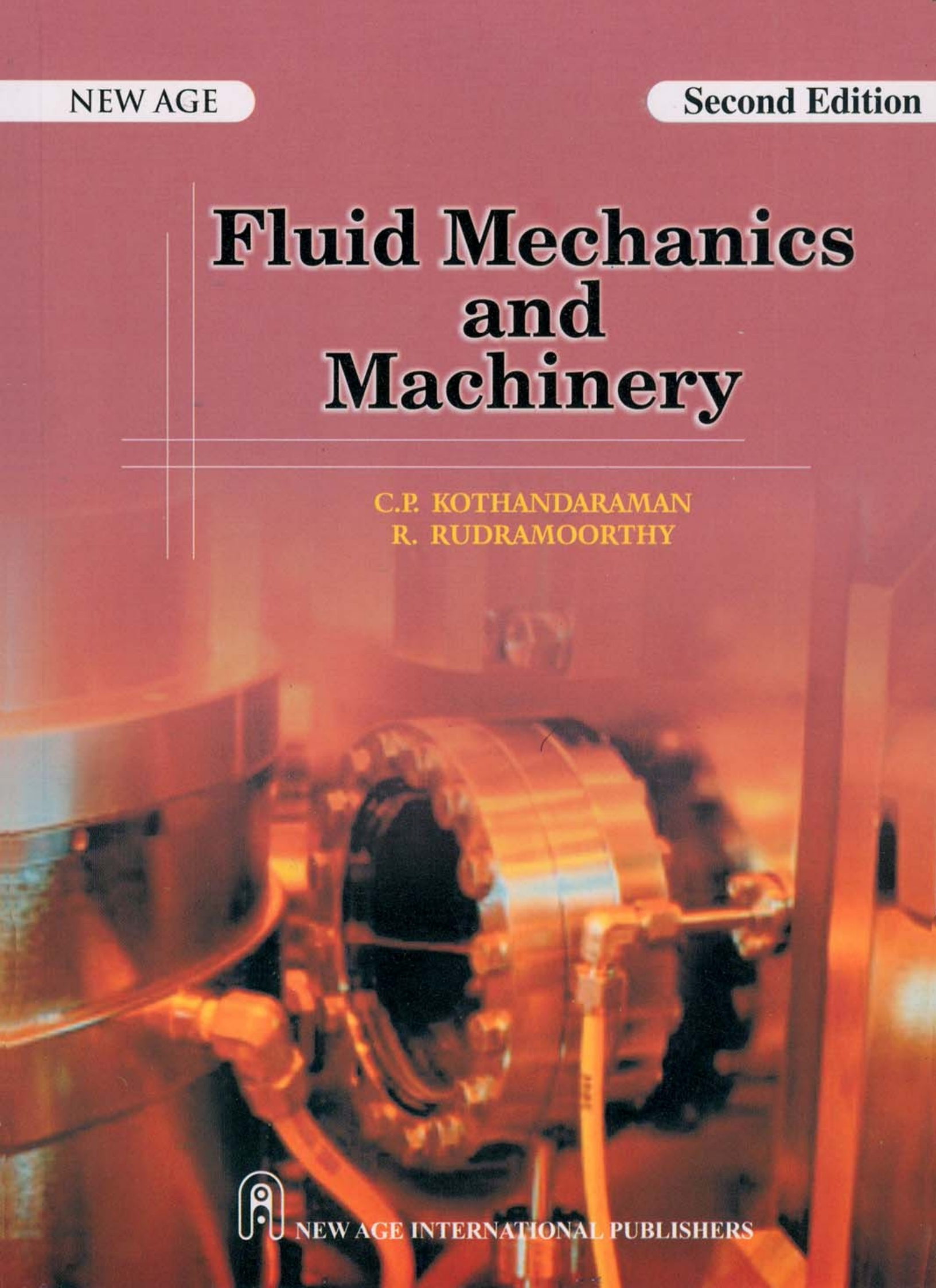
NEW AGE

Second Edition

Fluid Mechanics and Machinery

C.P. KOTHANDARAMAN
R. RUDRAMOORTHY



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Fluid Mechanics and Machinery

Fluid Mechanics and Machinery

(Second Edition)

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Preface to the Second Edition

This book Basic Fluid Mechanics is revised and enlarged by the addition of four chapters on Hydraulic Machinery and is now titled as Fluid Mechanics and Machinery. The authors hope this book will have a wider scope.

This book will be suitable for the courses on Fluid Mechanics and Machinery of the various branches of study of Anna University and also other Indian universities and the Institution of Engineers (India).

Professor Obert has observed in his famous treatise on Thermodynamics that concepts are better understood by their repeated applications to real life situations. A firm conviction of this principle has prompted the author to arrange the text material in each chapter in the following order.

In the first section after enunciating the basic concepts and laws, physical and mathematical models are developed leading to the formulation of relevant equations for the determination of outputs. Simple and direct numerical examples are included to illustrate the basic laws. More stress is on the model development as compared to numerical problems.

A section titled "SOLVED PROBLEMS" comes next. In this section more involved derivations and numerical problems of practical interest are solved. The investigation of the effect of influencing parameters for the complete spectrum of values is attempted here. Problems involving complex situations are shown solved in this section. It will also illustrate the range of values that may be expected under different situations. Two important ideas are stressed in this section. These are (1) checking for dimensional homogeneity in the case of all equations derived before these equations can be used and (2) The validation of numerical answers by cross checking. This concept of validation in professional practice is a must in all design situations.

In the next section a large number of objective type questions with answers are given. These are very useful for understanding the basics and resolving misunderstandings.

In the final section a large number of graded exercise problems involving simple to complex situations, most of them with answers, are included.

The material is divided into sixteen chapters. The first chapter deals in great detail with properties of fluids and their influence on the operation of various equipments. The next chapter discusses the determination of variation of pressure with depth in stationary and moving fluids. The third chapter deals with determination of forces on surfaces in contact with stationary fluids. Chapter four deals with buoyant forces on immersed or floating bodies and the importance of metacentric height on stability. In chapter five basic fluid flow concepts and hydrodynamics are discussed.

Energy equations and the variation of flow parameters along flow as well as pressure loss due to friction are dealt with in chapter six.

In chapter seven flow in closed conduits including flow in pipe net work are discussed.

Dimensional analysis and model testing and discussed in a detailed manner in chapters eight and nine. Boundary layer theory and determination of forces due to fluid flow on bodies are dealt with in chapter ten.

In chapter eleven various flow measuring methods and instruments are described. Flow in open channels is dealt with in detail in chapter twelve.

Chapter thirteen deals with dynamics of fluid flow in terms force exerted on surface due to change of momentum along the flow on the surface.

Chapter fourteen deals with the theory of turbo machines as applied to the different type of hydraulic turbines. The working of centrifugal and axial flow pumps is detailed in chapter fifteen. The last chapter sixteen discusses the working of reciprocating and other positive displacement pumps.

The total number of illustrative worked examples is around five hundred. The objective questions number around seven hundred. More than 450 exercise problems with answers are also included.

The authors thank all the professors who have given very useful suggestions for the improvement of the book.

Authors

Preface to the First Edition

This book is intended for use in B.E./B.Tech. courses of various branches of specialisation like Civil, Mechanical and Chemical Engineering. The material is adequate for the prescribed syllabi of various Universities in India and the Institution of Engineers. SI system of units is adopted throughout as this is the official system of units in India. In order to give extensive practice in the application of various concepts, the following format is used in all the chapters.

- Enunciation of Basic concepts
- Development of physical and mathematical models with interspersed numerical examples
- Illustrative examples involving the application and extension of the models developed
- Objective questions and exercise problems

The material is divided into 12 chapters. The first chapter deals in great detail with properties of fluids and their influence on the operation of various equipments. The next two chapters discuss the variation of pressure with depth in liquid columns, at stationary and at accelerating conditions and the forces on surfaces exerted by fluids. The fourth chapter deals with buoyant forces and their effect on floating and immersed bodies. The kinetics of fluid flow is discussed in chapter five.

Energy equations and the determination of pressure variation in flowing fluids and loss of pressure due to friction are discussed in chapters six and seven.

Dimensional analysis and model testing are discussed in a detailed manner in chapters eight and nine.

Boundary layer theory and forces due to flow of fluids over bodies are discussed in chapter ten. Chapter eleven details the methods of measurement of flow rates and of pressure in fluid systems. Open channel flow is analyzed in chapter twelve.

The total number of illustrative numerical examples is 426. The objective questions included number 669. A total number of 352 exercise problems, mostly with answers are available.

We wish to express our sincere thanks to the authorities of the PSG College of Technology for the generous permission extended to us to use the facilities of the college.

Our thanks are due to Mr. R. Palaniappan and Mr. C. Kuttumani for their help in the preparation of the manuscript.

C.P. Kothandaraman

R. Rudramoorthy

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1 Physical Properties of Fluids

1.0 INTRODUCTION

The flow of ideal non-viscous fluids was extensively studied and mathematical theories were developed during the last century. The field of study was called as 'Hydrodynamics'. However the results of mathematical analysis could not be applied directly to the flow of real fluids. Experiments with water flow resulted in the formulation of empirical equations applicable to engineering designs. The field was called Hydraulics. Due to the development of industries there arose a need for the study of fluids other than water. Theories like boundary layer theory were developed which could be applied to all types of real fluids, under various conditions of flow. The combination of experiments, the mathematical analysis of hydrodynamics and the new theories is known as 'Fluid Mechanics'. **Fluid Mechanics encompasses the study of all types of fluids under static, kinematic and dynamic conditions.**

The study of properties of fluids is basic for the understanding of flow or static condition of fluids. The important properties are **density, viscosity, surface tension, bulk modulus and vapour pressure.** Viscosity causes resistance to flow. Surface tension leads to capillary effects. Bulk modulus is involved in the propagation of disturbances like sound waves in fluids. Vapour pressure can cause flow disturbances due to evaporation at locations of low pressure. It plays an important role in cavitation studies in fluid machinery.

In this chapter various properties of fluids are discussed in detail, with stress on their effect on flow. Fairly elaborate treatment is attempted due to their importance in engineering applications. The basic laws used in the discussions are :

(*i*) Newton's laws of motion,

(*ii*) Laws of conservation of mass and energy,

(*iii*) Laws of Thermodynamics, and

(*iv*) Newton's law of viscosity.

A fluid is defined as a material which will continue to deform with the application of shear force however small the force may be.

1.1 THREE PHASES OF MATTER

Generally matter exists in three phases namely (*i*) Solid (*ii*) Liquid and (*iii*) Gas (includes vapour). The last two together are also called by the common term **fluids**.

In solids atoms/molecules are closely spaced and the attractive (cohesive) forces between atoms/molecules is high. The shape is maintained by the cohesive forces binding the atoms. When an external force is applied on a solid component, slight rearrangement in atomic positions balances the force. Depending upon the nature of force the solid may elongate or shorten or bend. When the applied force is removed the atoms move back to the original position and the former shape is regained. Only when the forces exceed a certain value (yield), a small deformation called plastic deformation will be retained as the atoms are unable to move to their original positions. When the force exceeds a still higher value (ultimate), the cohesive forces are not adequate to resist the applied force and the component will break.

In liquids the inter molecular distances are longer and the cohesive forces are of smaller in magnitude. The molecules are not bound rigidly as in solids and can move randomly. However, the cohesive forces are large enough to hold the molecules together below a free surface that forms in the container. Liquids will continue to deform when a shear or tangential force is applied. The deformation continues as long as the force exists. In fluids the **rate of deformation** controls the force (not deformation as in solids). More popularly it is stated that a fluid (liquid) cannot withstand applied shear force and will continue to deform. When at rest liquids will assume the shape of the container forming a free surface at the top.

In gases the distance between molecules is much larger compared to atomic dimensions and the cohesive force between atoms/molecules is low. So gas molecules move freely and fill the full volume of the container. If the container is open the molecules will diffuse to the outside. Gases also cannot withstand shear. The **rate of deformation** is proportional to the applied force as in the case of liquids.

Liquids and gases together are classified as fluids. Vapour is gaseous state near the evaporation temperature. The state in which a material exists depends on the pressure and temperature. For example, steel at atmospheric temperature exists in the solid state. At higher temperatures it can be liquefied. At still higher temperatures it will exist as a vapour.

A fourth state of matter is its existence as charged particles or ions known as plasma. This is encountered in MHD power generation. This phase is not considered in the text.

1.2 COMPRESSIBLE AND INCOMPRESSIBLE FLUIDS

If the density of a fluid varies significantly due to moderate changes in pressure or temperature, then the fluid is called compressible fluid. Generally gases and vapours under normal conditions can be classified as compressible fluids. In these phases the distance between atoms or molecules is large and cohesive forces are small. So increase in pressure or temperature will change the density by a significant value.

If the change in density of a fluid is small due to changes in temperature and or pressure, then the fluid is called incompressible fluid. All liquids are classified under this category.

When the change in pressure and temperature is small, gases and vapours are treated as incompressible fluids. For certain applications like propagation of pressure disturbances, liquids should be considered as compressible.

In this chapter some of the properties relevant to fluid mechanics are discussed with a view to bring out their influence on the design and operation of fluid machinery and equipments.

1.3 DIMENSIONS AND UNITS

It is necessary to distinguish clearly between the terms "Units" and "Dimensions". The word "dimension" is used to describe basic concepts like mass, length, time, temperature and force. "Large mass, long distance, high temperature" does not mean much in terms of visualising the quantity. Dimension merely describes the concept and does not provide any method for the quantitative expression of the same. Units are the means of expressing the value of the dimension quantitatively or numerically The term "second" for example is used to quantify time. "Ten seconds elapsed between starting and ending of an act" is the way of expressing the elapsed time in numerical form. The value of dimension should be expressed in terms of units before any quantitative assessment can be made.

There are three widely used systems of units in the world. These are (1) British or English system (it is not in official use now in Briton) (2) Metric system and (3) SI system (System International d'Unites or International System of Units). India has passed through the first two systems in that order and has now adopted the SI system of units.

The basic units required in Fluid Mechanics are for mass, length, time and temperature. These are kilogram (**kg**), metre (**m**), second (**s**) and kelvin (**K**). The unit of force is defined using Newton's second law of motion which states that applied force is proportional to the time rate of change of momentum of the body on which the force acts.

For a given mass **m**, subjected to the action of a force **F**, resulting in an acceleration **a**, Newton's law can be written in the form

$$\mathbf{F = (1/g_o)\ m\ a} \tag{1.3.1}$$

where g_o is a dimensional constant whose numerical value and units depend on those selected for **force, F, mass, m, and acceleration, a**. The unit of force is newton (N) in the SI system.

One newton is defined as the force which acting on a mass of one kilogram will produce an acceleration of 1 m/s^2. This leads to the relation

$$\mathbf{1\ N = (1/g_o) \times 1\ kg \times 1\ m/s^2} \tag{1.3.2}$$

Hence
$$\mathbf{g_o = 1\ kg\ m/N\ s^2} \tag{1.3.3}$$

The numerical value of g_o is unity (1) in the SI system and this is found advantageous in numerical calculations. However this constant should necessarily be used to obtain dimensional homogeneity in equations.

In metric system the unit of force is kg_f defined as the force acted on one kg mass by standard gravitational acceleration taken as 9.81 m/s^2. The value of g_o is 9.81 kg m/kg_fs^2.

In the English system the unit of force is lb_f defined as the force on one lb mass due to standard gravitational acceleration of 32.2 ft/s^2.

The value of g_o is 32.2 ft lb/lb_fs^2.

Some of the units used in this text are listed in the table below:

Quantity	Unit symbol	Derived units
mass	kg	ton (tonne) = 1000 kg
time	s	min (60s), hr (3600s)
length	m	mm, cm, km
temperature	K, (273 + °C)	°C
force	N (newton)	kN, MN (10^6 N)
energy, work, heat	Nm, J	kJ, MJ, kNm
power	W = (Nm/s, J/s)	kW, MW
pressure	N/m^2, (pascal, pa)	kPa, MPa, bar (10^5Pa)

Conversion constants between the metric and SI system of units are tabulated elsewhere in the text.

1.4 CONTINUUM

As gas molecules are far apart from each other and as there is empty space between molecules doubt arises as to whether a gas volume can be considered as a continuous matter like a solid for situations similar to application of forces.

Under normal pressure and temperature levels, gases are considered as a continuum (*i.e.,* as if no empty spaces exist between atoms). The test for continuum is to measure properties like density by sampling at different locations and also reducing the sampling volume to low levels. If the property is constant irrespective of the location and size of sample volume, then the gas body can be considered as a continuum for purposes of mechanics (application of force, consideration of acceleration, velocity etc.) and for the gas volume to be considered as a single body or entity. This is a very important test for the application of all laws of mechanics to a gas volume as a whole. When the pressure is extremely low, and when there are only few molecules in a cubic metre of volume, then the laws of mechanics should be applied to the molecules as entities and not to the gas body as a whole. In this text, only systems satisfying continuum requirements are discussed.

1.5 DEFINITION OF SOME COMMON TERMINOLOGY

***Density (mass density)*:** The mass per unit volume is defined as density. The unit used is kg/m^3. The measurement is simple in the case of solids and liquids. In the case of gases and vapours it is rather involved. The symbol used is ρ. The characteristic equation for gases provides a means to estimate the density from the measurement of pressure, temperature and volume.

***Specific Volume*:** The volume occupied by unit mass is called the specific volume of the material. The symbol used is v, the unit being m^3/kg. Specific volume is the reciprocal of density.

In the case of solids and liquids, the change in density or specific volume with changes in pressure and temperature is rather small, whereas in the case of gases and vapours, density will change significantly due to changes in pressure and/or temperature.

***Weight Density or Specific Weight*:** The force due to gravity on the mass in unit volume is defined as Weight Density or Specific Weight. The unit used is N/m^3. The symbol used is γ. At a location where g is the local acceleration due to gravity,

$$\textbf{Specific weight, } \gamma = g\rho \qquad (1.5.1)$$

In the above equation direct substitution of dimensions will show apparent non-homogeneity as the dimensions on the LHS and RHS will not be the same. On the LHS the dimension will be N/m^3 but on the RHS it is $kg/m^2\ s^2$. The use of g_o will clear this anomaly. As seen in section 1.1, $g_o = 1$ kg m/N s^2. The RHS of the equation 1.3.1 when divided by g_o will lead to perfect dimensional homogeneity. The equation should preferably be written as,

$$\textbf{Specific weight, } \gamma = (g/g_o)\,\rho \qquad (1.5.2)$$

Since newton (N) is defined as the force required to accelerate 1 kg of mass by $1/s^2$, it can also be expressed as $kg.m/s^2$. Density can also be expressed as Ns^2/m^4 (as kg = Ns^2/m). Beam balances compare the mass while spring balances compare the weights. The mass is the same (invariant) irrespective of location but the weight will vary according to the local gravitational constant. Density will be invariant while specific weight will vary with variations in gravitational acceleration.

***Specific Gravity or Relative Density*:** The ratio of the density of the fluid to the density of water—usually 1000 kg/m^3 at a standard condition—is defined as Specific Gravity or Relative Density δ of fluids. This is a ratio and hence no dimension or unit is involved.

Example 1.1. *The weight of an object measured on ground level where $g_e = 9.81\ m/s^2$ is 35,000 N. Calculate its weight at the following locations (i) Moon, $g_m = 1.62\ m/s^2$ (ii) Sun, $g_s = 274.68\ m/s^2$ (iii) Mercury, $g_{me} = 3.53\ m/s^2$ (iv) Jupiter, $g_j = 26.0\ m/s^2$ (v) Saturn, $g_{sa} = 11.2\ m/s^2$ and (vi) Venus, $g_v = 8.54\ m/s^2$.*

Mass of the object, m_e = weight $\times (g_o/g) = 35{,}000 \times (1/9.81) = 3567.8$ kg

Weight of the object on a planet, $p = m_e \times (g_p/g_o)$ where m_e is the mass on earth, g_p is gravity on the planet and g_o has the usual meaning, force conversion constant.

Hence the weight of the given object on,

(*i*)	Moon	=	3567.8×1.62	= 5,780 N
(*ii*)	Sun	=	3567.8×274.68	= 9,80,000 N
(*iii*)	Mercury	=	3567.8×3.53	= 12,594 N
(*iv*)	Jupiter	=	3567.8×26.0	= 92,762 N
(*v*)	Saturn	=	3567.8×11.2	= 39,959 N
(*vi*)	Venus	=	3567.8×8.54	= 30,469 N

Note that the mass is constant whereas the weight varies directly with the gravitational constant. Also note that the ratio of weights will be the same as the ratio of gravity values.

1.6 VAPOUR AND GAS

When a liquid is heated under a constant pressure, first its temperature rises to the boiling point (defined as saturation temperature). Then the liquid begins to change its phase to the

gaseous condition, with molecules escaping from the surface due to higher thermal energy level. **When the gas phase is in contact with the liquid or its temperature is near the saturation condition it is termed as vapour.**

Vapour is in gaseous condition but it does not follow the gas laws. Its specific heats will vary significantly. Moderate changes in temperature may change its phase to the liquid state.

When the temperature is well above the saturation temperature, vapour begins to behave as a gas. It will also obey the characteristic equation for gases. Then the specific heat will be nearly constant.

1.7 CHARACTERISTIC EQUATION FOR GASES

The characteristic equation for gases can be derived from Boyle's law and Charles' law. Boyle's law states that at constant temperature the volume of a gas body will vary inversely with pressure. Charles' law states that at constant pressure, the temperature will vary inversely with volume. Combining these two, the characteristic equation for a system containing **m** kg of a gas can be obtained as

$$\mathbf{PV = mRT} \tag{1.7.1}$$

This equation when applied to a given system leads to the relation 1.7.2 applicable for all equilibrium conditions irrespective of the process between the states.

$$\mathbf{(P_1V_1/T_1) = (P_2V_2/T_2) = (P_3\ V_3/T_3) = (PV/T) = Constant} \tag{1.7.2}$$

In the SI system, the units to be used in the equation are Pressure, $P \rightarrow N/m^2$, volume, $V \rightarrow m^3$, mass, m $\rightarrow$ kg, temperature, $T \rightarrow$ K and gas constant, $R \rightarrow$ Nm/kgK or J/kgK (Note: K = (273 + °C), J = Nm).

This equation defines the equilibrium state for any gas body. For a specified gas body with mass **m,** if two properties like **P, V** are specified then the third property **T** is automatically specified by this equation. The equation can also be written as,

$$\mathbf{Pv = RT} \tag{1.7.3}$$

where $v = V/m$ or specific volume. **The value for R for air is 287 J/kgK.**

Application of Avagadro's hypothesis leads to the definition of a new volume measure called molal volume. This is the volume occupied by the molecular mass of any gas at standard temperature and pressure. This volume as per the above hypothesis will be the same for all gases at any given temperature and pressure. Denoting this volume as V_m and the pressure as P and the temperature as T,

For a gas a, $\mathbf{PV_m = M_a\ R_a\ T}$ (1.7.4)

For a gas b, $\mathbf{PV_m = M_b\ R_b\ T}$ (1.7.5)

As P, T and V_m are the same in both cases.

$$\mathbf{M_aR_a = M_bR_b = M \times R = Constant} \tag{1.7.6}$$

The product $\mathbf{M \times R}$ is called **Universal gas constant** and is denoted by the symbol $\Re$. Its numerical value in SI system is **8314 J/kg mole K**. For any gas the value of gas constant R is obtained by dividing universal gas constant by the molecular mass in kg of that gas. The gas constant R for any gas (in the SI system, J/kg K) can be calculated using,

$$\mathbf{R = 8314/M} \tag{1.7.7}$$

The characteristic equation for gases can be applied for all gases with slight approximations, and for practical calculations this equation is used in all cases.

Example 1.2. *A balloon is filled with 6 kg of hydrogen at 2 bar and 20°C.* ***What will be the diameter of the balloon*** *when it reaches an altitude where the pressure and temperature are 0.2 bar and –60° C. Assume that the pressure and temperature inside are the same as that at the outside at this altitude.*

The characteristic equation for gases $PV = mRT$ is used to calculate the initial volume,

$V_1 = [(m\ RT_1)/P_1]$, For hydrogen, molecular mass = 2, and so

$R_H = 8314/2 = 4157$ J/kgK, $\quad \therefore \quad V_1 = 6 \times 4157 \times (273 + 20)/2 \times 10^5 = 36.54\ m^3$

Using the general gas equation the volume after the balloon has reached the altitude, V_2 is calculated. $[(P_1V_1)/T_1] = [(P_2V_2)/T_2]$

$[(2 \times 10^5 \times 36.54)/(273+20)] = [(0.2) \times 10^5 \times V_2)/(273 - 60)]$ solving,

$V_2 =$ **265.63 m^3**, Considering the shape of the balloon as a sphere of radius r,

Volume $= (4/3)\ \pi\ r^3 = 265.63\ m^3$, solving

Radius, r = 3.99 m and **diameter of the balloon = 7.98 m**

(The pressure inside the balloon should be slightly higher to overcome the stress in the wall material)

1.8 VISCOSITY

A fluid is defined as a material which will continue to deform with the application of a shear force. However, different fluids deform at different rates when the same shear stress (force/area) is applied.

Viscosity is that property of a real fluid by virtue of which it offers resistance to shear force. Referring to Fig. 1.8.1, it may be noted that a force is required to move one layer of fluid over another.

For a given fluid the force required varies directly as the rate of deformation. As the rate of deformation increases the force required also increases. This is shown in Fig. 1.8.1 (*i*).

The force required to cause the same rate of movement depends on the nature of the fluid. The resistance offered for the same rate of deformation varies directly as the viscosity of the fluid. As viscosity increases the force required to cause the same rate of deformation increases. This is shown in Fig. 1.8.1 (*ii*).

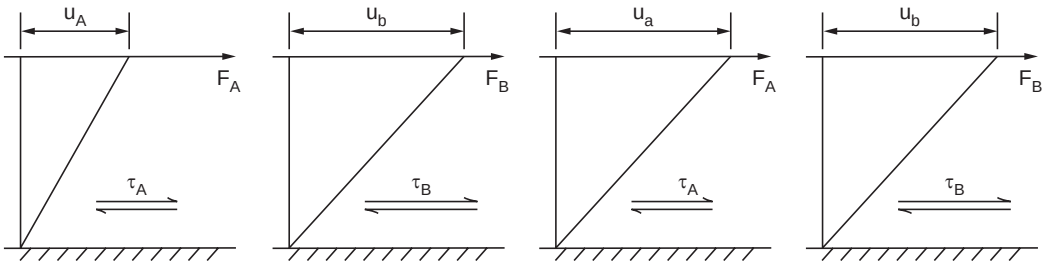
Newton's law of viscosity states that the shear force to be applied for a deformation rate of (du/dy) over an area A is given by,

$$\mathbf{F = \mu\ A\ (du/dy)} \tag{1.8.1}$$

or

$$\mathbf{(F/A) = \tau = \mu\ (du/dy) = \mu\ (u/y)} \tag{1.8.2}$$

where F is the applied force in N, A is area in m^2, du/dy is the velocity gradient (or rate of deformation), 1/s, perpendicular to flow direction, here assumed linear, and μ is the proportionality constant defined as the **dynamic or absolute viscosity** of the fluid.



$u_b > u_a$, $F_b > F_a$ $\mu_a = \mu_b$
(*i*) same fluid

$u_a = u_b$, $\mu_a < \mu_b$, $F_b > F_a$
(*ii*) same velocity

Figure 1.8.1 Concept of viscosity

The dimensions for dynamic viscosity μ can be obtained from the definition as Ns/m^2 or kg/ms. The first dimension set is more advantageously used in engineering problems. However, if the dimension of N is substituted, then the second dimension set, more popularly used by scientists can be obtained. The numerical value in both cases will be the same.

$$N = kg\ m/s^2\ ;\ \mu = (kg\ m/s^2)\ (s/m^2) = kg/ms$$

The popular unit for viscosity is Poise named in honour of Poiseuille.

$$\textbf{Poise} = 0.1\ \textbf{Ns/m}^2 \tag{1.8.3}$$

Centipoise (cP) is also used more frequently as,

$$\textbf{cP} = 0.001\ \textbf{Ns/m}^2 \tag{1.8.3}a$$

For water the viscosity at 20°C is nearly 1 cP. The ratio of dynamic viscosity to the density is defined as kinematic viscosity, ν, having a dimension of m^2/s. Later it will be seen to relate to momentum transfer. Because of this kinematic viscosity is also called momentum diffusivity. The popular unit used is stokes (in honour of the scientist Stokes). Centistoke is also often used.

$$\textbf{1 stoke} = 1\ \textbf{cm}^2\textbf{/s} = 10^{-4}\ \textbf{m}^2\textbf{/s} \tag{1.8.3}b$$

Of all the fluid properties, viscosity plays a very important role in fluid flow problems. The velocity distribution in flow, the flow resistance etc. are directly controlled by viscosity. In the study of fluid statics (*i.e.,* when fluid is at rest), viscosity and shear force are not generally involved. In this chapter problems are worked assuming linear variation of velocity in the fluid filling the clearance space between surfaces with relative movement.

Example 1.3. *The space between two large inclined parallel planes is 6mm and is filled with a fluid. The planes are inclined at 30° to the horizontal. A small thin square plate of 100 mm side slides freely down parallel and midway between the inclined planes with a constant velocity of 3 m/s due to its weight of 2N.* ***Determine the viscosity of the fluid.***

The vertical force of 2 N due to the weight of the plate can be resolved along and perpendicular to the inclined plane. The force along the inclined plane is equal to the drag force on both sides of the plane due to the viscosity of the oil.

Force due to the weight of the sliding plane along the direction of motion

$$= 2 \sin 30 = 1N$$

Viscous force, F = (A × 2) × μ × (*du*/*dy*) (both sides of plate). Substituting the values,

$$1 = \mu \times [(0.1 \times 0.1 \times 2)] \times [(3 - 0)/6/(2 \times 1000)\}]$$

Solving for viscosity, **μ = 0.05 Ns/m² or 0.5 Poise**

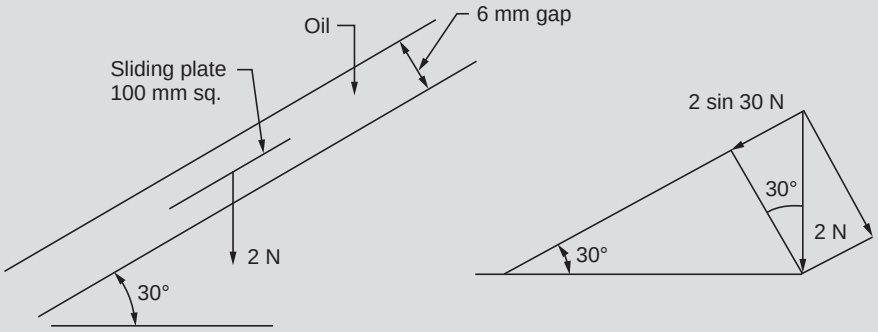


Figure Ex. 1.3

Example 1.4. *The velocity of the fluid filling a hollow cylinder of radius 0.1 m varies as u = 10 [1 – (r/0.1)²] m/s along the radius r. The viscosity of the fluid is 0.018 Ns/m². For 2 m length of the cylinder,* ***determine the shear stress and shear force over cylindrical layers of fluid*** *at r = 0 (centre line), 0.02, 0.04, 0.06 0.08 and 0.1 m (wall surface.)*

Shear stress = μ (*du*/*dy*) or μ (*du*/*dr*), *u* = 10 [1 – (*r*/0.1)²] m/s

$\therefore \quad du/dr = 10\ (-2r/0.1^2) = -2000\ r$

The – ve sign indicates that the force acts in a direction opposite to the direction of velocity, *u*. Shear stress = 0.018 × 2000 *r* = 36 rN/m²

Shear force over 2 m length = shear stress × area over 2*m*

$$= 36r \times 2\pi rL = 72\ \pi r^2 \times 2 = 144\ \pi r^2$$

The calculated values are tabulated below:

Radius, m	Shear stress, N/m²	Shear force, N	Velocity, m/s
0.00	0.00	0.00	0.00
0.02	0.72	0.18	9.60
0.04	1.44	0.72	8.40
0.06	2.16	1.63	6.40
0.08	2.88	2.90	3.60
0.10	3.60	4.52	0.00

Example 1.5. *The 8 mm gap between two large vertical parallel plane surfaces is filled with a liquid of dynamic viscosity 2 × 10⁻² Ns/m². A thin sheet of 1 mm thickness and 150 mm × 150 mm size, when dropped vertically between the two plates attains a steady velocity of 4 m/s.* ***Determine weight of the plate.*** *Assume that the plate moves centrally.*

$F = \tau\ (A \times 2) = \mu \times (du/dy)\ (A \times 2)$ = weight of the plate.

Substituting the values, *dy* = [(8 – 1)/(2 × 1000)] m and *du* = 4 m/s

$F = 2 \times 10^{-2}\ [4/\{(8 - 1)/(2 \times 1000)\}]\ [0.15 \times 0.15 \times 2]$ = **1.02 N (weight of the plate)**

Example 1.6. ***Determine the resistance offered to the downward sliding*** *of a shaft of 400 mm dia and 0.1 m length by the oil film between the shaft and a bearing of ID 402 mm. The kinematic viscosity is 2.4 × 10^{-4} m^2/s and density is 900 kg/m^3. The shaft is to move centrally and axially at a constant velocity of 0.1 m/s.*

Force, F opposing the movement of the shaft = shear stress × area

$F = \mu\,(du/dy)\,(\pi \times D \times L)$

$\mu = 2.4 \times 10^{-4} \times 900$ Ns/m^2, du = 0.1 m/s, L = 0.1 m, D= 0.4 m

dy = (402 – 400)/(2 × 1000)m, Substituting,

$F = 2.4 \times 10^{-4} \times 900 \times \{(0.1 - 0)/[(402 - 400)/(2 \times 1000)]\}\,(\pi \times 0.4 \times 0.1) =$ **2714 N**

1.8.1 Newtonian and Non Newtonian Fluids

An ideal fluid has zero viscosity. Shear force is not involved in its deformation. An ideal fluid has to be also incompressible. Shear stress is zero irrespective of the value of du/dy. Bernoulli equation can be used to analyse the flow.

Real fluids having viscosity are divided into two groups namely Newtonian and non Newtonian fluids. In Newtonian fluids a linear relationship exists between the magnitude of the applied shear stress and the resulting rate of deformation. It means that the proportionality parameter (in equation 1.8.2, $\tau = \mu\,(du/dy)$), viscosity, μ is constant in the case of Newtonian fluids (other conditions and parameters remaining the same). The viscosity at any given temperature and pressure is constant for a Newtonian fluid and is independent of the rate of deformation. The characteristics is shown plotted in Fig. 1.8.2. Two different plots are shown as different authors use different representations.

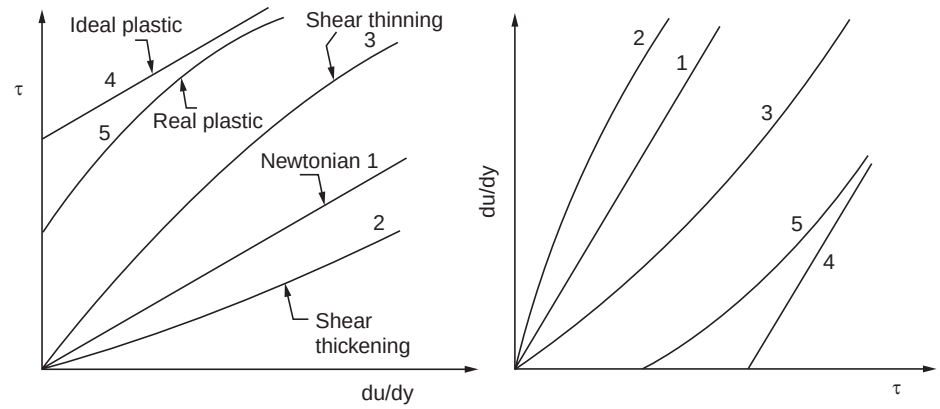


Figure 1.8.2 Rheological behaviour of fluids

Non Newtonian fluids can be further classified as simple non Newtonian, ideal plastic and shear thinning, shear thickening and real plastic fluids. In non Newtonian fluids the viscosity will vary with variation in the rate of deformation. Linear relationship between shear stress and rate of deformation (du/dy) does not exist. In plastics, up to a certain value of applied shear stress there is no flow. After this limit it has a constant viscosity at any given temperature. In shear thickening materials, the viscosity will increase with (du/dy) deformation rate. In shear thinning materials viscosity will decrease with du/dy. Paint, tooth paste, printers ink

are some examples for different behaviours. These are also shown in Fig. 1.8.2. Many other behaviours have been observed which are more specialised in nature. The main topic of study in this text will involve only Newtonian fluids.

1.8.2 Viscosity and Momentum Transfer

In the flow of liquids and gases molecules are free to move from one layer to another. When the velocity in the layers are different as in viscous flow, the molecules moving from the layer at lower speed to the layer at higher speed have to be accelerated. Similarly the molecules moving from the layer at higher velocity to a layer at a lower velocity carry with them a higher value of momentum and these are to be slowed down. Thus the molecules diffusing across layers transport a net momentum introducing a shear stress between the layers. The force will be zero if both layers move at the same speed or if the fluid is at rest.

When cohesive forces exist between atoms or molecules these forces have to be overcome, for relative motion between layers. A shear force is to be exerted to cause fluids to flow.

Viscous forces can be considered as the sum of these two, namely, the force due to momentum transfer and the force for overcoming cohesion. In the case of liquids, the viscous forces are due more to the breaking of cohesive forces than due to momentum transfer (as molecular velocities are low). In the case of gases viscous forces are more due to momentum transfer as distance between molecules is larger and velocities are higher.

1.8.3 Effect of Temperature on Viscosity

When temperature increases the distance between molecules increases and the cohesive force decreases. So, viscosity of liquids decrease when temperature increases.

In the case of gases, the contribution to viscosity is more due to momentum transfer. As temperature increases, more molecules cross over with higher momentum differences. Hence, in the case of gases, viscosity increases with temperature.

1.8.4 Significance of Kinematic Viscosity

Kinematic viscosity, $\nu = \mu/\rho$, The unit in SI system is m^2/s.

$$(Ns/m^2)\ (m^3/\ kg) = [(kg.m/s^2)\ (s/m^2)]\ [m^3/kg] = m^2/s$$

Popularly used unit is stoke $(cm^2/s) = 10^{-4}\ m^2/s$ named in honour of Stokes.

Centi stoke is also popular $= 10^{-6}\ m^2/s$.

Kinematic viscosity represents momentum diffusivity. It may be explained by modifying equation 1.8.2

$$\tau = \mu\ (du/dy) = (\mu/\rho) \times \{d\ (\rho u/dy)\} = \nu \times \{d\ (\rho u/dy)\} \qquad (1.8.4)$$

$d\ (\rho u/dy)$ represents momentum flux in the y direction.

So, $(\mu/\rho) = \nu$ kinematic viscosity gives the rate of momentum flux or momentum diffusivity.

With increase in temperature kinematic viscosity decreases in the case of liquids and increases in the case of gases. For liquids and gases absolute (dynamic) viscosity is not influenced significantly by pressure. But kinematic viscosity of gases is influenced by pressure due to change in density. In gas flow it is better to use absolute viscosity and density, rather than tabulated values of kinematic viscosity, which is usually for 1 atm.

1.8.5 Measurement of Viscosity of Fluids

1.8.5.1 Using Flow Through Orifices

In viscosity determination using Saybolt or Redwood viscometers, the time for the flow through a standard orifice, of a fixed quantity of the liquid kept in a cup of specified dimensions is measured in seconds and the viscosity is expressed as Saybolt seconds or Redwood seconds. The time is converted to poise by empirical equations. These are the popular instruments for industrial use. The procedure is simple and a quick assessment is possible. However for design purposes viscosity should be expressed in the standard units of Ns/m^2.

1.8.5.2 Rotating Cylinder Method

The fluid is filled in the interspace between two cylinders. The outer cylinder is rotated keeping the inner cylinder stationary and the reaction torque on the inner cylinder is measured using a torsion spring. Knowing the length, diameter, film thickness, rpm and the torque, the value of viscosity can be calculated. Refer Example 1.7.

Example 1.7. *In a test set up as in figure to measure viscosity, the cylinder supported by a torsion spring is 20 cm in dia and 20 cm long. A sleeve surrounding the cylinder rotates at 900 rpm and the torque measured is 0.2 Nm. If the film thickness between the cylinder and sleeve is 0.15 mm,* ***determine the viscosity of the oil.***

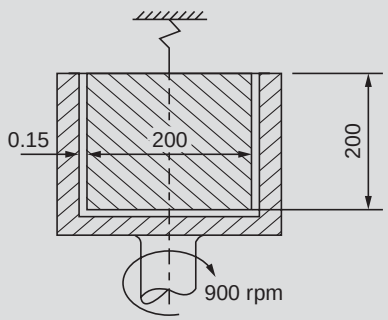


Figure Ex. 1.7 Viscosity test setup

The total torque is given by the sum of the torque due to the shear forces on the cylindrical surface and that on the bottom surface.

Torque due to shear on the cylindrical surface (eqn 1.9.1a), $T_s = \mu\pi^2 NLR^3/15\ h$,

Torque on bottom surface (eqn 1.9.3), $T_b = \mu\pi^2 NR^4/60\ h$

Where h is the clearance between the sleeve and cylinder and also base and bottom. In this case both are assumed to be equal. Total torque is the sum of values given by the above equations. In case the clearances are different then h_1 and h_2 should be used.

Total torque = $(\mu\ \pi^2 NR^3/\ 15.h)\ \{L + (R/4)\}$, substituting,

$$0.2 = [(\mu \times \pi^2\ 900 \times 0.1^3)/(15 \times 0.0015)] \times [0.2 + (0.1/4)]$$

Solving for viscosity, **$\mu = 0.00225\ Ns/m^2$ or 2.25 cP.**

This situation is similar to that in a Foot Step bearing.

1.8.5.3 Capillary Tube Method

The time for the flow of a given quantity under a constant head (pressure) through a tube of known diameter d, and length L is measured or the pressure causing flow is maintained constant and the flow rate is measured.

$$\Delta P = (32\ \mu\ VL)/d^2 \qquad (1.8.5)$$

This equation is known as Hagen-Poiseuille equation. The viscosity can be calculated using the flow rate and the diameter. Volume flow per second, $Q = (\pi d^2/4)\, V$. Q is experimentally measured using the apparatus. The head causing flow is known. Hence μ can be calculated.

1.8.5.4 Falling Sphere Method

A small polished steel ball is allowed to fall freely through the liquid column. The ball will reach a uniform velocity after some distance. At this condition, gravity force will equal the viscous drag. The velocity is measured by timing a constant distance of fall.

$$\mu = 2r^2g\ (\rho_1 - \rho_2)/9V \qquad (1.8.6)$$

(μ will be in poise. 1 poise = 0.1 Ns/m^2)

where r is the radius of the ball, V is the terminal velocity (constant velocity), ρ_1 and ρ_2 are the densities of the ball and the liquid. This equation is known as **Stokes equation.**

Example 1.8. *Oil flows at the rate of 3 l/s through a pipe of 50 mm diameter. The pressure difference across a length of 15 m of the pipe is 6 kPa.* ***Determine the viscosity of oil flowing through the pipe.***

Using Hagen-Poiesuille equation-1.8.5 , $\Delta P = (32\ \mu uL)/d^2$

$u = Q/(\pi d^2/4) = 3 \times 10^{-3}/(\pi \times 0.05^2/4) = 1.53$ m/s

$\mu = \Delta P \times d^2/32uL = (6000 \times 0.05^2)/(32 \times 1.53 \times 15) =$ **0.0204 Ns m^2**

Example 1.9. *A steel ball of 2 mm dia and density 8000 kg/m^3 dropped into a column of oil of specific gravity 0.80 attains a terminal velocity of 2mm/s.* ***Determine the viscosity of the oil****. Using Stokes equation, 1.8.6*

$\mu = 2r^2g\ (\rho_1 - \rho_2)/9u$

$= 2 \times (0.002/2)^2 \times 9.81 \times (8000 - 800)/(9 \times 0.002) = 7.85$ **Ns/m^2.**

1.9 APPLICATION OF VISCOSITY CONCEPT

1.9.1 Viscous Torque and Power—Rotating Shafts

Refer Figure 1.9.1

Shear stress, $\tau = \mu\ (du/dy) = \mu\ (u/y)$, as linearity is assumed

$u = \pi DN/60$, $y = h$, clearance in m

$\tau = \mu\ (\pi DN/60h)$, Tangential force $= \tau \times A$, $A = \pi DL$

Torque, T = tangential force $\times\ D/2 = \mu\ (\pi DN/60h)\ (\pi DL)\ (D/2)$

substituting $T = \mu\ \pi^2 NLD^3/\ 120\ h$ (1.9.1)

If radius is used, $T = \mu\pi^2 NLR^3/15\ h$ (1.9.1a)

As power, $P = 2\pi NT/60$,

$P = \mu\pi^3 N^2 LR^3/450\ h$ (1.9.2)

For equations 1.9.1 and 1.9.2, proper units are listed below:

L, R, D, h should be in meter and N in rpm. Viscosity μ should be in Ns/m^2 (or Pas). The torque will be obtained in Nm and the power calculated will be in W.

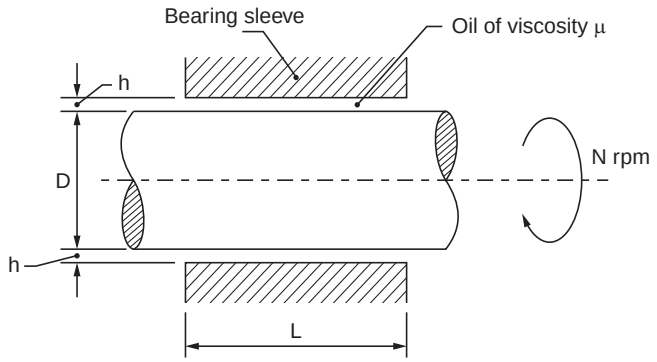


Figure 1.9.1 Rotating Shaft in Bearing

Note: Clearance h is also the oil film thickness in bearings. End effects are neglected. Linear velocity variation is assumed. Axial location is assumed.

Example 1.10. *Determine the power required to run a 300 mm dia shaft at 400 rpm in journals with uniform oil thickness of 1 mm. Two bearings of 300 mm width are used to support the shaft. The dynamic viscosity of oil is 0.03 Pas. (Pas = (N/m^2) × s).*

Shear stress on the shaft surface = $\tau = \mu\ (du/dy) = \mu(u/y)$

$u = \pi\ DN/60 = \pi \times 0.3 \times 400/60 = 6.28$ m/s

$\tau = 0.03\ \{(6.28 - 0)/\ 0.001\} = 188.4$ N/m^2

Surface area of the two bearings, $A = 2\ \pi\ DL$

Force on shaft surface $= \tau \times A = 188.4 \times (2 \times \pi \times 0.3 \times 0.3) = 106.6\ N$

Torque $= 106.6 \times 0.15 = 15.995$ Nm

Power required $= 2\ \pi\ NT/60 = 2 \times \pi \times 400 \times 15.995/60 =$ **670 W.**

(check using eqn. 1.9.2, $P = \mu\ \pi^3\ N^2LR^3/450$ h = 669.74 W)

1.9.2 Viscous Torque—Disk Rotating Over a Parallel Plate

Refer Figure 1.9.2.

Consider an annular strip of radius **r** and width **dr** shown in Figure 1.9.2. The force on the strip is given by,

$$F = A\mu\ (du/dy) = A\ \mu\ (u/y)$$

(as y is small linear velocity variation can be assumed)

$$u = 2\ \pi rN/60,\ y = h,\ A = 2\pi r\ dr$$

Torque = Force × radius, substituting the above values

torque dT on the strip is, $dT = 2\pi r\ dr\ \mu(2\pi rN/60h)r$

$$dT = 2\pi r.dr.\mu.\ 2\pi rN.r/60.h = [\mu\pi^2N/15.h]r^3dr$$

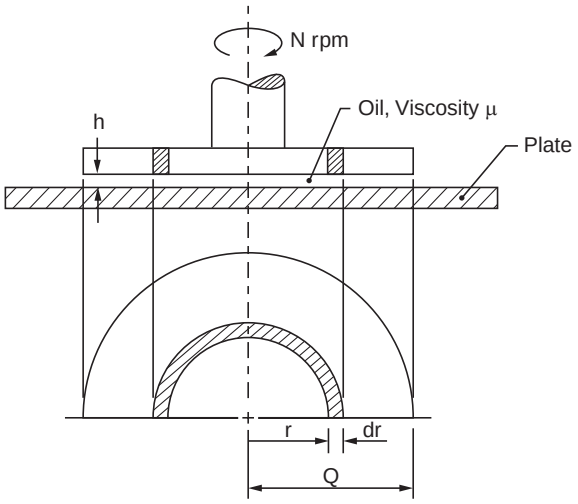


Figure 1.9.2 Rotating disk

Integrating the expression from centre to edge *i.e.*, 0 to R,

$$\mathbf{T = \mu\pi^2 NR^4/60\ h} \qquad (1.9.3)$$

If diameter is used, $R^4 = (1/16)D^4$

$$\mathbf{T = \mu\pi^2 ND^4/960\ h} \qquad (1.9.3a)$$

The power required, $P = 2\pi NT/60$

$$\mathbf{P = \mu\pi^3 N^2 R^4/1800\ h} \qquad (1.9.4)$$

use R in metre, N in rpm and μ in Ns/m^2 or Pa s.

For an annular area like a collar the integration limits are $\mathbf{R_o}$ and $\mathbf{R_i}$ and the torque is given by

$$\mathbf{T = \mu\pi^2 N(R_o^4 - R_i^4)/60\ h} \qquad (1.9.5)$$

Power, $$\mathbf{P = \mu\pi^3 N^2(R_o^4 - R_i^4)/1800\ h} \qquad (1.9.6)$$

Example 1.11. Determine the oil film thickness *between the plates of a collar bearing of 0.2 m ID and 0.3 m OD transmitting power, if 50 W was required to overcome viscous friction while running at 700 rpm. The oil used has a viscosity of 30 cP.*

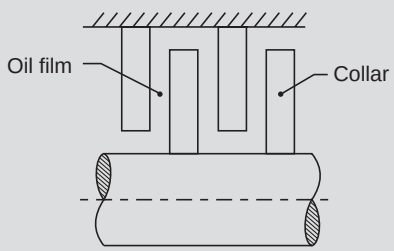


Figure Ex. 1.11

Power = $2\pi NT/60$ W, substituting the given values,

$50 = 2\pi \times 700 \times T/60$, Solving torque,

T = 0.682 Nm

This is a situation where an annular surface rotates over a flat surface. Hence, using equation 1.9.5, Torque, $T = \mu\pi^2 N\,(R_O^4 - R_i^4)/60.h$

$\mu = 30$ cP $= 30 \times .0001$ Ns/m^2, substituting the values,

$0.682 = (30 \times 0.0001) \times \pi^2 \times 700 \times (0.15^4 - 0.1^4)/60 \times h$

$\therefore$ **h = 0.000206m = 0.206 mm**

1.9.3 Viscous Torque—Cone in a Conical Support

Considering a small element between radius r and $r + dr$, as shown in figure 1.9.3. The surface width of the element in contact with oil is

$$dx = dr/\sin\theta$$

The surface area should be calculated with respect to centre O as shown in figure—the point where the normal to the surface meets the axis—or the centre of rotation, the length OA being $r/\cos\theta$.

Hence contact surface area = $2\pi r.dr/\sin\theta.\cos\theta$.

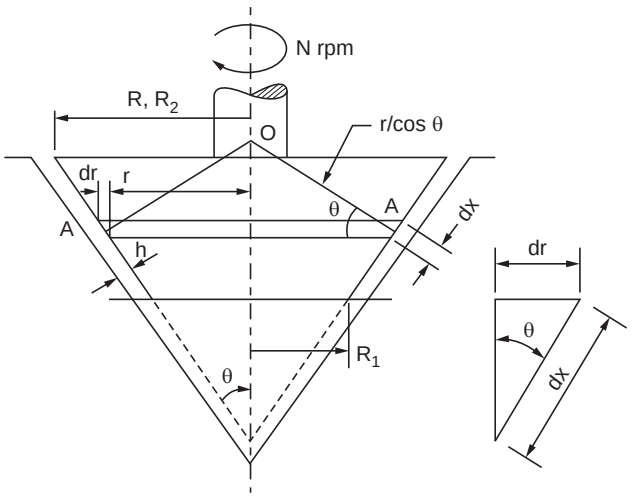


Figure 1.9.3 Rotating cone or conical bearing

The velocity along the surface is $(2\pi rN/60).\cos\theta$ and the film thickness is h.

$$F = A\mu\,(du/dy) = \{(2\pi r./\sin\theta.\cos\theta)\}\,\mu(2\pi rN.\cos\theta/60)\,(1/h)$$

$$F = (\pi^2\mu Nr^2dr)/(15.h.\sin\theta), \qquad \text{Torque} = F.r$$

Torque on element, $dT = \pi^2\mu Nr^2dr.r/15.h.\sin\theta = (\pi\mu N/15\ \text{h}\ \sin\theta)r^3\,dr$

Integrating between $r = 0$ and $r = R$

$$\mathbf{T = \pi^2\,\mu NR^4/60.h\ \sin\theta} \qquad (1.9.7)$$

Using μ in Ns/m^2, h and R in metre the torque will be in N.m. When semicone angle $\theta = 90°$, this reduces to the expression for the disk—equation 1.9.3. For contact only between R_1 and R_2.

$$\mathbf{T = \mu\pi^2\,N(R_2^4 - R_1^4)/60.h.\ \sin\theta} \qquad (1.9.8)$$

Power required, $$\mathbf{P = 2\pi NT/60 = \mu^3N^2[R_2^4 - R_1^4]/1800\ h\ \sin\theta} \qquad (1.9.9)$$

Exmaple 1.12. ***Determine the power required*** *to overcome viscous friction for a shaft running at 700 rpm fitted with a conical bearing. The inner and outer radius of the conical bearing are 0.3 m and 0.5 m. The height of the cone is 0.3 m. The 1.5 mm uniform clearance between the bearing and support is filled with oil of viscosity 0.02 Ns/m².*

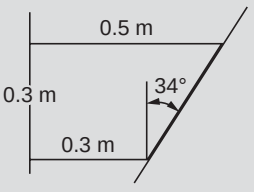


Figure Ex. 1.12

Equation 1.9.8 is applicable in this case.

$$\tan\theta = (0.5 - 0.3)/0.3 = 0.667, \quad \therefore \quad \theta = 34°$$

$T = \pi^2\mu\ N\ (R_o^4 - R_i^4)/\ 60.\ \text{h.sin}\ \theta$, substituting the values

$$T = \pi^2 \times 0.03 \times 700 \times (0.5^4 - 0.3^4)/60 \times 0.0015 \times \sin 34 = 149.36 \text{ Nm}$$

Power required $= 2\pi NT/60 = 2\pi \times 700 \times 149.36/60 =$ **10948 W**

Check using equation 1.9.9 also,

$$P = \mu \times \pi^3 \times 700^2 \times [0.5^4 - 0.3^4]/ [1800 \times 0.0015 \times \sin 34] = 10948 \text{ W}.$$

Note the high value of viscosity

1.10 SURFACE TENSION

Many of us would have seen the demonstration of a needle being supported on water surface without it being wetted. This is due to the surface tension of water.

All liquids exhibit a free surface known as meniscus when in contact with vapour or gas. Liquid molecules exhibit cohesive forces binding them with each other. The molecules below the surface are generally free to move within the liquid and they move at random. When they reach the surface they reach a dead end in the sense that no molecules are present in great numbers above the surface to attract or pull them out of the surface. So they stop and return back into the liquid. A thin layer of few atomic thickness at the surface formed by the cohesive bond between atoms slows down and sends back the molecules reaching the surface. This cohesive bond exhibits a tensile strength for the surface layer and this is known as surface tension. Force is found necessary to stretch the surface.

Surface tension may also be defined as the work in Nm/m^2 or N/m required to create unit surface of the liquid. The work is actually required for pulling up the molecules with lower energy from below, to form the surface.

Another definition for surface tension is the force required to keep unit length of the surface film in equilibrium (N/m). The formation of bubbles, droplets and free jets are due to the surface tension of the liquid.

1.10.1 Surface Tension Effect on Solid-Liquid Interface

In liquids cohesive forces between molecules lead to surface tension. The formation of droplets is a direct effect of this phenomenon. So also the formation of a free jet, when liquid flows out of an orifice or opening like a tap. The pressure inside the droplets or jet is higher due to the surface tension.

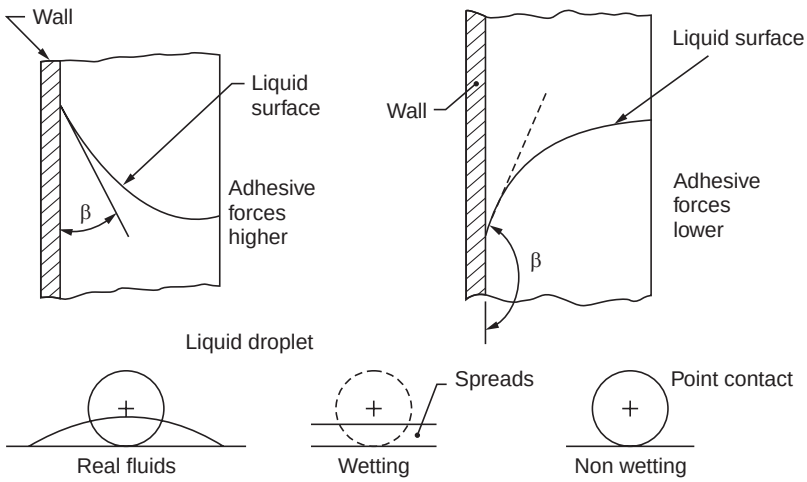


Figure 1.10.1 Surface tension effect at solid-liquid interface

Liquids also exhibit adhesive forces when they come in contact with other solid or liquid surfaces. At the interface this leads to the liquid surface being moved up or down forming a curved surface. When the adhesive forces are higher the contact surface is lifted up forming a concave surface. Oils, water etc. exhibit such behaviour. These are said to be surface wetting. When the adhesive forces are lower, the contact surface is lowered at the interface and a convex surface results as in the case of mercury. Such liquids are called nonwetting. These are shown in Fig. 1.10.1.

The angle of contact "β" defines the concavity or convexity of the liquid surface. It can be shown that if the surface tension at the solid liquid interface (due to adhesive forces) is σ_{s1} and if the surface tension in the liquid (due to cohesive forces) is σ_{11} then

$$\cos \beta = [(2\sigma_{s1}/\sigma_{11}) - 1] \quad (1.10.1)$$

At the surface this contact angle will be maintained due to molecular equilibrium. The result of this phenomenon is capillary action at the solid liquid interface. The curved surface creates a pressure differential across the free surface and causes the liquid level to be raised or lowered until static equilibrium is reached.

Example 1.13. ***Determine the surface tension*** *acting on the surface of a vertical thin plate of 1m length when it is lifted vertically from a liquid using a force of 0.3N.*

Two contact lines form at the surface and hence, Force = 2 × 1 × Surface tension

0.3 = 2 × 1 × Surface tension. Solving, Surface tension, **σ = 0.15 N/m.**

1.10.2 Capillary Rise or Depression

Refer Figure 1.10.2.

Let D be the diameter of the tube and β is the contact angle. The surface tension forces acting around the circumference of the tube = $\pi \times D \times \sigma$.

The vertical component of this force = $\pi \times D \times \sigma \times \cos \beta$

This is balanced by the fluid column of height, h, the specific weight of liquid being γ.

Equating, $h \times \gamma \times A = \pi \times D \times \sigma \cos \beta$, $A = \pi D^2/4$ and so

$$\mathbf{h = (4\pi \times D \times \sigma \times \cos \beta)/(\gamma \pi D^2) = (4\sigma \times \cos \beta)/\rho g D} \quad (1.10.2)$$

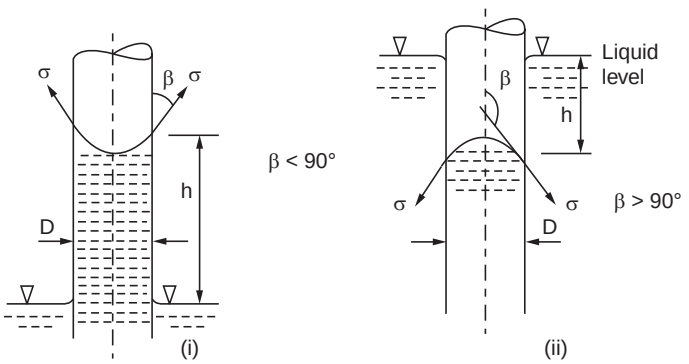


Figure 1.10.2 Surface tension, (*i*) capillary rise (*ii*) depression

This equation provides the means for calculating the capillary rise or depression. The sign of cos β depending on β > 90 or otherwise determines the capillary rise or depression.

Example 1.14. ***Determine the capillary depression*** *of mercury in a 2 mm ID glass tube. Assume* $\sigma = 0.5$ *N/m and* $\beta = 130°$.

Specific weight of mercury, $\gamma = 13600 \times 9.81$ N/m^3

Using eqn. 1.10.2, $h = (4\ \sigma \times \cos\beta)/\rho g/D$

$= (4 \times 0.5 \times \cos 130)/(13600 \times 9.81 \times 0.002)$

$= -4.82 \times 10^{-3}$ m $= -$ **4.82 mm**

Example1.15. *In a closed end single tube manometer, the height of mercury column above the mercury well shows 757 mm against the atmospheric pressure. The ID of the tube is 2 mm. The contact angle is 135°.* ***Determine the actual height*** *representing the atmospheric pressure if surface tension is 0.48 N/m. The space above the column may be considered as vacuum.*

Actual height of mercury column = Mercury column height + Capillary depression

Specific weight of mercury $= \rho g = 13600 \times 9.81$ N/m^3

Capillary depression, $\mathbf{h} = (4\ \sigma \times \cos\beta)/\gamma D$

$= (4 \times 0.48 \times \cos 135)/(0.002 \times 13600 \times 9.81)$

$= -5.09 \times 10^{-3}$m **= – 5.09 mm (depression)**

Corrected height of mercury column = 757 + 5.09 = **762.09 mm**

1.10.3 Pressure Difference Caused by Surface Tension on a Doubly Curved Surface

Consider the small doubly curved element with radius r_1 and included angle $d\phi$ in one direction and radius r_2 and $d\theta$ in the perpendicular direction referred to the normal at its center.

For equilibrium the components of the surface tension forces along the normal should be equal to the pressure difference.

The sides are $r_1\, d\phi$ and $r_2\, d\theta$ long. Components are $\sigma r_1 \sin(d\theta/2)$ from θ direction sides and $\sigma r_2 \sin(d\phi/2)$ from the φ direction sides.

$$2\sigma r_1 d\phi \sin(d\theta/2) + 2\sigma r_2\, d\theta \sin(d\phi/2) = (p_i - p_o) r_1 r_2\, d\theta d\phi$$

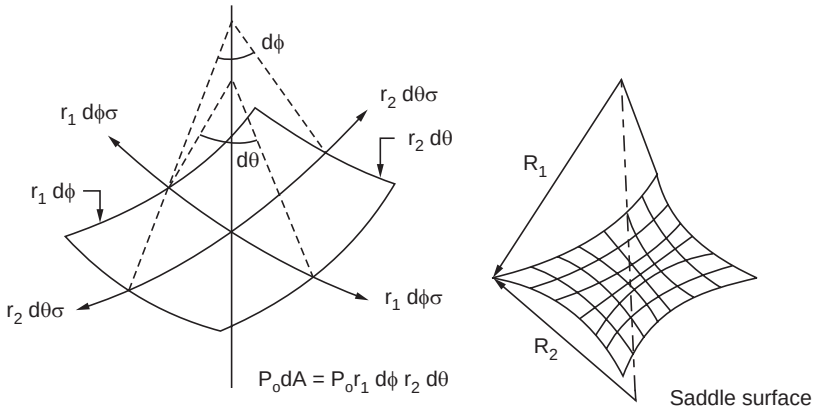


Figure 1.10.3 Pressure difference, doubly curved surface

For small values of angles, sin θ = θ, in radians. Cancelling the common terms

$$\sigma [r_1 + r_2] = (p_i - p_o) \times r_1 r_2. \text{ Rearranging,} \quad (1.10.3)$$

$$\mathbf{(p_i - p_o) = [(1/r_1) + (1/r_2)] \times \sigma}$$

For a spherical surface, $r_1 = r_2 = R$

So, $$\mathbf{(p_i - p_o) = 2\sigma/R} \quad (1.10.4)$$

where R is the radius of the sphere.

For cylindrical shapes one radius is infinite, and so

$$\mathbf{(p_i - p_o) = \sigma/R} \quad (1.10.4a)$$

These equations give the pressure difference between inside and outside of droplets and free jets of liquids due to surface tension. The pressure inside air bubbles will be higher compared to the outside pressure. The pressure inside a free jet will be higher compared to the outside.

The pressure difference can be made zero for a doubly curved surface if the curvature is like that of a saddle (one positive and the other negative). This situation can be seen in the jet formed in tap flow where internal pressure cannot be maintained.

1.10.4 Pressure Inside a Droplet and a Free Jet

Refer Figure 1.10.4.

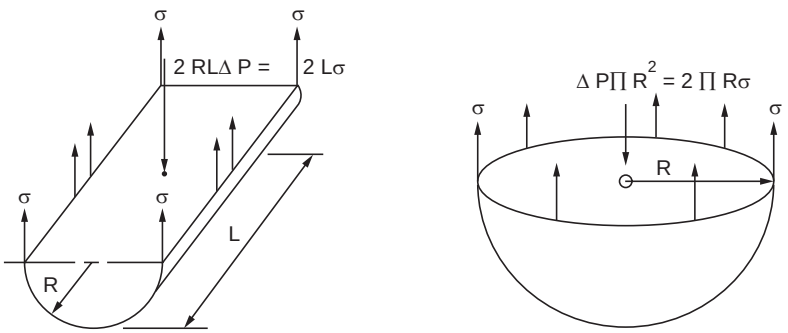


Figure 1.10.4 Surface tension effects on bubbles and free jets

Considering the sphere as two halves or hemispheres of diameter D and considering the equilibrium of these halves,

Pressure forces = Surface tension forces, $(p_i - p_o)(\pi D^2/4) = \sigma \times \pi \times D$

$$\mathbf{(p_i - p_o) = 4(\sigma/D) = 2(\sigma/R)} \quad (1.10.5)$$

Considering a cylinder of length L and diameter D and considering its equilibrium, taking two halves of the cylinder.

pressure force = $DL(p_i - p_o)$, surface tension force = $2\sigma L$

$$\mathbf{(p_i - p_o) = 2\ (\sigma/D) = (\sigma/R)} \quad (1.10.6)$$

Example 1.16. ***Determine the pressure difference*** *across a nozzle if diesel is sprayed through it with an average diameter of 0.03mm.* The surface tension is 0.04N/m.
The spray is of cylindrical shape

$$\mathbf{P = } \sigma/R = 0.04/(0.03 \times 10^{-3}/2) = 2666.67 \text{ N/m}^2 = \mathbf{2.67 \text{ kpa}}$$

Example 1.17. ***Calculate the surface tension*** *if the pressure difference between the inside and outside of a soap bubble of 3mm dia is 18 N/m^2.*

Referring equation 1.10.5, $\Delta P = 4\sigma/D$

Surface tension, $\sigma = \Delta P \times D/4 = 18 \times (0.003/4) = \mathbf{0.0135 \text{ N/m}}$

1.11 COMPRESSIBILITY AND BULK MODULUS

Bulk modulus, E_v is defined as the ratio of the change in pressure to the rate of change of volume due to the change in pressure. It can also be expressed in terms of change of density.

$$\mathbf{E_v = - dp/(dv/v) = dp/(d\rho/\rho)} \qquad (1.11.1)$$

where **dp** is the change in pressure causing a change in volume **dv** when the original volume was **v**. The unit is the same as that of pressure, obviously. Note that $dv/v = - d\rho/\rho$.

The negative sign indicates that if *dp* is positive then *dv* is negative and vice versa, so that the bulk modulus is always positive (N/m^2). The symbol used in this text for bulk modulus is E_v (K is more popularly used).

This definition can be applied to liquids as such, without any modifications. In the case of gases, the value of compressibility will depend on the process law for the change of volume and will be different for different processes.

The bulk modulus for liquids depends on both pressure and temperature. The value increases with pressure as *dv* will be lower at higher pressures for the same value of *dp*. With temperature the bulk modulus of liquids generally increases, reaches a maximum and then decreases. For water the maximum is at about 50°C. The value is in the range of 2000 MN/m^2 or 2000×10^6 N/m^2 or about 20,000 atm. Bulk modulus influences the velocity of sound in the medium, which equals $(g_o \times E_v/\rho)^{0.5}$.

Example 1.18. ***Determine the bulk modulus of*** *a liquid whose volume decreases by 4% for an increase in pressure of 500×10^5 pa. Also* ***determine the velocity of sound*** *in the medium if the density is 1000 kg/m^3.*

Bulk modulus is defined as $E_v = - dp/(dv/v)$, substituting the values,

$$\boldsymbol{E_v} = (-500 \times 10^5)/(-4/100) = \mathbf{1.25 \times 10^9 \text{ N/m}^2}$$

Velocity of sound c is defined as $= (g_o \times E_v/\rho)^{0.5}$

$$\therefore \quad \mathbf{c} = [1 \times 1.25 \times 10^9/100]^{0.05} = \mathbf{1118 \text{ m/s.}}$$

Example 1.19. *The pressure of water in a power press cylinder is released from 990 bar to 1 bar isothermally. If the average value of bulk modulus for water in this range is 2430×10^6 N/m^2.* ***What will be the percentage increase in specific volume?***

The definition of bulk modulus, $E_v = - dp/(dv/v)$ is used to obtain the solution. Macroscopically the above equation can be modified as

$$E_v = - \{P_1 - P_2\}\{(v_2 - v_1)/v_1\}, \text{ Rearranging,}$$

Change in specific volume $= (v_2 - v_1)/v_1 = -(P_2 - P_1)/E_v$

$= (990 \times 10^5 - 1 \times 10^5)/2430 \times 10^6 = 0.0407$

% change in specific volume = 4.07%

Example 1.20. *Density of sea water at the surface was measured as 1040 kg/m³ at an atmospheric pressure of 1 bar. At certain depth in water, the density was found to be 1055 kg/m³.* ***Determine the pressure at that point.*** *The bulk modulus is 2290 × 10⁶ N/m².*

Bulk modulus, $E_v = -dp/(dv/v) = -(P_2 - P_1)/[(v_2 - v_1)v_1]$

As $v = 1/\rho, -(P_2 - P_1) = E_v \times [\{1/\rho_2) - (1/\rho_1)\}/(1/\rho_1)]$

$= E_v \times [(\rho_1 - \rho_2)/\rho_2]$

$P_2 = P_1 - E_v \times [(\rho_1 - \rho_2)/\rho_2] = 1 \times 10^5 - 2290 \times 10^6 \{(1040 - 1055)/1055\}$

$=$ **32.659 × 10⁶ N/m² or about 326.59 bar.**

1.11.1 Expressions for the Compressibility of Gases

The expression for compressibility of gases for different processes can be obtained using the definition, namely, **compressibility = – dp/(dv/v)**. In the case of gases the variation of volume, **dv**, with variation in pressure, **dp**, will depend on the process used. The relationship between these can be obtained using the characteristic gas equation and the equation describing the process.

Process equation for gases can be written in the following general form

$$\mathbf{Pv^n = constant} \tag{1.11.2}$$

where n can take values from 0 to ∞. If $n = 0$, then P = constant or the process is a constant pressure process. If $n = \infty$, then v = constant and the process is constant volume process. These are not of immediate interest in calculating compressibility. If $dp = 0$, compressibility is zero and if $dv = 0$, compressibility is infinite.

The processes of practical interest are for values of $n = 1$ to $n = c_p/c_v$ (the ratio of specific heats, denoted as k). The value $n = 1$ means Pv = constant or isothermal process and $n = c_p/c_v = k$ means isentropic process.

Using the equation Pv^n = constant and differentiating the same,

$$\mathbf{nPv^{(n-1)}dv + v^n dp = 0} \tag{1.11.3}$$

rearranging and using the definition of E_v,

$$\mathbf{E_v = -dp/(dv/v) = n \times P} \tag{1.11.4}$$

Hence compressibility of gas varies as the product $n \times P$.

For isothermal process, $n = 1$, compressibility = P.

For isentropic process, compressibility = $k \times P$.

For constant pressure and constant volume processes compressibility values are zero and ∞ respectively.

In the case of gases the velocity of propagation of sound is assumed to be isentropic. From the definition of velocity of sound as $[g_o \times E_v/\rho]^{0.5}$ it can be shown that

$$\mathbf{c = [g_o \times k\ P/\rho]^{0.5} = [g_o \times k \times R \times T]^{0.5}} \tag{1.11.5}$$

It may be noted that for a given gas the velocity of sound depends only on the temperature. As an exercise the velocity of sound at 27°C for air, oxygen, nitrogen and hydrogen may be calculated as 347.6 m/s, 330.3 m/s, 353.1 m/s and 1321.3 m/s.

1.12 VAPOUR PRESSURE

Liquids exhibit a free surface in the container whereas vapours and gases fill the full volume. Liquid molecules have higher cohesive forces and are bound to each other. In the gaseous state the binding forces are minimal.

Molecules constantly escape out of a liquid surface and an equal number constantly enter the surface when there is no energy addition. The number of molecules escaping from the surface or re-entering will depend upon the temperature.

Under equilibrium conditions these molecules above the free surface exert a certain pressure. This pressure is known as vapour pressure corresponding to the temperature. As the temperature increases, more molecules will leave and re-enter the surface and so the vapour pressure increases with temperature. All liquids exhibit this phenomenon. Sublimating solids also exhibit this phenomenon.

The vapour pressure is also known as saturation pressure corresponding to the temperature. The temperature corresponding to the pressure is known as saturation temperature. If liquid is in contact with vapour both will be at the same temperature and under this condition these phases will be in equilibrium unless energy transaction takes place.

The vapour pressure data for water and refrigerants are available in tabular form. The vapour pressure increases with the temperature. For all liquids there exists a pressure above which there is no observable difference between the two phases. This pressure is known as critical pressure. Liquid will begin to boil if the pressure falls to the level of vapour pressure corresponding to that temperature. Such boiling leads to the phenomenon known as cavitation in pumps and turbines. In pumps it is usually at the suction side and in turbines it is usually at the exit end.

1.12.1 Partial Pressure

In a mixture of gases the total pressure P will equal the sum of pressures exerted by each of the components if that component alone occupies the full volume at that temperature. The pressure exerted by each component is known as its partial pressure.

$$\mathbf{P = p_1 + p_2 + p_3 + ...} \tag{1.12.1}$$

where $p_1 = (m_1 R_1 T)/V$; $p_2 = (m_2 R_2 T)/V$ in which T and V are the common temperature and volume.

For example air is a mixture of various gases as well as some water vapour. The atmospheric pressure is nothing but the sum of the pressures exerted by each of these components. Of special interest in this case is the partial pressure of water vapour. This topic is studied under Psychrometry. The various properties like specific heat, gas constant etc. of the mixture can be determined from the composition.

$$\mathbf{c_m = \Sigma (c_i \times m_i)/\Sigma m_i} \tag{1.12.2}$$

where c_m is the specific heat of the mixture and c_i and m_i are the specific heat and the mass of component i in the mixture.

SOLVED PROBLEMS

Problem 1.1. *A liquid with kinematic viscosity of 3 centi stokes and specific weight 9 kN/m³ fills the space between a large stationary plate and a parallel plate of 475 mm square, the film thickness being 1 mm. If the smaller plate is to be pulled with uniform velocity of 4 m/s,* ***determine the force required*** *if the liquid film is maintained all through.*

The force required (eqn 1.8.2), $\tau \times A = A \times \mu \times (du/dy)$, where τ is shear stress, and μ is dynamic viscosity. In this problem kinematic viscosity and specific weight are given.

Stoke = 10^{-4} m²/s. Density = specific weight/g. So, $\mu = 0.03 \times 10^{-4} \times 9000/9.81$ Ns/m²

Force = $[0.03 \times 10^{-4} \times 9000/9.81] \times (4.0/0.001) \times 0.475 \times 0.475$ = **2.484 N.**

Problem 1.2. *A small thin plane surface is pulled through the liquid filled space between two large horizontal planes in the parallel direction. Show that the force required will be minimum if the plate is located midway between the planes.*

Let the velocity of the small plane be u, and the distance between the large planes be h.

Let the small plane be located at a distance of y from the bottom plane. Assume linear variation of velocity and unit area. Refer Fig. P 1.2.

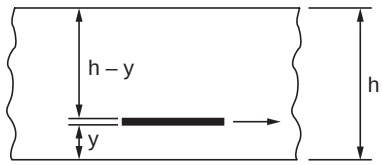


Figure P.1.2 Problem model

Velocity gradient on the bottom surface = u/y

Velocity gradient on the top surface = $u/(h - y)$,

Considering unit area,

Force on the bottom surface = $\mu \times (u/y)$, Force on the top surface = $\mu \times u/(h - y)$

Total force to pull the plane = $\mu \times u \times \{(1/y) + [1/(h - y)]\}$ ...(A)

To obtain the condition for minimisation of the force the variation of force with respect to y should be zero, or $dF/dy = 0$, Differentiating the expression A,

$$dF/dy = \mu \times u \{(-1/y^2) + [1/(h - y)^2]\}, \text{ Equating to zero}$$

$$y^2 = (h - y)^2 \text{ or } y = h/2$$

or the plane should be located at the mid gap position for the force to be minimum.

The force required for different location of the plate is calculated using the following data and tabulated below.

$\mu = 0.014$ Ns/m² , $u = 5$ m/s, $h = 0.1$ m.

Equation A is used in the calculation.

Model calculation is given for $y = 0.002$ m.

$$F = 0.014 \times 5 \times \{(1/0.002) + [1/0.01 - 0.002)]\} = 43.75 \text{ N/m}^2$$

Note that the minimum occurs at mid position

Distance, y mm	2	3	4	5	6
Force, N/m²	43.75	33.33	29.17	28.00	29.17

Problem 1.3. *A small plane is pulled along the centre plane of the oil filled space between two large horizontal planes with a velocity u and the force was measured as F. The viscosity of the oil was μ_1. If a lighter oil of viscosity μ_2 fills the gap what should be the location of the plate for the force to be the same when pulled with the same velocity u.*

If the plane is located centrally in the case where the oil is lighter the force will be smaller.

So the plane should now be located away from the central plane. Let it be located at a distance, y from the lower plane as shown in Fig. P1.2 :

Case 1: The velocity gradient is equal on both sides = $u/(h/2) = 2 \times u/h$

Total force = $\mu_1 \times \{(2u/h) + (2u/h)\} = \mathbf{4 \times \mu_1 \times u/h}$

Case 2: Velocity gradient on the top surface = $u/(h–y)$

Velocity gradient on the bottom surface = u/y

Total force = $\mu_2 \times u \times \{(1/y) + [1/(h - y)]\} = \mu_2 \times u \times \{h/[y \times (h - y)]\}$

Equating and solving, $(\mu_2/\mu_1) = 4 \times y \times (h - y)/h^2 = \mathbf{4[y/h] \times [1 - (y/h)]}$

Solve for (y/h). A quadratic equation.

Problem 1.4. *A large thin plate is pulled through a narrow gap filled with a fluid of viscosity μ on the upper side and a fluid of viscosity $c\mu$ on the lower side. Derive an expression for the location of the plate in the gap for the total force to be minimum.*

The force will not be minimum if the plate is centrally located as the viscosity are not equal. Let the plate be located at a distance of y from the lower surface on the side where the viscosity is $c\mu$. Let the gap size be h, the total force for unit area will be

$$F = c\mu \times (u/y) + \mu \times u/(h - y) = \mu \times u \{(c/y) + [1/(h - y)]\}$$

At the minimum conditions the slope *i.e.*, the derivative dF/dy should be zero.

$dF/dy = \mu \times u \{[1/(h - y)^2] - [c/y^2]\}$, Equating to zero yields, $y^2 = c \times (h - y)^2$

Taking the root,

$$\sqrt{c} \times (h - y) = y \quad \text{or} \quad y = (h \times \sqrt{c})/(1 + \sqrt{c}) = h/[1 + (1/\sqrt{c})]$$

Consider the following values for the variables and calculate the force for different locations of the plate.

$$u = 5 \text{ m/s}, \ \mu = 0.014 \text{ N/m}^2, \ h = 4 \text{ mm} \quad \text{and} \quad c = 0.49 \quad \text{or} \quad \sqrt{c} = 0.7$$

For optimum conditions

$$y = (0.004 \times 0.7)/(1 + 0.7) = 0.001647 \text{ m}$$

Using $F = 5 \times 0.014 \times \{(0.49/y) + [1/(0.004 - y)]\}$, the force for various locations is calculated and tabulated below:

y, mm	1.0	1.5	1.65	2.0	2.5
Force, N/m^2	57.63	50.87	50.58	52.15	60.39

Problem 1.5. *A hydraulic lift shaft of 450 mm dia moves in a cylinder of 451 mm dia with the length of engagement of 3 m. The interface is filled with oil of kinematic viscosity of* 2.4×10^{-4} m^2/s *and density of 900* kg/m^3. *Determine the uniform velocity of movement of the shaft if the drag resistance was 300 N.*

The force can be determined assuming that the sliding is between the developed surfaces, the area being $\pi \times D \times L$, $\mu = \rho v = 2.4 \times 10^{-4} \times 900 = 0.216$ Ns/m^2,

Clearance = $(D_o - D_i)/2 = 0.5$ mm. Using equations 1.8.1 and 1.8.2

Drag resistance = $300 = \mu \times 0.45 \times 3 \times 0.216 \times (u / 0.0005)$

Solving for u, velocity = **0.16374 m/s.**

Problem 1.6. *A shaft of 145 mm dia runs in journals with a uniform oil film thickness of 0.5 mm. Two bearings of 20 cm width are used. The viscosity of the oil is 19 cP.* ***Determine the speed if the power absorbed is 15 W.***

The equation that can be used is, 1.9.2 *i.e.,* (n is used to denote rpm)

$$P = [\mu \pi^3 n^2 L R^3 / 450\ h]$$

The solution can be obtained from basics also. Adopting the second method,

$\tau = \mu\ (du/dy) = \mu\ (u/y)$, $\mu = 19$ cP $= 0.019$ Ns/m^2,

$y = 0.5$ mm $= 0.0005$ m, let the rpm be n

$u = \pi Dn / 60 = \pi \times 0.145 \times n/60 = 7.592 \times 10^{-3} \times n$

$\tau = 0.019\ (7.592 \times 10^{-3} \times n/0.0005) = 0.2885 \times n$ N/m^2,

$A = 2 \times \pi DL = 0.182$ m^2, Force F = A × τ = $0.2885 \times n \times 0.182 = 0.0525 \times n$,

Torque = force × radius,

$T = 0.0525 \times n \times 0.145/2 = 3.806 \times 10^{-3} \times n$ Nm

Power, $P = 2\pi nT/60 = 15 = 2 \times \pi \times n\ 3.806 \times 10^{-3} \times n / 60$

Solving, speed, n = **194 rpm.** (Check using the equation 1.9.2)

$$15 = [0.019 \times \pi^3 \times n^2\ (2 \times 0.20) \times 0.0725^3 / (450 \times 0.0005)]$$

Solving speed, **n = 194 rpm.**

Problem 1.7. *A circular disc of 0.3 m dia rotates over a large stationary plate with 1 mm thick fluid film between them.* ***Determine the viscosity of the fluid*** *if the torque required to rotate the disc at 300 rpm was 0.1 Nm.*

The equation to be used is 1.9.3, (n denoting rpm)

Torque $T = (\mu \times \pi^2 \times n \times R^4)/(60 \times h)$, ($h$ – clearance),

$n = 300$ rpm, $R = 0.15$ m, $h = 0.001$ m, Substituting the values,

$0.1 = \mu \times \pi^2 \times 300 \times 0.15^4 / (60 \times 0.001)$, Solving for μ

Viscosity **$\mu = 4 \times 10^{-3}$ Ns/m^2 or 4 cP.**

(care should be taken to use radius value, check from basics.)

Problem 1.8. ***Determine the viscous drag torque and power*** *absorbed on one surface of a collar bearing of 0.2 m ID and 0.3 m OD with an oil film thickness of 1 mm and a viscosity of 30 cP if it rotates at 500 rpm.*

The equation applicable is 1.9.5. $T = \mu \times \pi^2 \times n \times (R_o^4 - R_i^4)/60 \times h$

$\mu = 30 \times 0.001$ Ns/m^2, $n = 500$ rpm, $R_o = 0.15$ m, $R_i = 0.1$ m, $h = 0.002$ m

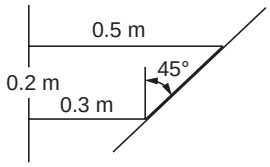
substituting the values

$$\mathbf{T} = 30 \times 0.001 \times \pi^2 \times 500 \times \{0.15^4 - 0.1^4\}/\{(60 \times 0.002)\} = \mathbf{0.5012\ Nm}$$

$$\mathbf{P} = 2\pi nT/60 = 2 \times \pi \times 500 \times 0.5012/60 = \mathbf{26.243\ W.}$$

Problem 1.9. *A conical bearing of outer radius 0.5 m and inner radius 0.3 m and height 0.2 m runs on a conical support with a uniform clearance between surfaces. Oil with viscosity of 30 cP is used. The support is rotated at 500 rpm.* ***Determine the clearance*** *if the power required was 1500 W.*

The angle θ is determined using the difference in radius and the length.



$$\tan\theta = (0.5 - 0.3)/0.2 = 1.0;\ \text{So}\ \theta = 45°.$$

Using equation 1.9.9 *i.e.*,

$$P = \pi^3 \times \mu \times n^2 \times (R_2^4 - R_1^4)/1800 \times h \times \sin\theta$$

$(\mu = 30\ \text{cP} = 0.03\ \text{Ns/m}^2,\ n = 500\ \text{rpm},\ R_2 = 0.5\ \text{m},\ R_1 = 0.3\ \text{m})$

$$1500 = \pi^3 \times 0.03 \times 500^2 \times (0.5^4 - 0.3^4)/1800 \times h \times \sin 45°$$

Solving for clearance, $\mathbf{h = 6.626 \times 10^{-3}\ m\ or\ 6.63\ mm}$

Problem 1.10. If the variation of velocity with distance from the surface, y is given by $u = 10\ y^{0.5}$ *whre u is in m/s and y is in m in a flow field up to y = 0.08 m,* **determine the wall shear stress** *and the shear stress at y = 0.04 and 0.08 m from the surface.*

$$u = 10y^{0.5},\ (du/dy) = 5/y^{0.5}.$$

The substitution $y = 0$ in the above will give division by zero error. It has to be approximated as $(u_2 - u_1)/(y_2 - y_1)$ for near zero values of y.

Considering layers $y = 0$ and $y = 10^{-6}$, the velocities are 0.0 and 0.01 m/s

(using $u = 10\ y^{0.5}$), the difference in y value is 10^{-6}.

So $(u_2 - u_1)/(y_2 - y_1) = 0.01/10^{-6} = 10000$,

At the wall, $(du/dy) = 10000$, $\tau = \mu\ (du/dy) = \mathbf{10000 \times \mu}$

At $\boldsymbol{y = 0.04}$, $(du/dy) = 5/0.04^{0.5} = 25$, $\tau = \mathbf{25 \times \mu}$

At $\boldsymbol{y = 0.08}$, $(du/dy) = 5/0.08^{0.5} = 17.68$, $\tau = \mathbf{17.68 \times \mu}$

In this case the clearance considered is large and so the assumption of linear velocity variation may lead to larger error. The concept that the torque along the radius should be constant can be used to determine the torque more accurately.

Problem 1.11. *A hollow cylinder of 12 cm ID filled with fluid of viscosity 14 cP rotates at 600 rpm. A shaft of diameter 4 cm is placed centrally inside.* ***Determine the shear stress*** *on the shaft wall.*

The hollow cylinder rotates while the shaft is stationary. Shear stress is first calculated at the hollow cylinder wall (Assume 1 m length).

Solution is obtained from basics. Linear velocity variation is assumed. Clearance,

$$h = 0.04\ \text{m},\ \mu = 14 \times 0.001 = 0.014\ \text{N/m}^2$$

At the inside wall of the hollow cylinder,

$$u = 2\,\pi Rn/60 = 3.77 \text{ m/s}$$

$$(du/dr) = u/h = 3.77/0.04 = 94.25/\text{s},\ \tau = \mu\,(du/dr)$$

$$= 0.014 \times 94.25 = 1.32 \text{ N/m}^2$$

$$F = \pi \times D \times L \times \tau = \pi \times 0.12 \times 1 \times 1.31 = 0.498 \text{ N}$$

$$\text{torque} = F \times R = 0.498 \times 0.06 = 29.86 \times 10^{-3} \text{ Nm}$$

Torque at all radii should be the same. At mid radius $R = 0.04$ m, the velocity gradient is obtained by using this concept.

$$29.86\ 10^{-3} = \left.\frac{du}{dr}\right|_{0.04} \times 0.014 \times \pi \times 0.08 \times 1 \times 0.04,$$

Solving, $$\left.\frac{du}{dr}\right|_{0.04} = 212.06/\text{s},$$

This can be checked using equation, (see problem 1.13)

$$\left.\frac{du}{dr}\right|_{R_1} = \left.\frac{du}{dr}\right|_{R_2} \times (R_2^2/R_1^2) \text{ at } 0.04,\ \left.\frac{du}{dr}\right|_{0.04} \times 25 \times 0.06^2/0.04^2 = 212.06/\text{s}$$

The velocity gradient at the shaft surface = $94.25 \times 0.06^2/0.02^2 = 848.25/\text{s}$

Shear stress at the shaft wall = 848.25×0.014 = **11.88 N/m².**

Problem 1.12. *The velocity along the radius of a pipe of 0.1 m radius varies as* $u = 10 \times [1 - (r/0.1)^2]$ *m/s. The viscosity of the fluid is 0.02 Ns/m².* ***Determine the shear stress*** *and the shear force over the surface at r = 0.05 and r = 0.1 m.*

$$\tau = \mu\,(du/dr),\ u = 10 \times [1 - (r/0.1)^2],$$

$$du/dr = -\,10 \times (2 \times r/0.1^2) = -\,2000\,r$$

(the –ve sign indicates that the force acts opposite to the flow direction.)

$\tau = 0.02 \times (-\,2000) \times r = -\,40\,r$, Shear force $F = 2\pi rL\tau$, Considering $L = 1$

At r = 0.05, τ = – 2 N/m², F = 0.628 N

At r = 0.1, τ = – 4 N/m², F = 2.513 N.

Problem 1.13. *A sleeve surrounds a shaft with the space between them filled with a fluid. Assuming that when the sleeve rotates velocity gradient exists only at the sleeve surface and when the shaft rotates velocity gradient exists only at the shaft surface, determine the ratio of these velocity gradients.*

The torque required for the rotation will be the same in both cases. Using general notations,

$$\tau_i\,[2\pi\,r_i \times L] \times r_i = \tau_o\,[2\pi\,r_o \times L] \times r_o$$

$$\tau_i = \mu\,(du/dr)_{ri},\ \tau_o = \mu\,(du/dr)_{ro}$$

Substituting in the previous expression and solving

$$(du/dr)_i = (du/dr)_o \times [r_o^2/r_i^2]$$

This will plot as a second degree curve. When the gap is large % error will be high if linear variation is assumed.

Problem 1.14. *Derive an expression for the force required for axial movement of a shaft through a taper bearing as shown in figure. The diameter of the shaft is D m and the length is L m. The clearance at the ends are t_1m and t_2m. The oil has a viscosity of μ and the shaft moves axially at a velocity u.*

In this case the clearance varies along the length and so the velocity gradient will vary along the length. Hence the shear stress also will vary along the length. The total force required can be determined by integrating the elemental force over a differential length *dX*. The clearance, *t* at location *X* is obtained, assuming $t_1 > t_2$,

$$t = t_1 - (t_1 - t_2) \times (X/L) = \{(t_1 \times L) - (t_1 - t_2)\, X\}/L$$

$$du/dy = u/t = u \times L/\{(t_1 \times L) - (t_1 - t_2) \times X\}$$

The velocity gradient at this location is *u/t*, assumed linear.

$$\tau = \mu\,(du/dy),\ dF = \tau\, dA = \tau \times \pi \times D \times dX, \text{ substituting}$$

$$dF = [\{L \times \mu \times u \times \pi \times D\}] \times [dX/\{(t_1 \times L) - (t_1 - t_2) \times X\}]$$

Integrating between the limits $X = 0$ to $X = L$

$$\mathbf{F = [\{\pi \times D \times u \times L \times \mu\}/\{t_1 - t_2\}] \times [\ln(t_1/t_2)]}$$

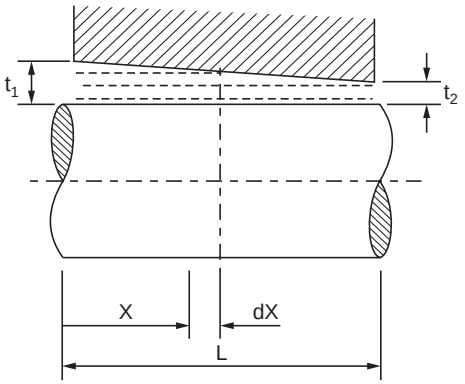


Figure P.1.14

Problem 1.15. *The clearance between the shaft of 100 mm dia and the bearing varies from 0.2 mm to 0.1 mm over a length of 0.3 m. The viscosity of the oil filling the clearance is 4.8×10^{-2} Ns/m^2. The axial velocity of the shaft is 0.6 m/s.* ***Determine the force required.***

Using the equation derived in the previous problem as given below and substituting the values $F = [\{\pi \times D \times u \times L \times \mu\}/\{t_1 - t_2\}]\ [\ln(t_1/t_2)]$

$\mathbf{F} = [\{\pi \times 0.1 \times 0.6 \times 0.3 \times 4.8 \times 10^{-2}\}/\{0.0002 - 0.0001\}] \times [\ln(0.0002/0.0001)] = \mathbf{18.814\ N}$

If the clearance was uniform, $F = \pi \times D \times L \times u \times \mu/t$

For $t = 0.2$ mm, $F_{0.2} = 13.572$ N, For $t = 0.1$ mm, $F_{0.1} = 27.143$ N

The arithmetic average is 20.36 N, while the logarithmic average is what is determined in this problem, 18.814 N.

Problem 1.16. ***Derive an expression for the torque required*** *to overcome the viscous resistance when a circular shaft of diameter D rotating at N rpm in a bearing with the clearance t varying uniformly from t_1 m at one end to t_2m at the other end. The distance between the ends is L m. The oil has a viscosity of* μ.

In this case the clearance varies along the length and so the velocity gradient (du/dr) will vary along the length. Hence the shear stress and the torque also will vary along the length. The total torque required can be determined by integrating the elemental torque over a differential length dX.

The clearance, t at location X is obtained, assuming $t_1 > t_2$,

$$t = t_1 - (t_1 - t_2) \times (X/L) = \{(t_1 \times L) - (t_1 - t_2) \times X\}/L$$

The velocity gradient at this location X is u/t, as linear profile is assumed.

$$\therefore \quad du/dy = u/t = u \times L/\{(t_1 \times L) - (t_1 - t_2) \times X\}$$

$$\tau = \mu\,(du/dy),\ dF = \tau\, dA = \tau \times \pi \times D \times dX, \text{ substituting}$$

$$dF = [\{L \times \mu \times u \times \pi \times D\}] \times [dX/\{(t_1 \times L) - (t_1 - t_2) \times X\}]$$

Torque = $dF \times (d/2)$ and $u = (\pi DN)/60$. Substituting and Integrating between the limits

$X = 0$ to $X = L$, Torque $\mathbf{= [\{\pi^2 \times D^3 \times L \times N \times \mu\}/\{120(t_1 - t_2)\}] \times [\ln (t_1/t_2)]}$

Power = $2\pi NT/60$, hence

$$\mathbf{P = [\{\pi^3 \times D^3 \times L \times N^2 \times \mu\}/\{3600(t_1 - t_2)\}] \times [\ln (t_1/t_2)].}$$

Problem 1.17 *The clearance between the shaft of 100 mm dia and the bearing varies from 0.2 mm to 0.1 mm over a length of 0.3 m. The viscosity of the oil filling the clearance is 7.1 × 10^{-2} Pa.s (Ns/m^2). The shaft runs at 600 rpm.* ***Determine the torque and power required.***

Using the equations derived in the previous problem as given below and substituting the values

$$T = [\{\pi^2 \times D^3 \times L \times N \times \mu\}/\{120(t_1 - t_2)\}] \times [\ln(t_1/t_2)]$$

$$P = [\{\pi^3 \times D^3 \times L \times N^2 \times \mu\}/\{3600(t_1 - t_2)\}] \times [\ln(t_1/t_2)]$$

$$\mathbf{T} = [\{\pi^2 \times 0.1^3 \times 0.3 \times 600 \times 7.1 \times 10^{-2}\}/\{120(0.0002 - 0.0001)\}] \times [\ln (0.0002/0.0001)]$$

$$= \mathbf{7.29\ Nm.}$$

$$\mathbf{P} = [\{\pi^3 \times 0.1^3 \times 0.3 \times 600^2 \times 7.1 \times 10^{-2}\}/\{3600(0.0002 - 0.0001)\}] \times [\ln (0.0002/0.0001)]$$

$$= \mathbf{457.8\ W.}$$

Check: $P = 2\pi \times 600 \times 7.29/60 = 458$W.

Problem 1.18. ***Determine the capillary depression*** *of mercury in a 4 mm ID glass tube. Assume surface tension as 0.45 N/m and* $\beta = 115°$.

The specific weight of mercury = 13550 × 9.81 N/m^3, Equating the surface force and the pressure force, $[h \times \gamma \times \pi D^2/4] = [\pi \times D \times \sigma \times \cos \beta]$, Solving for h,

$$h = \{4 \times \sigma \times \cos \beta\}/\{\gamma \times D\} = [4 \times 0.45 \times \cos 115]/[13550 \times 9.81 \times 0.004]$$

$$= \mathbf{-\ 1.431 \times 10^{-3}\ m \quad or \quad -\ 1.431\ mm,\ (depression)}$$

Problem 1.19. *A ring 200 mm mean dia is to be separated from water surface as shown in figure. The force required at the time of separation was 0.1005 N.* ***Determine the surface tension of water.***

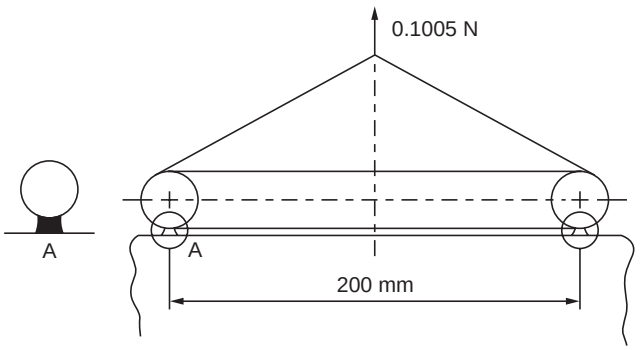


Figure P.1.19

The total length of contact just before lifting from the surface will be twice the circumference or $2\pi D$. The force will equal the product of surface tension and the length of contact.

$\sigma \times 2 \times \pi \times 0.2 = 0.1005$ N. Solving $\boldsymbol{\sigma = 0.08}$ **N/m**

The surface tension of a liquid can be measured using this principle provided the fluid wets the surface.

Problem 1.20. *A thin plate 1 m wide is slowly lifted vertically from a liquid with a surface tension of 0.1 N/m. Determine what force will be required to overcome the surface tension. Assume* $\beta = 0$.

The total length of contact just before separation from the surface will be twice the width of the plate or $2L$. The force will equal the product of surface tension and the length of contact.

$$\mathbf{F = 2 \times 1 \times 0.1 = 0.02\ N.}$$

Problem 1.21. *Diesel injection nozzle sprays fuel with an average diameter of 0.0254 mm. The surface tension is 0.0365 N/m. Determine the pressure difference between the inside and outside of the nozzle. Also determine the pressure difference if the droplet size is reduced to 10 μm.*

A droplet forms at the mouth of the nozzle. The pressure inside the droplet will be higher compared to that at outside.

The equation applicable is $(P_i - P_o) = 2\sigma/R$.

So $(P_i - P_o) = \{2 \times 0.0365 \times 2\}/\{0.0254 \times 10^{-3}\} =$ **5748 N/m^2 = 5.748 kN/m^2**

When the droplet size is reduced to 10 μm the pressure difference is

$(P_i - P_o) = \{2 \times 0.0365 \times 2\}/\{10 \times 10^{-6}\} =$ **14600 N/m^2 = 14.6 kN/m^2.**

Problem 1.22. *A glass tube of 8 mm ID is immersed in a liquid at 20°C. The specific weight of the liquid is 20601 N/m^3. The contact angle is 60°. Surface tension is 0.15 N/m.* ***Calculate the capillary rise*** *and* ***also the radius of curvature of the meniscus****.*

Capillary rise, $h = \{4 \times \sigma \times \cos \beta\}/\{\gamma \times D\} = \{4 \times 0.15 \times \cos 60\}/\{20601 \times 0.008\}$

$= \mathbf{1.82 \times 10^{-3}\ m \quad or \quad 1.82\ mm.}$

The meniscus is a doubly curved surface with **equal radius** as the section is circular.

(using equation 1.10.3)

$$(P_i - P_o) = \sigma \times \{(1/R_1) + (1/R_2)\} = 2\,\sigma/R$$

$$R = 2\sigma/(P_i - P_o),\ (P_i - P_o) = \text{specific weight} \times h$$

So, $R = [2 \times 0.15]/\ [1.82 \times 10^{-3} \times 2060] = \mathbf{8 \times 10^{-3}\ m \quad or \quad 8\ mm.}$

Problem 1.23. *A mercury column is used to measure the atmospheric pressure. The height of column above the mercury well surface is 762 mm. The tube is 3 mm in dia. The contact angle is 140°.* ***Determine the true pressure in mm*** *of mercury if surface tension is 0.51 N/m. The space above the column may be considered as vacuum.*

In this case capillary depression is involved and so the true pressure = mercury column + capillary depression.

The specific weight of mercury = 13550 × 9.81 N/m^3, equating forces,

$[h \times \gamma \times \pi D^2/4] = [\pi \times D \times \sigma \times \cos \beta]$.

So $h = \{4 \times \sigma \times \cos \beta\}/\{\gamma \times D\}$

$h = (4 \times 0.51) \times \cos 140]/[13550 \times 9.81 \times 0.003]$

$= \mathbf{-3.92 \times 10^{-3}\ m \quad or \quad -3.92\ mm,\ (depression)}$

Hence actual pressure indicated = 762 + 3.92 = **765.92 mm** of mercury.

Problem 1.24. ***Calculate the pressure difference*** *between the inside and outside of a soap bubble of 2.5 mm dia if the surface tension is 0.022 N/m.*

The pressure difference in the case of a sphere is given by, equation 1.10.5

$(P_i - P_o) = 2\sigma/R = \{2 \times 0.022\}/\{0.0025\} = \mathbf{17.5\ N/m^2.}$

Problem 1.25. *A hollow cylinder of 150 mm OD with its weight equal to the buoyant forces is to be kept floating vertically in a liquid with a surface tension of 0.45 N/m^2. The contact angle is 60°. Determine the additional force required due to surface tension.*

In this case a capillary rise will occur and this requires an additional force to keep the cylinder floating.

Capillary rise, $h = \{4 \times \sigma \times \cos \beta\}/\{\gamma \times D\}$.

As $(P_i - P_o) = h \times$ specific weight, $(P_i - P_o) = \{4 \times \sigma \times \cos \beta\}/D$

$(P_i - P_o) = \{4 \times 0.45 \times \cos 60\}/\{0.15\} = 6.0$ N/m^2

Force = Area × $(P_i - P_o) = \{\pi \times 0.015^2/4\} \times 6 = 0.106$ N

As the immersion leads to additional buoyant force the force required to kept the cylinder floating will be double this value.

So the additional force = 2 × 0.106 = **0.212 N.**

Problem 1.26. *The volume of liquid in a rigid piston—cylinder arrangement is 2000 cc. Initially the pressure is 10 bar. The piston diameter is 100 mm. Determine the distance through which the piston has to move so that the pressure will increase to 200 bar. The temperature remains constant. The average value of bulk modulus for the liquid is 2430 × 10^{-6} N/m^2.*

By definition—refer eqn 1.11.1

$$E_v = -dP/(dv/v) = -(P_2 - P_1)/[(v_2 - v_1)/v_1]$$

So $2430 \times 10^6 = -190 \times 10^5/(dv/0.002)$, Solving,

$$dv = -0.002 \times 190 \times 10^5/2430 \times 10^6 = 15.64 \times 10^{-6}\ \text{m}^3$$

Piston movement, $L = dv/\text{area}$

$$L = dv \times 4/\pi D^2 = 15.64 \times 10^{-6} \times 4/\pi \times 0.1^2 = 1.991 \times 10^{-3}\ \text{m} = \mathbf{1.991\ mm}$$

(the piston-cylinder arrangement is assumed to be rigid so that there is no expansion of the container)

Problem 1.27. *The pressure of water increases with depth in the ocean. At the surface, the density was measured as 1015 kg/m^3. The atmospheric pressure is 1.01 bar. At a certain depth, the pressure is 880 bar.* ***Determine the density of sea water at the depth.*** *The average value of bulk modulus is 2330 × 10^6 N/m^2.*

The density will increase due to the pressure increase.

Bulk modulus is defined in eqn 1.11.1 as $E_v = -dP/(dv/v) = -(P_2 - P_1)/[v_2 - v_1)/v_1]$,

$$[(v_2 - v_1)/v_1] = -(P_2 - P_1)/E_v = -[880 \times 10^5 - 1.01 \times 10^5]/2330 \times 10^6 = -0.03772$$

$v_1 = 1/1015\ \text{m}^3/\text{kg}$, substituting the values in

$v_2 = [v_1 \times \{-(P_2 - P_1)/E_v\}] + v_1$,

$v_2 = [-0.03772 \times (1/1015)] + (1/1015) = 9.48059 \times 10^{-4}\ \text{m}^3/\text{kg}$

Density $= 1/(9.48059 \times 10^{-4}\ \text{m}^3/\text{kg}) =$ **1054.79 kg/m^3** an increase of 4%.

The density increases by 4.0% due to the increase in pressure.

$[(v_2 - v_1)/v_1]$ also equals $[(\rho_1 - \rho_2)/\rho_2] = [(P_2 - P_1)/E_v]$

Use of this equation should also give the same answer.

Problem 1.28. *A diesel fuel pump of 10 mm ID is to deliver against a pressure of 200 bar. The fuel volume in the barrel at the time of closure is 1.5 cc. Assuming rigid barrel determine the plunger movement before delivery begins. The bulk modulus of the fuel is 1100 × 10^6 N/m^2.*

By definition—eqn 1.11.1—the bulk modulus is $E_v = -dP/(dv/v)$,

$$1100 \times 10^6 = -200 \times 10^5/(dv/1.5 \times 10^{-6}), \text{ Solving } dv = -2.77 \times 10^{-8}\ \text{m}^3$$

Plunger movement $= dv/\text{area} = -2.77 \times 10^{-8} \times 4/(\pi \times 0.0015^2)$

$$= 3.47 \times 10^{-4}\ \text{m} = \mathbf{0.347\ mm}$$

(the pressure rise will also be affected by the expansion of the pipe line).

OBJECTIVE QUESTIONS

O Q 1.1 Fill in the blanks with suitable words:

1. Cohesive forces between molecules/atoms are highest in the _____ phase.
2. When the applied load is released solids _____.
3. Solids _____ applied shear while liquids _____.

4. In solids _____ is proportional to the applied stress.
5. The mobility of atoms is least in _____.
6. The distance between atoms is least in _____.
7. _____ have specific shape that does not change by itself.
8. When heated the atoms in solids _____.
9. In some solids molecules come out when heated. The phenomenon is called _____.
10. For solids the proportionality limit between deformation and stress is called _____.

Answers

(1) Solid (2) regain their original shape (3) resist, continue to deform (4) deformation (5) Solids (6) Solids (7) Solids (8) vibrate more (9) Sublimation (10) Elastic limit.

O Q 1.2 Fill in the blanks with suitable words:

1. Fluids cannot withstand _____.
2. Fluids _____ to deform when a shear force is applied.
3. The atoms/molecules are _____ to move in fluids.
4. In liquids _____ is proportional to shear stress.
5. The difference between liquids and gases is _____.
6. Liquids form a _____ when in a container.
7. Gases _____ the container.
8. The distance between molecules is highest in _____.
9. Cohesive faces between atoms is least in _____.
10. Vapour is the gaseous state of matter when the temperature is near the _____.

Answers

(1) shear force (2) continue (3) free (4) rate of deformation (5) that the atomic molecular spacing is much larger in gas and atoms move all over the container filling it (6) free surface (7) completely fill (8) gases (9) gases (10) saturation conditions (Boiling conditions)

O Q 1.3 Fill in the blanks with suitable words:

1. The three phases of matter are _____ .
2. A special state of matter at very high temperatures is _____.
3. Density is defined as _____.
4. Specific weight is defined as _____.
.5. Specific gravity is defined as _____.
6. A fluid is defined as _____.
7. A liquid is defined as _____.
8. A vapour is defined as _____.
9. A gas is defined as _____.
10. A mole is defined as _____.

Answers

(1) Solid, liquid & gas (2) plasma (3) mass per unit volume (4) force due to gravity on mass in unit volume (5) ratio of mass of substance/mass of water at 10°C per unit volume (6) Material which cannot resist shear stress or material which will continuously deform under applied shear stress (7) A material which will exhibit a free surface in a container (8) gaseous state

very near the formation temperature at that pressure (9) material with low cohesive force with large distance between molecules which will occupy the full volume of the container (10) Molecular mass of a substance.

O Q 1.4 Fill in the blanks with suitable words:

1. Bulk modulus is defined as _____.
2. Bulk modulus of gases depend on _____.
3. Bulk modulus of liquid will _____ with pressure.
4. Liquids have _____bulk modulus.
5. Unit of bulk modulus is the same as that of _____.
6. The concept of bulk modulus is used in the analysis of _____ propagation in the medium.
7. Viscosity is defined as _____.
8. Kinematic viscosity is defined as _____.
9. An ideal fluid is defined as _____.
10. A Newtonian fluid is defined as one having _____.

Answers

(1) $-dp/(dv/v)$ (2) the process of change (3) increase (4) high (5) pressure (6) sound (7) $\mu = \tau/(du/dy)$ the proportionality constant between shear stress and velocity gradient (8) μ/ρ (9) one with no viscosity or compressibility (10) A constant viscosity irrespective of the velocity gradient.

O Q 1.5 Fill in the blanks with suitable words:

1. A non Newtonian fluid is defined as _____.
2. An ideal plastic is defined as _____.
3. A thixotropic fluid is defined as _____.
4. Surface tension is defined as _____.
5. Vapour pressure is defined as _____.
6. Surface tension is due to _____ forces.
7. Capillary rise is caused by _____ forces.
8. Capillary rise is when _____ forces predominate.
9. Capillary depression is when _____ forces predominate.
10. Droplet formation and free circular jet formation is due to _____.

Answers

(1) a fluid whose viscosity varies with the velocity gradient (2) a material which requires a definite shear to cause the first deformation but then the stress is proportional to the velocity gradient (3) A substance whose viscosity increases with increase in velocity gradient (4) Work required to create a unit area of free surface in a liquid/force required to keep unit length of free surface in equilibrium (5) The pressure over the fluid due to the vapour over a liquid under equilibrium conditions of temperature (6) Cohesive (7) Adhesive forces (8) Adhesive (9) Cohesive (10) Surface tension

O Q 1.6 Fill in the blanks with "increasing " or "decreasing" or "remains constant":

1. When gravitational force increases specific weight _____.
2. When gravitational force decreases specific weight _____.
3. When gravitational force increases density _____.
4. When gravitational force decreases density _____.

5. The specific gravity _____ when density increases.
6. As molecular weight of a gas increases its gas constant _____.
7. The product of gas constant and molecular weight _____.
8. At constant temperature, the pressure exerted by a gas in a container _____ when the volume increases.
9. Bulk modulus of liquids _____ with increase in pressure at constant temperature.
10. At constant pressure the bulk modulus of liquids (*a*) _____ and then (*b*) _____ with increase in temperature.

Answers

Increases 1, 5, 9, 10a **Decrease 2, 6, 8, 10b** **Remains constant 3, 4, 7**

O Q 1.7 Fill in the blanks with "increases", "decreases" or "remains constant"

1. Viscosity of liquids _____ with increase of temperature.
2. Viscosity of gases _____ with increase of temperature.
3. As tube diameter decreases the capillary rise _____.
4. As tube diameter increases the capillary rise _____.
5. As the diameter of a bubble increases the pressure difference between inside and outside _____.
6. As the diameter decreases the pressure difference between inside and outside of a free jet _____.
7. At a given temperature the vapour pressure for a liquid _____.
8. As temperature increases, the vapour pressure _____.
9. The vapour pressure over a liquid _____ when other gases are present in addition to the vapour.
10. As cohesive force _____ compared to adhesive forces, the capillary will rise.

Answers

Increases 2, 3, 6, 8 **Decreases 1, 4, 5, 10** **Remains constant 7, 9**

O Q 1.8 Indicate whether the statements are correct or incorrect.

1. Density is the ratio of mass of unit volume of liquid to the mass of unit volume of water.
2. In the gas equation temperature should be used in Kelvin scale.
3. Specific weight is the mass of unit volume.
4. The cohesive forces are highest in gases.
5. The shear force in solid is proportional to the deformation.
6. In fluids the shear force is proportional to the rate of deformation.
7. Newtonian fluid is one whose viscosity will increase directly with rate of deformation.
8. The vapour pressure will vary with temperature.
9. Ideal fluid has zero viscosity and is incompressible.
10. Gases can be treated as incompressible when small changes in pressure and temperature are involved.

Answers

Correct 2, 5, 6, 8, 9, 10 Incorrect 1, 3, 4, 7

O Q 1.9 Indicate whether the statements are correct or incorrect.

1. Specific weight of a body will vary from place to place.
2. Mass is measured by a spring balance.
3. Specific weight is measured by a spring balance.
4. The weight of man will be lower on the moon.
5. The weight of a man will be higher in Jupiter.
6. Dynamic viscosity is a measure of momentum diffusivity.
7. Viscosity of liquids increases with temperature.
8. Viscosity of gases increases with temperature.
9. Higher the surface tension higher will be the pressure inside a bubble.
10. The head indicated by a mercury manometer is lower than the actual value.
11. The head indicated by a water manometer is lower than the actual value.

Answers

Correct 1, 3, 4, 5, 8, 9, 10 Incorrect 2, 6, 7, 11

O Q 1.10 The following refer either to viscosity effects or surface tension effects. Classify them accordingly.

1. Capillary rise
2. Drag
3. Liquid bubble
4. Free jet
5. Heating of lubricating oil in bearings
6. Free surface of liquids
7. Momentum transfer
8. Gas flow

Answers

Surface tension 1, 3, 4, 6 Viscosity 2, 5, 7, 8

O Q 1.11 Choose the correct answer.

1. The mass of an object is 10 kg. The gravitational acceleration at a location is 5 m/s^2. The specific weight is

   (*a*) 2 N (*b*) 15 N (*c*) 5 N (*d*) 50 N

2. The dynamic viscosity is 1.2×10^{-4} Ns/m^2. The density is 600 kg/m^3. The kinematic viscosity in m^2/s is

   (*a*) 72×10^{-3} (*b*) 20×10^{-8} (*c*) 7.2×10^{3} (*d*) 70×10^{6}

3. The velocity gradient is 1000/s. The viscosity is 1.2×10^{-4} Ns/m^2. The shear stress is

   (*a*) 1.2×10^{-1} N/m (*b*) 1.2×10^{-7} N/m^2 (*c*) 1.2×10^{2} N/m^2 (*d*) 1.2×10^{-10} N/m^2

4. The velocity distribution in a flow through a tube is given by $u = (-10/\mu)(0.01 - r^2)$. The pipe radius $R = 0.1$ m. The shear stress at the wall in N/m^2 is

   (*a*) $10/\mu$ (*b*) 0 (*c*) 2μ (*d*) $2/\mu$

5. The excess pressure in a droplet of 0.002 m dia a fluid with surface tension of 0.01 N/m is

   (*a*) 10 (*b*) 20 (*c*) 4π (*d*) 0.00004 π

Answers

(1) *d* (2) *b* (3) *a* (4) *d* (5) *b*

O Q 1.12 Match the pairs.

	Set A	Set B
(*i*)	1. density	A. N/m
	2. surface tension	B. N/m^3
	3. kinematic viscosity	C. kg/m^3
	4. specific weight	D. m^2/s
(*ii*)	1. dynamic viscosity	A. surface tension
	2. capillary rise	B. momentum transfer
	3. kinematic viscosity	C. liquid
	4. free surface	D. shear stress
(*iii*)	1. droplet formation	A. zero viscosity
	2. weight	B. constant viscosity
	3. ideal fluid	C. gravitational acceleration
	4. Newtonian fluid	D. surface tension

Answers

(*i*) 1-C, 2-A, 3-D, 4-B (*ii*) 1-D, 2-A, 3-B, 4-C (*iii*) 1-D, 2-C, 3-A, 4-B

REVIEW QUESTIONS

1. Differentiate between the three states of matter.
2. Distinguish between compressible and incompressible fluids.
3. Distinguish between vapour and gas.
4. Explain the concept of "Continuum".
5. Define density, specific volume, weight density and specific gravity.
6. Define "Compressibility" and "Bulk Modulus".
7. State the characteristic equation for gases and explain its significance.
8. Derive the general expression for compressibility of gases.
9. Define the term viscosity and explain the significance of the same.
10. Distinguish between Newtonian and non Newtonian Fluids.
11. Explain from microscopic point of view the concept of viscosity and momentum transfer. Explain how viscosity of liquids and gases behave with temperature.
12. Define kinematic viscosity and explain the significance of the same.
13. Derive an expression for the torque and power required to overcome the viscous drag for a shaft running at a particular rpm.
14. Derive an expression for the torque required to rotate a collar bearing (disc over a parallel plate).
15. Derive an expression for the torque required to rotate a conical bearing.
16. Describe some methods to determine the viscosity of a fluid.
17. Explain the concepts of (*i*) vapour pressure (*ii*) partial pressure and (*iii*) surface tension.
18. Explain how liquid surface behaves by itself and when it is in contact with other surfaces.
19. Derive an expression for the capillary rise or depression, given the value of the contact angle β and the density and surface tension of the liquid.
20. Derive an expression for the pressure difference caused by surface tension on a doubly curved surface.
21. Derive expressions from basics for the pressure inside a droplet and a free jet.

EXERCISE PROBLEMS

E1.1. Determine the density, specific weight and specific volume of air if the specific gravity (with water as reference fluid) is 0.011614. **(11.614 kg/m^3, 113.94 N/m^3, 0.0861 m^3/kg)**

E1.2. A liquid with kinematic viscosity of 2.7 centistokes fills the space between a large stationary plate and a parallel plate of 500 mm square, the film thickness being 1 mm. If the force required to pull the smaller plate with a uniform velocity of 3 m/s was 1.734 N, determine specific weight of the liquid. Assume that the liquid film is maintained all over. **(8.4 kN/m^3)**

E1.3. Two large plates are 6 mm apart and the space in-between in filled with a fluid. A plate of 1 mm thickness and 10 cm square is pulled parallel to the planes and midway between them with a velocity of 2 m/s. Assume linear velocity profile on either side. The force required was 0.32 N. Determine the viscosity of the fluid. **(2 × 10^{-2} Ns/m^2)**

E1.4. Two large vertical plane parallel surfaces are 5 mm apart and the space between them is filled with a fluid. A thin plate of 12.5 cm square falls freely between the planes along the central plane and reaches a steady velocity of 2 m/s. Determine the weight of the plate if the viscosity of the fluid filling the space is 0.02 Ns/m^2. **(0.5 N)**

E1.5. Two large planes are parallel to each other and are inclined at 30° to the horizontal with the space between them filled with a fluid of viscosity 20 cp. A small thin plate of 0.125 m square slides parallel and midway between the planes and reaches a constant velocity of 2 m/s. The weight of the plate is 1 N. Determine the distance between the plates. **(5 mm)**

E1.6. A hydraulic lift shaft of 500 mm dia moves in a cylindrical sleeve the length of engagement being 2 m. The interface is filled with oil of kinematic viscosity of 2.4 × 10^{-4} m^2/s and density of 888 kg/m^3 . The drag resistance when the shaft moves at 0.2 m/s is 267.81 N. Determine the *ID* of the cylinder.

E1.7 A shaft of 150 mm dia rotates in bearings with a uniform oil film of thickness 0.8 mm. Two bearings of 15 cm width are used. The viscosity of the oil is 22 cP. Determine the torque if the speed is 210 rpm. **(10.58 Nm)**

E1.8 A circular disc rotates over a large stationary plate with a 2 mm thick fluid film between them, the viscosity of the fluid being 40 cp. The torque required to rotate the disc at 200 rpm was 0.069 Nm. Determine the diameter of the disc. **(200 mm)**

E1.9 The torque to overcome viscous drag of the oil film of viscosity of 28 cp in collar bearing of 0.16 m *ID* and 0.28 m *OD* running at 600 rpm was 0.79 Nm. Determine the film thickness. **(1.2 mm)**

E1.10 A conical bearing of outer radius 0.4 m and inner radius 0.2 m and height 0.2 m runs on a conical support with a clearance of 1 mm all around. The support is rotated at 600 rpm. Determine the viscosity of the oil used if the torque required was 21.326 Nm. Also determine the power dissipated (Fig. 1.9.3). **(20.0 cP, 1340 W)**

E1.11 If $u = 10\, y^{1.5}$ where u is in m/s and y is in m in a flow field up to $y = 0.08$ m, determine the wall shear stress and the shear stress at $y = 0.04$ and 0.08.

E1.12 Determine the pressure difference between two points 10 m apart in flow of oil of viscosity 13.98 cp in a pipe, pipe of 40 mm diameter, the flow velocity being 1.8 m/s. **(5 kN/m^2)**

E1.13 The viscosity of an oil of density 820 kg/m^3 is 30.7 poise. What will be the terminal velocity of a steel ball of density 7800 kg/m^3 and dia 1.1 mm when dropped in the oil? **(90 mm/min)**

E1.14 As shown in figure, a block of mass M slides on a horizontal table on oil film of thickness h and viscosity μ. The mass m causes the movement. Derive an expression for the viscous force on the block when it moves at a velocity u. Also obtain an expression for the maximum speed of the block. **($F = \mu u A/h$, $u_{max} = mgh/\mu A$)**

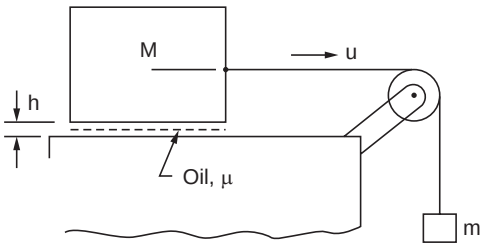


Figure E.1.13

E1.15 A viscous clutch as shown in figure transmits torque. Derive an expression for the torque and power transmitted. **($T = \pi\mu(\omega_1 - \omega_2) R^4/2a$, $P = \pi\mu\omega_2(\omega_1 - \omega_2) R^4/2a$)**

E1.16 Derive an expression for the torque in the case of a spherical bearing as shown in figure, in terms of h, r, ω, α and μ.

$$T = \frac{2\pi\mu\omega R^4}{h}\left[\frac{\cos^2\alpha}{3} - \cos\alpha + \frac{2}{3}\right]$$

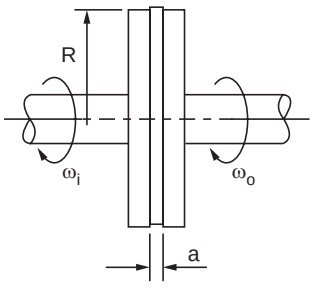


Figure E.1.15

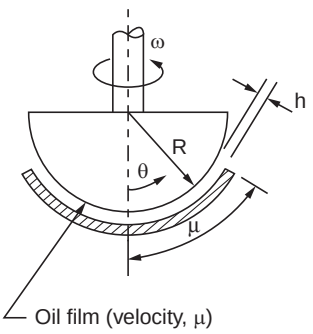


Figure E.1.16

E1.17 Calculate the shear stress due to fluid flow at the plate and at 10 mm above it if the velocity distribution along y direction is $u = 2y - 2y^3 + y^4$. μ = 0.001 Ns/m^2.

(2×10^{-3} N/m^2, 1.94 N/m^2)

E1.18 The capillary depression of mercury in a 3.25 mm *ID* glass tube was found as 2.99 mm. Determine the value of surface tension. β = 129°. **(0.511 N/m)**

E1.19 In order to separate a ring of 160 mm mean dia from water surface the force required just at the point of separation was 0.0732 N. Determine the surface tension of water at that temperature. **(0.0728 N/m)**

E1.20 In manometers an error in measurement will occur when a small bore tube is used. Capillary rise adds to the column height and capillary depression reduces the column height. The height of water column (at 20°C) in a tube of 8 mm *ID* is 12 mm, out of this 3.17 mm is due to capillary action. Determine the value of surface tension. β = 0. **(0.0728 N/m)**

E1.21 In order to lift a thin plate 1 m width slowly and vertically from a liquid, a force of 0.2 N was required at the instant of separation to overcome surface tension forces. Assume β = 0. Determine the value of surface tension. **(0.1 N/m)**

E1.22 Diesel injection nozzle sprays fuel of surface tension 0.0365 N/m. The pressure difference between the inside and outside of the nozzle was 2.874 kN/m^2. Determine the droplet size.

(0.0254 mm)

E1.23 Determine the droplet size if the pressure difference is increased to 7300 N/m^3 in the nozzle of a diesel engine. Assume the value of surface tension as 0.0365 N/m. **(10 μm)**

E1.24 A glass tube of 8 mm *ID* is immersed in a liquid at 20°C. The specific weight of the liquid is 20601 N/m^3. The contact angle is 60°. The capillary rise was 1.82 mm. Determine the value of surface tension and also the radius of curvature of the meniscus. **(0.15 N/m, 8 mm)**

E1.25 The actual atmospheric pressure was 765.92 mm of mercury column. Determine the height of column above the mercury well in a Fortins barometer using a tube of 3 mm dia. The contact angle is 140°. The surface tension is 0.51 N/m. The space above the column may be considered as vacuum. **(762 mm)**

E1.26 If the pressure difference between the inside and outside of a soap bubble of 2.5 mm dia is 17.6 N/m^2, determine the value of surface tension of the soap solution. **(0.022 N/m)**

E1.27 An additional force of 0.212 N was required to keep a cylinder of 150 mm OD with weight equal to the buoyant forces, floating in a liquid with contact angle $\beta = 60°$ due to surface tension effects. Determine the value of surface tension. **(0.45 N/m)**

E1.28 Show that the capillary rise in an annulus is given by $2\sigma \cos \beta/\gamma (r_o - r_i)$, where r_o and r_i are the radii and σ is the surface tension, γ is the specific weight and β is the contact angle.

E1.29 In case a capillary of diameter 3×10^{-6} m is used, determine the capillary rise in water. σ = 0.0735 N/m. **(10 m)**

E1.30 Bubbles are to be blown using a glass tube of 2 mm diameter immersed in oil to a depth of 10 mm. The specific gravity of oil is 0.96. If the surface tension of the oil is 0.0389 N/m. Determine the pressure inside the bubble at formation. **(172 N/m^2)**

E1.31 When 1000 cc of water is heated in a cylindrical vessel of 100 mm diameter from 20°C to 50°C the increase in the water level was 0.76 mm. Determine the coefficient of linear expansion for the vessel material. For water the coefficient of cubical expansion is 2.1×10^{-4} m^3/m^3 per °C. **(3.6×10^{-6} m/m per °C)**

E1.32 When the pressure of water in a press cylinder is released from 1000×10^5 N/m^2 to 1 bar, there was a 4.11 percentage increase in specific volume while the temperature remained constant. Determine the average value of bulk modulus for water in this range. **(2430×10^6 N/m^2)**

E133 The pressure of water increases with depth in the ocean. At the surface, the density was measured as 1024.5 kg/m^3. The atmospheric pressure is 1.01 bar. At a certain depth where the pressure was 900 bar the density was measured as 1065.43 kg/m^3. Determine the average value of bulk modulus. **(2340×10^6 N/m^2)**

E1.34 When water was heated in a rigid vessel the pressure rise was 14.49×10^6 N/m^2. Assuming that the vessel volume did not increase due to the increase in temperature or due to the stress induced, determine the percentage change in density. Assume $E_v = 2300 \times 10^6$ N/m^2. **(0.63%)**

E1.35 Due to an increase in pressure the volume of a liquid increases by 2.7%. Determine the pressure increase. The bulk modulus of the liquid is 37.04×10^9 N/m^2. **(1000 bar)**

E1.36 Determine the diameter of a spherical balloon at an altitude where pressure and temperature are 0.1 bar and –50°C, if 5.65 kg of hydrogen was charged into the balloon at ground level where the pressure and temperature were 1 bar and 30°C**.** **(10 m)**

2 Pressure Distribution in Fluids

2.0 INTRODUCTION

Fluids are generally found in contact with surfaces. Water in the sea and in reservoirs are in contact with the ground and supporting walls. Atmospheric air is in contact with the ground. Fluids filling vessels are in contact with the walls of the vessels. **Fluids in contact with surfaces exert a force on the surfaces**. The force is mainly due to the specific weight of the fluid in the case of liquids. In the case of gases molecular activity is the main cause of force exerted on the surfaces of the containers. Gas column will also exert a force on the base, but this is usually small in magnitude. **When the whole mass of a fluid held in a container is accelerated or decelerated without relative motion between layers inertia forces also exert a force on the container walls**. This alters the force distribution at stationary or atatic conditions. Surfaces may also be immersed in fluids. A ship floating in sea is an example. In this case the force exerted by the fluid is called buoyant force. This is dealt with in a subsequent chapter. The force exerted by fluids vary with location. The variation of force under static or dynamic condition is discussed in this chapter.

This chapter also deals with pressure exerted by fluids due to the weight and due to the acceleration/deceleration of the whole mass of the fluid without relative motion within the fluid.

Liquids held in containers may or may not fill the container completely. When liquids partially fill a container a free surface will be formed. Gases and vapours always expand and fill the container completely.

2.1 PRESSURE

Pressure is a measure of force distribution over any surface associated with the force. **Pressure is a surface phenomenon and it can be physically visualised or calculated only if the surface over which it acts is specified**. Pressure may be defined as the force acting along the normal direction on unit area of the surface. However a more precise definition of pressure, P is as below:

$$P = \lim_{A \to a} (\Delta F/\Delta A) = dF/dA \quad (2.1.1)$$

F is the resultant force acting normal to the surface area **A**. **'a'** is the limiting area which will give results independent of the area. This explicitly means that **pressure is the ratio of the elemental force to the elemental area normal to it.**

The force *dF* in the normal direction on the elemental area *dA* due to the pressure *P* is

$$dF = P\,dA \quad (2.1.2)$$

The unit of pressure in the SI system is N/m^2 also called Pascal (Pa). As the magnitude is small kN/m^2 (kPa) and MN/m^2 (Mpa) are more popularly used. **The atmospheric pressure is approximately 10^5 N/m^2 and is designated as "bar"**. This is also a popular unit of pressure. In the metric system the popular unit of pressure is kgf/cm^2. This is approximately equal to the atmospheric pressure or 1 bar.

2.2 PRESSURE MEASUREMENT

Pressure is generally measured using a sensing element which is exposed on one side to the pressure to be measured and on the other side to the surrounding atmospheric pressure or other reference pressure. The details of some of the pressure measuring instruments are as shown in Fig. 2.2.1.

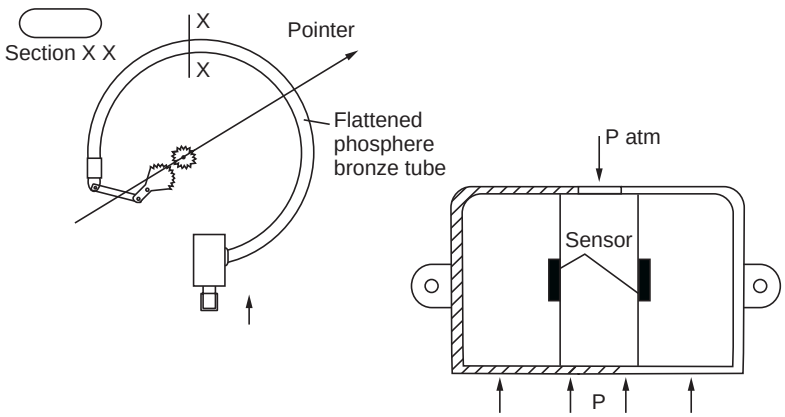


Figure 2.2.1 Pressure gauges

In the Borden gauge a tube of elliptical section bent into circular shape is exposed on the inside to the pressure to be measured and on the outside to atmospheric pressure. The tube will tend to straighten under pressure. The end of the tube will move due to this action and will actuate through linkages the indicating pointer in proportion to the pressure. Vacuum also can be measured by such a gauge. Under vacuum the tube will tend to bend further inwards and as in the case of pressure, will actuate the pointer to indicate the vacuum pressure. The scale is obtained by calibration with known pressure source.

The pressure measured by the gauge is called gauge pressure. The sum of the gauge pressure and the outside pressure gives the absolute pressure which actually is the pressure measured.

The outside pressure is measured using a mercury barometer (Fortins) or a bellows type meter called Aneroid barometer shown in Fig. 2.2.2. The mercury barometer and bellow type meter have zero as the reference pressure. The other side of the measuring surface in these cases is exposed to vacuum. Hence these meters provide the absolute pressure value.

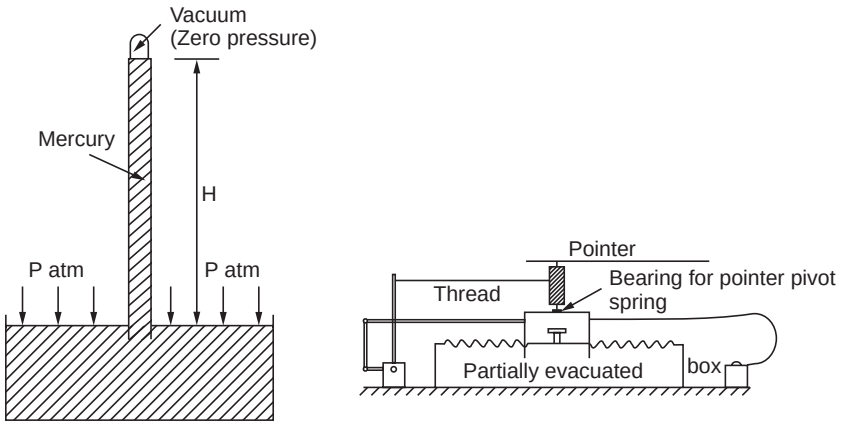


Figure 2.2.2 Barometer

When the pressure measured is above surroundings, then

Absolute pressure = gauge pressure + surrounding pressure

The surrounding pressure is usually the atmospheric pressure.

If the pressure measured is lower than that of surrounding pressure then

Absolute pressure = surrounding pressure – gauge reading

This will be less than the surrounding pressure. This is called Vacuum.

Electrical pressure transducers use the deformation of a flexible diaphragm exposed on one side to the pressure to be measured and to the surrounding pressure or reference pressure on the other side. The deformation provides a signal either as a change in electrical resistance or by a change in the capacitance value. An amplifier is used to amplify the value of the signal. The amplified signal is generally calibrated to indicate the pressure to be measured.

In this text the mension pressure means absolute pressure. Gauge pressure will be specifically indicated.

Example 2.1. *A gauge indicates 12 kPa as the fluid pressure while, the outside pressure is 150 kPa. Determine the absolute pressure of the fluid. Convert this pressure into* kgf/cm^2

Absolute pressure = Gauge pressure + Outside pressure

= 150 + 12 = **162 kPa** or 1.62 bar.

1.62 bar = 1.62×10^5 N/m^2

As $1 \text{ kgf/cm}^2 = 9.81 \text{ N/cm}^2 = 9.81 \times 10^4 \text{ N/m}^2 = 98100 \text{ N/m}^2$

$1.62 \times 10^5 \text{ N/m}^2 = 1.62 \times 10^5/98100 =$ **1.651 kgf/cm²**

Example 2.2. *A vacuum gauge fixed on a steam condenser indicates 80 kPa vacuum. The barometer indicates 1.013 bar. Determine the absolute pressure inside the condenser. Convert this pressure into head of mercury.*

Barometer reading = 1.013 bar = 101.3 kPa.

Absolute pressure = atmospheric pressure – vacuum gauge reading

Absolute pressure in the condenser = 101.3 – 80 = **21.3 kPa**

101.3 kPa = 760 mm of Hg. (standard atmosphere)

∴ 21.3 kPa = (21.3/101.3) × 760 = **159.8 mm of Hg**

2.3 PASCAL'S LAW

In fluids under static conditions pressure is found to be independent of the orientation of the area. This concept is explained by **Pascal's law** which **states that the pressure at a point in a fluid at rest is equal in magnitude in all directions**. Tangential stress cannot exist if a fluid is to be at rest. This is possible only if the pressure at a point in a fluid at rest is the same in all directions so that the resultant force at that point will be zero.

The proof for the statement is given below.

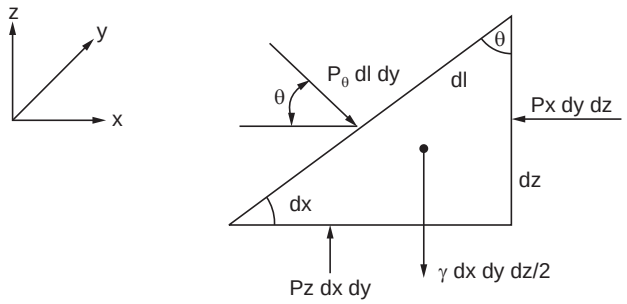


Figure 2.3.1 Pascals law demonstration

Consider a wedge shaped element in a volume of fluid as shown in Fig. 2.3.1. Let the thickness perpendicular to the paper be **dy**. Let the pressure on the surface inclined at an angle θ **to vertical be** P_θ and its length be **dl**. Let the pressure in the x, y and z directions be P_x, P_y, P_z.

First considering the x direction. For the element to be in equilibrium,

$$P_\theta \times dl \times dy \times \cos\theta = P_x \times dy \times dz$$

But, $dl \times \cos\theta = dz$ So, $P_\theta = P_x$

When considering the vertical components, the force due to specific weight should be considered.

$$P_z \times dx \times dy = P_\theta \times dl \times dy \times \sin\theta + 0.5 \times \gamma \times dx \times dy \times dz$$

The second term on RHS of the above equation is negligible, its magnitude is one order less compared to the other terms.

Also, $dl \times \sin\theta = dx$, So, $P_z = P_\theta$

Hence, $\mathbf{P_x = P_z = P_\theta}$

Note that the angle has been chosen arbitrarily and so this relationship should hold for all angles. By using an element in the other direction, it can be shown that

$$\mathbf{P_y = P_\theta \quad and \quad so \; P_x = P_y = P_z}$$

Hence, the pressure at any point in a fluid at rest is the same in all directions. The pressure at a point has only one value regardless of the orientation of the area on which it is measured. This can be extended to conditions where fluid as a whole (like a rotating container) is accelerated like in forced vortex or a tank of water getting accelerated without relative motion between layers of fluid. Surfaces generally experience compressive forces due to the action of fluid pressure.

2.4 PRESSURE VARIATION IN STATIC FLUID (HYDROSTATIC LAW)

It is necessary to determine the pressure at various locations in a stationary fluid to solve engineering problems involving these situations. **Pressure forces are called surface forces. Gravitational force is called body force as it acts on the whole body of the fluid.**

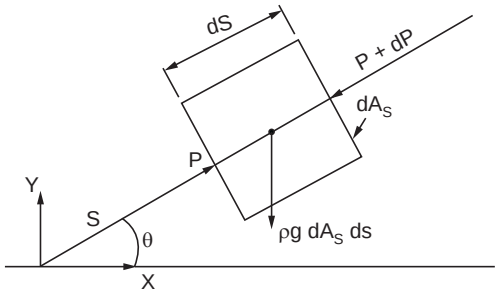


Figure 2.4.1 Free body diagram to obtain hydrostatic law

Consider an element in the shape of a small cylinder of constant area dA_s along the s direction inclined at angle θ to the horizontal, as shown in Fig. 2.4.1. The surface forces are P at section s and $P + dp$ at section $s + ds$. The surface forces on the curved area are balanced. The body force due to gravity acts vertically and its value is $\gamma \times ds \times dA_s$. A force balance in the s direction (for the element to be in equilibrium) gives

$$P \times dA_s - (P + dp) \times dA_s - \gamma \times dA_s \times ds \times \sin\theta = 0$$

Simplifying,

$$dp/ds = -\gamma \times \sin\theta \text{ or, } dp = -\gamma \times ds \times \sin\theta \qquad (2.4.1)$$

This is the fundamental equation in fluid statics. The variation of specific weight γ with location or pressure can also be taken into account, if these relations are specified as (see also section 2.4.2).

$$\gamma = \gamma (P, s)$$

For x axis, $\theta = 0$ and $\sin \theta = 0$.

$$\therefore \quad dP/dx = 0 \tag{2.4.2}$$

In a static fluid with no acceleration, the pressure gradient is zero along any horizontal line i.e., planes normal to the gravity direction.

In y direction, $\theta = 90$ and $\sin \theta = 1$,

$$dP/dy = -\gamma = -\rho g/g_o \tag{2.4.4}$$

Rearranging and integrating between limits y_1 and y

$$\int_{p_1}^{p} dp = -\gamma \int_{y_1}^{y} dy \tag{2.4.5}$$

If γ is constant as in the case of liquids, these being incompressible,

$$P - P_1 = -\gamma \times (y - y_1) = -\rho\, g\, (y - y_1)/g_o \tag{2.4.6}$$

As P_1, y_1 and γ are specified for any given situation, P will be constant if y is constant. This leads to the statement,

The pressure will be the same at the same level in any connected static fluid whose density is constant or a function of pressure only.

A consequence is that the free surface of a liquid will seek a common level in any container, where the free surface is everywhere exposed to the same pressure.

In equation 2.4.6, if $y = y_1$ then $P = P_1$ and $dp = 0$. This result is used very extensively in solving problems on manometers.

2.4.1 Pressure Variation in Fluid with Constant Density

Consider the equation 2.4.6,

$$P - P_1 = -\gamma \times (y - y_1) = \gamma \times (y_1 - y) = \rho\, g\, (y_1 - y)/g_o \tag{2.4.7}$$

As y increases, the pressure decreases and vice versa **(y is generally measured in the upward direction)**. In a static fluid, the pressure increases along the depth. If the fluid is incompressible, then **the pressure at any y location is the product of head and specific weight, where head is the y distance** of the point from the reference location.

Example 2.3. *An open cylindrical vertical container is filled with water to a height of 30 cm above the bottom and over that an oil of specific gravity 0.82 for another 40 cm. The oil does not mix with water. If the atmospheric pressure at that location is 1 bar,* ***determine the absolute and gauge pressures*** *at the oil water interface and at the bottom of the cylinder.*

This has to be calculated in two steps, first for oil and then for water.

Density of the oil $= 1000 \times 0.82 = 820$ kg/m^3

Gauge pressure at interface $= (\rho \times g \times h)^*_{\text{oil}}$

$= 820 \times 9.81 \times 0.4 =$ **3217.68 N/m^2**

Absolute pressure at interface $= 3217.68 + 1 \times 10^5$ N/m^2

$= 103217.68$ N/m^2 = **1.0322 bar**

Pressure due to water column $= \rho \times g \times h = 1000 \times 9.81 \times 0.3 = 2943$ N/m^2

Gauge pressure at the bottom = gauge pressure at the interface + $(\rho \times g \times h)_{water}$

= 3217.68 + 1000 × 9.81 × 0.3 = **6160.68 N/m²**

Absolute pressure at bottom

= 6160.68 + 1 × 10^5

= **106160.68 N/m² or 1.0616 bar**

This value also equals the sum of absolute pressure at interface and the pressure due to water column.

*__Note:__ g_o is left out as g_o = 1 in SI units

Example 2.4. *The gauge pressure at the surface of a liquid of density 900 kg/m³ is 0.4 bar. If the atmospheric pressure is 1 × 10^5 Pa, calulate the absolute pressure at a depth of 50 m.*

$\mathbf{P_{50}}$ = atmospheric pressure + pressure at top surface + ρgh

= 1 × 10^5 + 0.4 × 10^5 + 900 × 9.81 × 50 N/m²

= 5.8145 × 10^5 Pa = **5.8145 bar (absolute)**

2.4.2 Pressure Variation in Fluid with Varying Density

Consider equation 2.4.4,

$$dP/dy = -\gamma$$

Gamma can be a function of either P or y or both, If $\gamma = \gamma(y)$ then

$$\int dP = -\gamma(y)\, dy, \quad \text{If } \gamma = \gamma(P) \text{ then } \int \gamma(P)\, dP = \int dy,$$

If $\gamma = \gamma(P, y)$ then the variables should be separated and integrated.

Example 2.5. *The local atmospheric pressure at a place at 30° C is 1 bar. Determine the pressure at an altitude of 5 km if (i) the air density is assumed to be constant (ii) if the temperature is assumed to be constant and (iii) if with altitude the temperature decreases linearly at a rate of 0.005°C per metre. Gas constant R = 287 J / kg K*

(*i*) constant air density, using equation 2.4.2, 2.4.3 and 2.4.5

$$\int_{p_1}^{p} dp = -\int_{y_1}^{y} \gamma dy$$

$$\gamma = (P/RT) \times g = \{1 \times 10^5/[287 \times (273 + 30)]\} \times 9.81 = 11.28 \text{ N/m}^3$$

Integrating between 0 and 5000 m

$$P - 1 \times 10^5 = -11.28 \times (5000 - 0), \text{ Solving, } P = 43{,}600 \text{ N/m}^2 = \mathbf{0.436 \text{ bar}}$$

(*ii*) isothermal

$P \times v$ = constant or (P/ρ) = constant or $(P/\rho g)$ = constant or (P/γ) = constant, at any location

i.e., $(P/\gamma) = (P_o/\gamma_o)$; $\gamma = (P \times \gamma_o)/P_o$

As $(dP/dy) = -\gamma = -(P \times \gamma_o)/P_o$, separating variables

$(dP/P) = -(\gamma_o/P_o)\, dy$

Integrating from zero altitude to y m

$$\int_{p_1}^{p} (dp/p) = -(\gamma_o/p_o) \int_0^y dy$$

In $$(P/P_o) = -(\gamma_o/P_o) \times y \qquad (2.4.7)$$

$$P = P_o \exp [-(\gamma_o \times y/P_o)], \text{ Now } y = 5000, \quad (2.4.8)$$

$$P = 1 \times 10^5 \exp [-(11.28 \times 5000)/(1 \times 10^5)] = 56{,}893 \text{ N/m}^2$$

$$= \mathbf{0.56893 \text{ bar}}$$

(*iii*) The condition reduces to the form, $T = T_o - cy$

$$Pv = RT;\ (P/\rho) = RT;\ \rho = (P/RT);\ \rho \times g = (P \times g/RT);$$

$$\gamma = Pg/RT = (g/R) \times [P/(T_o - c \times y)]$$

$$(dP/dy) = -\gamma = -(g/R) \times [P/(T_o - c \times y)]$$

or, $$(dP/P) = -(g/R) \times [dy/(T_o - c \times y)], \text{ Integrating,}$$

$$\int_{p_1}^{p} (dp/p) = -(g/R) \int_0^y dy/(T_1 - cy)$$

In $$(P/P_o) = -(g/R)\{1/(-c)\} \text{ In } \{(T_o - cy)/(T_o - c \times 0) \quad (2.4.9)$$

$$= (g/Rc) \ln \{(T_o - cy)/T_o\}$$

or $$P/P_o = [(T_o - cy)/T_o]^{-g/RC} \quad (2.4.10)$$

Substituting for $T_o = 303$ and $c = 0.005$, $y = 5000$ m and solving,

$$P = 55{,}506 \text{ N/m}^2 = \mathbf{0.55506 \text{ bar}}$$

2.5 MANOMETERS

Manometer is a device to measure pressure or mostly difference in pressure using a column of liquid to balance the pressure. It is a basic instrument and is used extensively in flow measurement. It needs no calibration. Very low pressures can be measured using micromanometers. The basic principle of operation of manometers is that at the same level in contigues fluid at rest, the pressure is the same. The pressure due to a constant density liquid (ρ) column if height h is equal to $\rho gh/g_o$. g_o in SI system of units has a numerical value of unity. Hence it is often left out in the equations. For dimensional homogenity g_o should be used. The principle of operation is shown in Fig. 2.5.1 (*a*) and some types of manometers are shown in Fig. 2.5.1 (*b*). In Fig. 2.5.1 (*a*), the pressure inside the conduit is higher than atmospheric pressure. The column of liquid marked *AB* balances the pressure existing inside the conduit. The pressure at point *C* above the atmospheric pressure (acting on the open limb) is given by $h \times (\gamma_1 - \gamma_2)$ where γ_1 and γ_2 are the specific weights of fluids 1 and 2, and h is the height of the column of liquid (*AB*). The pressure at the centre point *D* can be calculated as

$$P_d = P_c - \gamma_2 \times h'$$

Generally the pressure at various points can be calculated using the basic hydrostatic equation $dP/dy = -\gamma$ and continuing the summation from the starting point at which pressure is known, to the end point, where the pressure is to be determined.

Another method of solving is to start from a point of known pressure as datum and adding $\gamma \times \Delta y$ when going downwards and subtracting of $\gamma \times \Delta y$ while going upwards. The pressure at the end point will be the result of this series of operations.

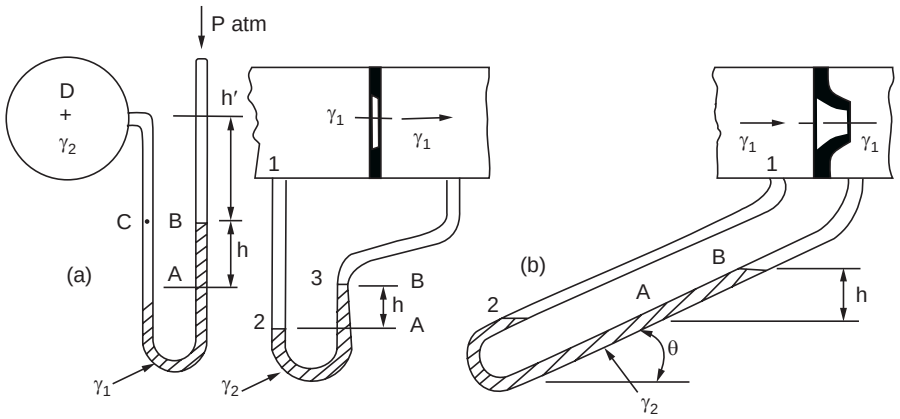


Figure 2.5.1 Types of manometers

$$\Delta P_{1-5} = \gamma_1 \Delta y_1 + \gamma_2 \Delta y_2 + \gamma_3 \Delta y_3 + \gamma_4 \Delta y_4$$

with proper sign for Δy values.

The advantages of using manometers are (*i*) their simplicity (*ii*) reliability and (*iii*) ease of operation and maintenance and freedom from frequent calibration needed with other types of gauges. As only gravity is involved, horizontal distances need not be considered in the calculation.

The sensitivity of simple manometers can be improved by using inclined tubes (at known angle) where the length of the column will be increased by (1/sin θ) where θ is the angle of inclination with the horizontal (Fig. 2.5.1 (*b*)).

Example 2.6. *A manometer is fitted as shown in Fig. Ex. 2.6. Determine the pressure at point A.*

With respect to datum at *B*, pressure at left hand side = pressure at right hand side

$P_C = P_B$ Consider the left limb

$P_C = P_a + 0.125 \times 900 \times 9.81 + 0.9 \times 13600 \times 9.81$

$= P_a + 121178$ N/m^2

Consider the right limb $P_A = P_B - 0.9 \times 1000 \times 9.81 = P_a + 121178 - 0.9 \times 1000 \times 9.81$

$= P_a + 112349$ N/m^2 Expressed as gauge pressure

$\mathbf{P_A} = 112349$ N/m^2

$=$ **112.35 kPa** gauge

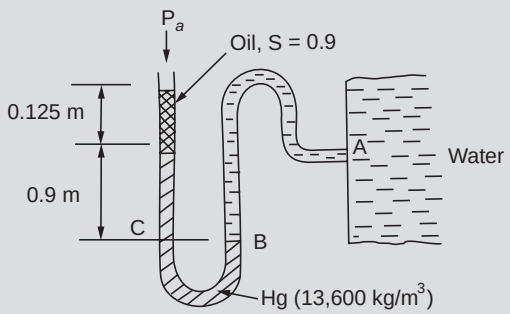


Figure Ex. 2.6

Example 2.7. *An inverted U-tube manometer is fitted between two pipes as shown in Fig.Ex.2.7. Determine the pressure at E if P_A = 0.4 bar (gauge)*

$$P_B = P_A - [(0.9 \times 1000) \times 9.81 \times 1.2]$$
$$= 40000 - [(0.9 \times 1000) \times 9.81 \times 1.2] = 29{,}405.2 \text{ N/m}^2$$
$$P_C = P_B - [(0.9 \times 1000) \times 9.81 \times 0.8] = 22342 \text{ N/m}^2$$
$$P_C = P_D = 22342 \text{ N/m}^2$$
$$\mathbf{P_E} = P_D + [1000 \times 9.81 \times 0.8] = 30190 \text{ N/m}^2 = \mathbf{30.19 \text{ kPa (gauge)}}$$

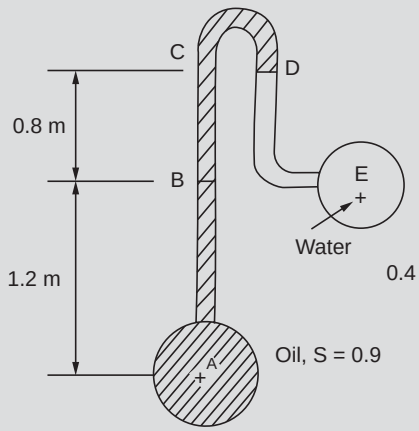


Figure Ex. 2.7

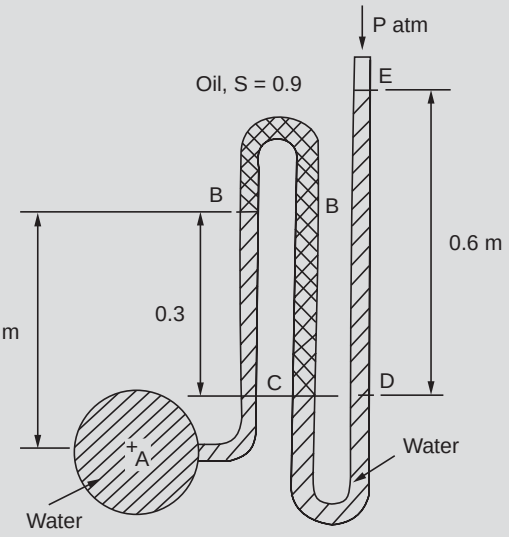


Figure Ex. 2.8

Example 2.8. *A multiple U-tube manometer is fitted to a pipe with centre at A as shown in Fig. Ex.2.8. Determine the pressure at A.*

Pressure at E = atmospheric pressure, P_{atm}

$$P_D = P_{atm} + (1000 \times 9.81 \times 0.6) = P_{atm} + 5886 \; P_a \text{ As } P_C = P_D$$
$$P_B = P_C - [0.9 \times 1000 \times 9.81 \times 0.3]$$
$$= P_{atm} + 5886 - 2648.7 = P_{atm} + 3237.3 \text{ Pa}$$
$$\mathbf{P_A} = P_B + [1000 \times 9.81 \times 0.4] = P_{atm} + 7161.3$$
$$= P_{atm} + 7161.3 \text{ N/m}^2 \text{ or } \mathbf{7161.3 \text{ kPa (gauge)}}$$

2.5.1 Micromanometer

Small differences in liquid levels are difficult to measure and may lead to significant errors in reading. Using an arrangement as shown in Fig. 2.5.1, the reading may be amplified. For improved accuracy the manometer fluid density should be close to that of the fluid used for measurement.

Chambers A and B are exposed to the fluid pressures to be measured. $P_A - P_B$ is the required value. These chambers are connected by a U tube having a much smaller area compared to the chambers A and B. The area ratio is the significant parameter. The volumes above this manometric fluid is filled with a fluid of slightly lower density.

Let pressure $P_A > P_B$ and let it cause a depression of Δy in chamber A. The fluid displaced goes into the U tube limb of area a. The displacement in the limb will therefore by $(y \times A/a)$ which becomes better readable.

Let the original level of manometric fluid in the U tube be at 2-2 and let the fluid levels originally in the chambers be 1-1. After connecting to the pressure sides let the level of manometric fluid be 3-3 on the high pressure side. Let the displacement in the chamber A be Δy. Let the specific weight of the pressure side fluid be γ_1 and that of the other fluid be γ_2 and that of the manometric fluid be γ_3. The fall in level of the manometric fluid from 2–3 on the left limb will equal the rise of the level from 3 to 4 in the right limb.

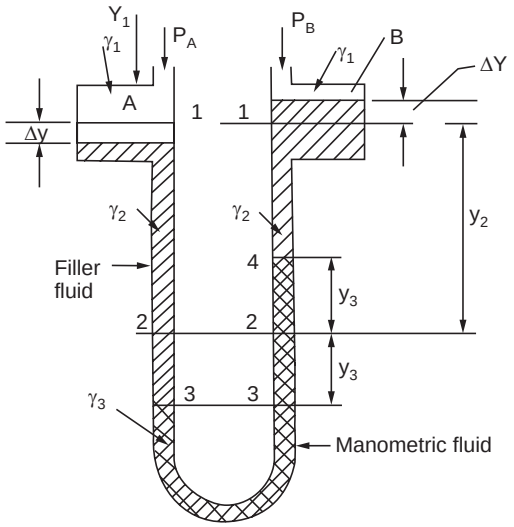


Figure 2.5.2 Micromanometer

Starting from level in chamber A and level 3 as datum

$$P_B = P_A + \{(y_1 + \Delta y) \times \gamma_1\} + \{(y_2 + y_3 - \Delta y) \times \gamma_2\} - \{2y_3 \times \gamma_3\} - \{(y_2 - y_3 + \Delta y) \times \gamma_2\} - \{(y_1 - \Delta y) \times \gamma_1\}$$

$$= P_A - [2 \times y_3 \times (\gamma_3 - \gamma_2) + 2 \times \Delta y \times (\gamma_2 - \gamma_1)]$$

As $$\Delta y = (a/A) \times y_3$$

$$P_A - P_B = 2 \times y_3 \times [\gamma_3 - \gamma_2 \times \{1 - (a/A)\}] - [2 \times y_3 \times (a/A) \times \gamma_1] \quad (2.5.1)$$

Very often γ_1 is small (because gas is generally the medium) and the last term is negligible. So

$$P_A - P_B = 2 \times y_3 \times [\gamma_3 - \gamma_2 \times \{1 - (a/A)\}] \quad (2.5.2)$$

For a given instrument y_3 is a direct measure of $\Delta P \rightarrow (P_A - P_B)$.

To facilitate improved reading accuracy or increased value of y_3, it is necessary that $(\gamma_3 - \gamma_2)$ is small.

Example 2.9. *A micromanometer is to be used to find the pressure difference of air flowing in a pipeline between two points A and B. The air density is 1.2 kg/m^3. The micromanometer fluid is having a specific gravity of 1.1 and the filler fluid is water. Under measuring conditions, the manometric fluid movement on the pressure side is 5 cm. Determine the pressure difference between the two points A and B, if the area of the well chamber is 10 times that of the tube.*

Refer Fig. 2.5.1

$y_3 = 5$ cm $= 0.05$ m;

$\gamma_1 = 1.2 \times 9.81$ N/m^3, $\gamma_2 = 1000 \times 9.81$ N/m^3;

$\gamma_3 = 1.1 \times 1000 \times 9.81$ N/m^3; and $(a/A) = 1/10$,

Using equation 2.5.1

$$\mathbf{P_A - P_B} = 2 \times y_3 \times [\gamma_3 - \gamma_2 \times \{1 - (a/A)\}] - \{2 \times y_3 \times (a/A) \times \gamma_1\}$$

$$= 2 \times 0.05 \times [1.1 \times 1000 \times 9.81) - 1000 \times 9.81) \times (1 - 1/10)] - \{2 \times 0.05 \times (1/10) \times 1.2 \times 9.81\}$$

$$= 196.2 - 0.11772 = \mathbf{196.08\ N/m^2}$$

The second term due to air is negligible as it does not contribute even 0.1%. The advantage of this micromanometer is that the deflection is as high as 5 cm even for a pressure difference of 196.08 Pa. This helps to measure very low pressure differences with sufficient accuracy. In case ordinary manometer is used the deflection will be 5 mm only.

Example 2.10. *Determine the fluid pressure at a tapping connected with an inclined manometer if the rise in fluid level is 10 cm along the inclined tube above the reservoir level. The tube is inclined at 20° to horizontal as shown in figure. The density of manometric fluid is 800 kg / m^3.*

The actual head, $y = 0.1 \times \sin 20 = 0.0342$ m

Pressure at the tapping point $= \gamma \times y = 800 \times 9.81 \times 0.0342$

$= 268.42$ N/m^2 (gauge)

Reading accuracy is improved as 3.42 cm is amplified to 10 cm.

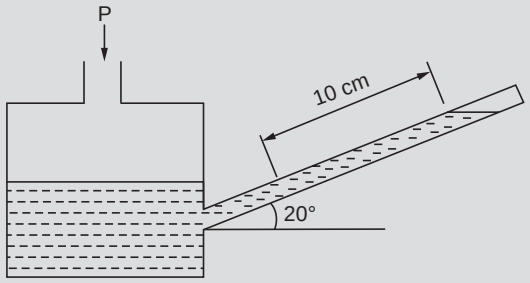


Figure Ex. 2.10

2.6 DISTRIBUTION OF PRESSURE IN STATIC FLUIDS SUBJECTED TO ACCELERATION, a_s

Consider the small cylindrical element of sectional area dA_s and length s inside the fluid, which is accelerated at a_s along the s direction. For equilibrium along s direction,

Surface forces + Body forces = Inertia forces

The net force in the s direction = rate of change of momentum is s direction.

Pressure force + Body force along s direction

$$= \{P \times dA_s - (P + dP) \times dA_s\} - \gamma \times dA_s \times ds \times \sin \theta$$

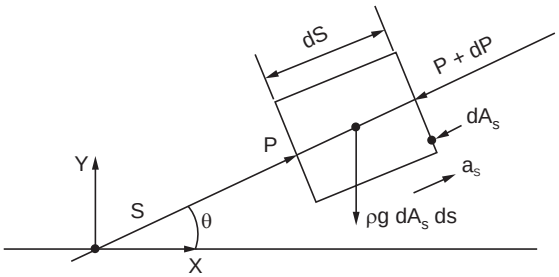


Figure 2.6.1 Free body diagram for accelerating fluid element

Inertial force = The rate of change of momentum = $\rho \times dA_s \times ds \times a_s$

Equating and simplifying,

$$dP/ds = -(\gamma \times \sin\theta + \rho \times a_s) \tag{2.6.1}$$

For the y direction, $\theta = 90°$

$$dP/dy = -(\gamma + \rho \times a_y) \tag{2.6.2}$$

dP/dy will be zero when, $\gamma = -\rho \times a_y$

For the x direction, $\theta = 0°$

$$dP/dx = -\rho \times a_x \tag{2.6.3}$$

This shows that when there is acceleration, a pressure gradient in x direction (horizontal direction) is also possible. The above three equations are to be used to determine the pressure distribution in cases where the fluid as a whole is accelerated without flow or relative motion in the fluid.

These equations can be integrated if a_s, γ, ρ are specified as functions of P or s. However, variable density problems are more involved in this situation and solutions become more complex.

2.6.1 Free Surface of Accelerating Fluid

The pressure gradient along any free surface is zero, as this surface is exposed to the same pressure all over. If the direction of free surface is s then $dP/ds = 0$. Using equation 2.6.1

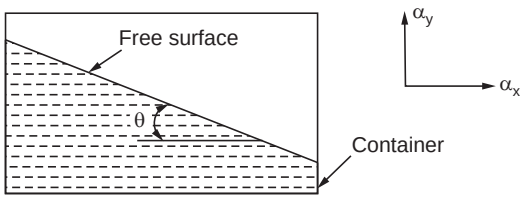


Figure 2.6.2 Free surface of accelerating fluid

$$\gamma \times \sin\theta = -\rho \times a_s \tag{2.6.5}$$

or
$$\theta = \sin^{-1}(-\rho \times a_s/\gamma) = \sin^{-1}(-a_s/g) \tag{2.6.6}$$

In general, for acceleration, in direction s inclined at θ to x direction, (two dimensional)

$$a_s = a_y \times \sin\theta + a_x \times \cos\theta$$

Substituting in equation 2.6.5 and rearranging

$$\tan\theta = -[a_x/(g + a_y)] \tag{2.6.7}$$

The consequence of these equations are

***(i)* If $a_x = 0$, the free surface will be horizontal**

***(ii)* If $g = 0$, $\tan\theta = -a_x/a_y$**

The constant pressure surface (free surface) will be normal to the resultant acceleration.

(*iii*) In general, the free surface angle will depend on a_x, a_y and g.

(*iv*) The free surfaces of liquids are constant pressure surfaces and hence follow equations 2.6.5–2.6.7.

When an open container filled with liquid accelerates, a free surface will be formed as specified by the above equations. **When gravity is not present, liquids may not assume a free surface but will be influenced only by surface tension. In space liquid spilling poses problems because of this condition**. When a closed container completely filled with liquid is accelerated a free surface cannot form. But the pressure at the various locations will be governed by these equations.

2.6.2 Pressure Distribution in Accelerating Fluids along Horizontal Direction

Using the general expression for the model (fluid under acceleration) and the equation 2.6.1

$$(dP/ds) = -(\gamma \times \sin\theta + \rho \times a_s)$$

$\theta = 0$ for x direction, $ds = dx$, $a_s = a_x$ (x directional acceleration)

$$(dP/dx) = -(\rho \times a_x) \tag{2.6.8}$$

***(i)* For constant density conditions:**

$$\int_{p_1}^{p_2} dp = -(\rho\, a_s) \int_{x_1}^{x_2} dx$$

$$(P_2 - P_1) = -(\rho \times a_x)(x_2 - x_1)$$

$$P_2 = P_1 - (\rho \times a_x)(x_2 - x_1) \tag{2.6.9}$$

a_x is positive in x direction (towards right) and negative in the $-x$ direction (left).

***(ii)* If density varies with pressure as, $\rho = AP + B$** (A, B are constants):

Using equation 2.6.8, $[dP/(AP + B)] = -a_x \times dx$

Integrating, between the locations x_1 and x_2

$$(1/A) \times [\ln(AP + B)]_{p_1}^{p_2} = -a_x(x_2 - x_1) \text{ or}$$

$$\ln[(AP_2 + B)/(AP_1 + B)] = -A \times a_x \times (x_2 - x_1)$$

$$(AP_2 + B) = (AP_1 + B) \exp[-A \times a_x \times (x_2 - x_1)]$$

or
$$P_2 = (1/A)[(AP_1 + B) \exp\{-A \times a_x \times (x_2 - x_1)\} - B] \tag{2.6.10}$$

This equation provides solution for pressure variation in the x direction when density varies linearly with pressure.

Example 2.11. *A cylinder (Figure Ex. 2.11) containing oil of specific gravity 0.8 is accelerated at 5 m/s² towards (i) right and (ii) left. Under this condition the pressure gauge fitted at the right end shows a reading of 150 kPa.* ***Determine the pressure at the left end*** *if the tube is 2 m long.*

Since the specific gravity of the oil is constant, equation 2.6.9 can be used to solve this problem.

$$P_2 = P_1 - \rho \times a_x \times (x_2 - x_1)$$

Case (i) a_x is towards right and so +ve and $(x_2 - x_1) = 2$ m

$$1{,}50{,}000 = P_1 - 800 \times 5 \times 2,$$

$$\therefore \quad \mathbf{P_1} = 1{,}58{,}000 \text{ N/m}^2 = \mathbf{158 \text{ kPa.}}$$

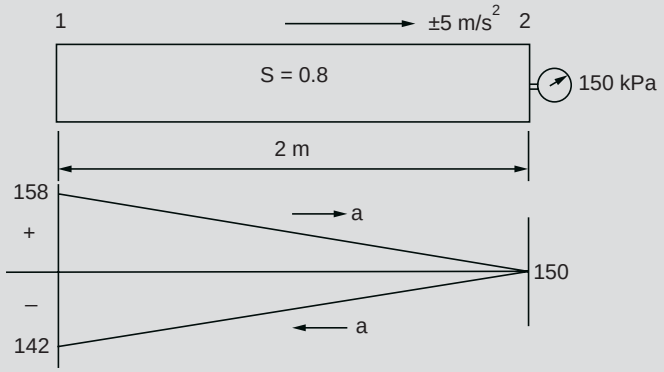


Figure Ex. 2.11

Case (ii) a_x is towards left and so –ve.

$$1{,}50{,}000 = P_1 + 800 \times 5 \times 2$$

$$\mathbf{P_1} = 1{,}42{,}000 \text{ Pa} = \mathbf{142 \text{ kPa}}. \text{ (note the unit of pressure used is N/m}^2\text{)}$$

Example 2.12. *A horizontal long cylinder containing fluid whose density varies as* $= 1.2 \times 10^{-5} \times P$ *is accelerated towards right at 15 m/s².* ***Determine the pressure at a point which is 5 m to the left*** *of a point where the pressure gauge shows a reading of 250 kPa.*

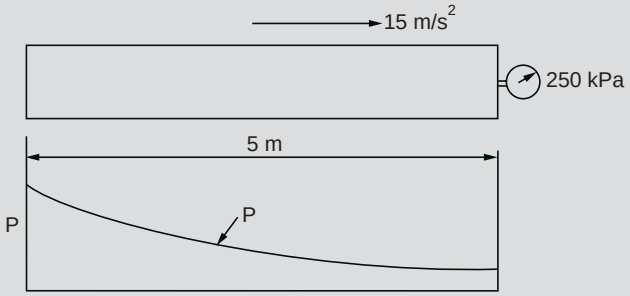


Figure Ex. 2.12

Equation 2.6.10 has to be used as density varies with pressure

$$P_2 = \{1/A\}\ \{AP_1 + B) \times \exp\ [-A \times a_x(x_2 - x_1)] - B\}$$

Here, $B = 0$, $A = 1.2 \times 10^{-5}$, $a_x = 15$ m/s², $x_2 - x_1 = 5$ m

$$2{,}50{,}000 = \{1/1.2 \times 10^{-5}\} \times \{(1/1.2 \times 10^{-5} \times P_1) \times \exp\ [(-1.2 \times 10^{-5} \times 15 \times 5)]\}$$

$$\mathbf{P_1 = 250225 \text{ Pa} = 250.225 \text{ kPa.}}$$

Example 2.13. *A fluid of specific gravity 0.8 is filled fully in a rectangular open tank of size 0.5 m high, 0.5 m wide and 0.8 m long. The tank is uniformly accelerated to the right at 10 m/s². **Determine the volume of fluid spilled from the tank**.*

Since the fluid tank is accelerated in the horizontal direction $a_y = 0$.

Using equation 2.6.7, $\tan\theta = -a_x/(g + a_y) = -10/(9.81 + 0)$

With reference to the figure,

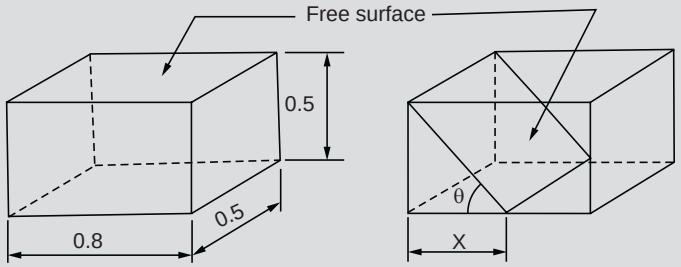


Figure Ex. 2.13

$\tan\theta = -0.5/x = -10/9.81$, So $x = 0.4905$ m

Remaining volume of fluid $= (1/2) \times 0.4905 \times 0.5 \times 0.5 = 0.0613125\ m^3$

Fluid tank volume or initial volume of fluid

$= 0.5 \times 0.5 \times 0.8 = 0.2\ m^3$

Fluid spilled $= 0.2 - 0.0613125 =$ **0.1386875 m³**

Example 2.14. *A U-tube as shown in figure filled with water to mid level is used to measure the acceleration when fixed on moving equipment. Determine the acceleration a_x as a function of the angle θ and the distance A between legs.*

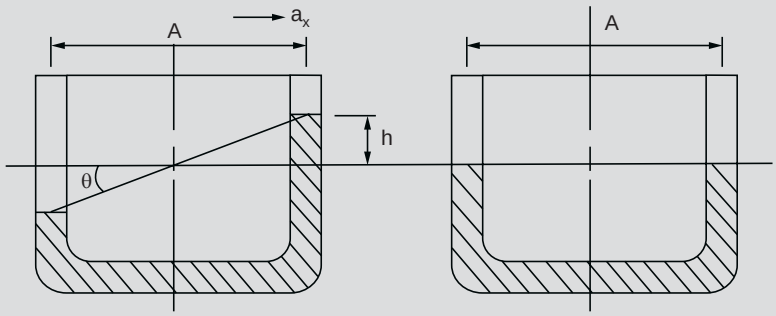


Figure Ex. 2.14

This is similar to the formation of free surface with angle θ, using eqn. 2.6.7

$\tan\theta = -a_x/(g + a_y)$. As $a_y = 0$, $\tan\theta = -a_x/g$

The acute angle θ will be given by, $\theta = \tan^{-1}(a_x/g)$

$\mathbf{a_x} = g \times \tan\theta$, As $\tan\theta = 2h/A$, $\mathbf{h = A\ a_x/2g}$

Example 2.15. *Water is filled in a rectangular tank of 0.5 m high, 0.5 m wide and 0.8 m long to a depth of 0.25 m. **Determine the acceleration** which will cause water to just start to spill and also when half the water has spilled.*

Since the tank is half full, at the time of spill the free surface will be along the left top and right bottom. The angle of the free surface with horizontal at the time of starting of spill is a_x/g = tan θ = 0.5/0.8 = 0.625

$\therefore$ $\mathbf{a_x}$ = 0.625 × 9.81 = **6.13 m/s²**

When half the water has spilled, the water will be at 0.4 m at bottom

tan θ = 0.5/0.4 = 1.25

$\therefore$ a_x = 1.25 × 9.81 = **12.26 m/s²**

Example 2.16. *A tank containing 1.5 m height of water in it is accelerating downwards at 3.5 m/s². **Determine the pressure** at the base of the tank above the atmospheric pressure. **What should be the acceleration** if the pressure on the base to be atmospheric?*

Using equation 2.6.2,

$$(dP/dy) = -(\gamma + \rho \times a_y);\ dP = -(\gamma + \rho \times a_y)\, dy$$

$$(P_2 - P_1) = (y_1 - y_2)(\gamma + \rho \times a_y)$$

$(y_1 - y_2)$ = 1.5 m, as a_y is downwards and hence negative

$$(P_2 - P_1) = 1.5\,(\gamma - \rho \times a_y)$$

$$(P_2 - P_1) = 1.5\,(9810 - 1000 \times 3.5)/1000 = \mathbf{9.465\ kN/m^2}\ \text{(above atmospheric)}$$

At static conditions, the pressure would have been

1.5 × 1000 × 9.81/1000 = 14.715 kN/m^2 (above atmospheric)

for the pressure at base to be atmospheric,

$P_2 - P_1 = 0 = 1.5\,[9810 - 1000\ a_y]$. i.e., $\mathbf{a_y = 9.81\ m/s^2}$

This is the situation of weightlessness, where $\mathbf{a_y = -g}$, the weight of water is zero

2.7 FORCED VORTEX

When a cylindrical container filled with a liquid is rotated about its axis, the liquid as a whole rotates. The angular velocity is the same at all points, but the linear velocity varies along the radius. The variation of the linear speed with radius causes a concave free surface to form, with fluid moving away from the centre.

The fluid rotates as a rigid body with velocity of ω × r at a radius r (ω being the angular velocity). Fluid particles rotating in concentric circle with velocities of r × ω along the tangent to the circles form a forced vortex. It is assumed that there is no relative sliding between layers.

The pressure variations and gradients caused by the rotation can be determined using equations 2.6.1 – 2.6.3. An element of fluid as shown in Fig. 2.7.1 is considered. The radius r is taken as positive along the outward direction.

Equation 2.6.1 gives

$$dP/dr = -\rho \times a_s = \rho \times r \times \omega^2 \text{ as } \theta = 0, \text{ and } a_r = -r \times \omega^2$$

$$dP/dy = -\gamma \text{ as } a_y = 0$$

Using the first equation, the pressure change along r_1 and r_2 is obtained as

$$(P_{r_2} - P_{r_1}) = \rho \times (\omega^2/2) \times (r_2^{\,2} - r_1^{\,2}) \qquad (2.7.1)$$

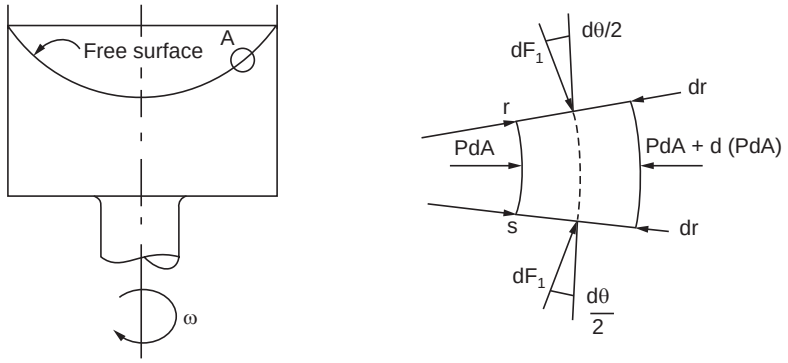


Figure 2.7.1 Free body diagram of rotating fluid element

From centre to any radius r,

$$(P_r - P_o) = \rho \times (\omega^2/2) \times r^2 = \rho \times (\omega r^2)/2 \tag{2.7.2}$$

If the pressure at the centre of the base or any radius is known, the pressure at all other points on the base can be calculated.

$$(P - P_b) = -\gamma \times (y - y_o) = -\gamma \times y, \text{ taking } y_o \text{ as the datum}$$

Here P is the pressure at the surface at any radius and P_b is the pressure at the base at the same radius and y is the height of liquid at that location. This gives

$$P_b = P + \gamma \times y \tag{2.7.3}$$

In order to determine the value of slope at any radius equation 2.6.7 is used. The surface profile is shown in Fig. 2.7.2.

$$\tan \theta = -a_x / (g + a_y)$$

For a rotating cylinder,

$$a_y = 0,\ a_x = -r \times \omega^2 \text{ and } \tan \theta = dy/dr = r\omega^2/g$$

Hence, $dy/dr = r \times \omega^2/g \quad \therefore \quad dy = r\,dr \times \omega^2/g$

Integrating from centre to radius r and rearranging,

$$y_2 - y_1 = \frac{\omega^2}{2g} [r_2^2 - r_1^2] \tag{2.7.4}$$

$$y = y_o + [(\omega \times r)^2/(2 \times g)] \tag{2.7.4 (a)}$$

where y_o is the height of liquid at the centre. This shows that the free surface is a paraboloid. The height y at any radius depends on the angular velocity, radius and g.

In forced vortex, v/r = constant as $v = \omega.r$ and ω is constant

If $g = 0$, (space application) then $y \to \infty$ and the free surface becomes cylindrical or the liquid adheres to the surface in a layer.

A free vortex forms when the container is stationary and the fluid drains at the centre as in the case of draining a filled sink. Here the fluid velocity is inversely proportional to the radius (volume flow depends on area), the velocity near the centre being the highest ($v \times r$ = constant).

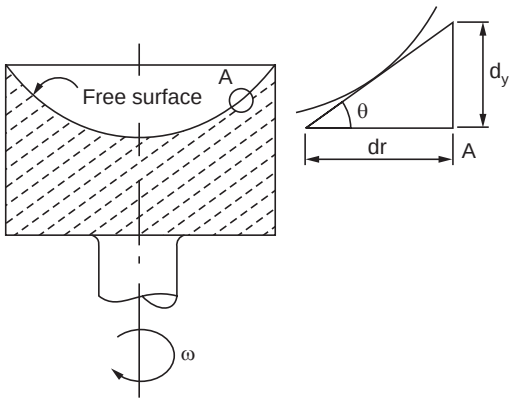
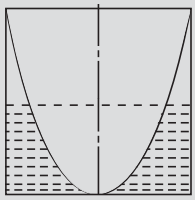


Figure 2.7.2 Forced Vortex—Free surface

Example 2.17. *A tall cylinder of 1 m dia is filled with a fluid to a depth of 0.5 m and rotated at a speed such that the height at the centre is zero. Determine the speed of rotation.*



It is to be noted here that the volume of a paraboloid of height h is equal to the volume of cylinder of half its height and the same radius. Hence the height at the outer radius is 1 m. Using equation 2.7.4

$y = y_o + [(\omega^2 \times r^2)/(2 \times g)]$, substituting the values,

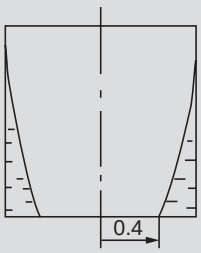
$1 = 0 + (\omega^2 \times 0.5^2)/(2 \times 9.81)$,

$\therefore \quad \omega = 8.86$ rad/s

$\boldsymbol{\omega} = 2\pi$ N/60; N = $(8.86 \times 60)/(2 \times \pi)$ = **84.6 rpm**

Example 2.18. *Water is filled partially in a cylinder of 1 m dia and rotated at 150 rpm. The cylinder is empty at the bottom surface up to a radius of 0.4 m.* ***Determine the pressure at the extreme bottom edge****. Also calculate the height of liquid at the edge.*

Equation 2.7.1 is applicable.



$$(P_{r_2} - P_{r_1}) = \rho \times (\omega^2/2) \times (r_2^2 - r_1^2)$$

$P_{r_1} = 0$ (gauge) at $r_1 = 0.4$, $r_2 = 0.5$ m

$\boldsymbol{\omega} = 2\pi$ N/60 = $2 \times \pi \times 150/60$ = **15.71 rad/s.**

$P_{r_2} - 0 = 1000 \times (15.71^2/2) \times (0.5^2 - 0.4^2)$ N/m^2

$P_{r_2} = 11106.184$ N/m^2 (gauge),

Using equation 2.7.4,

$$\mathbf{y_2 - y_1} = \frac{\omega^2}{2g} [r_2^2 - r_1^2]$$

$= 15.71^2 (0.5^2 - 0.4^2)/2 \times 9.81$ = **1.132 m**

SOLVED PROBLEMS

Problem 2.1. *Four pressure gauges A, B, C and D are installed as shown in figure in chambers 1 and 2. The outside pressure is 1.01 bar. The gauge A reads 0.2 bar, while the gauge*

B reads – 0.1 bar. Determine the pressures in chamber 1 and chamber 2 and the reading of gauge C and D.

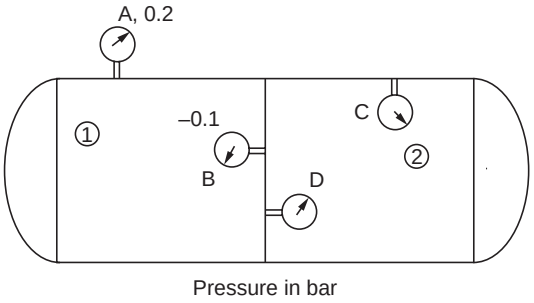


Figure P. 2.1

pressure in chamber 1 = atmospheric *pr* + reading of gauge *A*

= 1.01 + 0.2 = 1.21 bar

pressure in chamber 2 = pressure in chamber 1 + reading of gauge *B*

= 1.21 – 0.1 = 1.11 bar

atmospheric pressure = pressure in chamber 2 + reading of gauge *C*

reading of gauge $C = 1.01 - 1.11 = -0.1$ bar

Gauge *D* reads the pressure in chamber 1 as compared to chamber 2

gauge reading *D* = pressure in chamber 1 – pressure in chamber 2

= 1.21 – 1.11 = 0.1 bar (opposite of gauge *B*)

Problem 2.2. *The pressures in chambers A, B, C and D as shown in Fig. P.2.2 are 3.4, 2.6, 1.8 and 2.1 bar respectively. Determine the readings of gauges 1 to 6.*

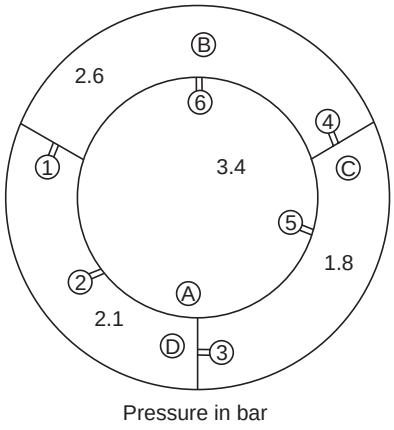


Figure P. 2.2

Gauge 1. This gauge measures the pressure in chamber *B* and the gauge is situated in chamber *D*, denoting the gauge reading by the corresponding suffix,

$$P_B = P_1 + P_D,\ 2.6 = P_1 + 2.1$$

$$\therefore \quad P_1 = 0.5 \text{ bar}$$

gauge 1 should show 0.5 bar

Gauge 2. This gauge measures the pressure in chamber *A*. The gauge is in chamber *D*.

$$P_A = P_2 + P_D, \qquad 3.4 = P_2 + 2.1$$

$\therefore$ $P_2 = 1.3$ bar,

By similar procedure the reading of gaug 3, 4, 5, 6 are obtained as below:

P_3: $P_D = P_3 + P_C$, $2.1 = 1.8 + P_3$ $\therefore$ $P_3 = 0.3$ bar,

P_4: $P_C = P_4 + P_B$, $1.8 = 2.6 + P_4$ $\therefore$ $P_4 = -0.8$ bar,

P_5: $P_C = P_5 + P_A$, $1.8 = 3.4 + P_5$ $\therefore$ $P_5 = -1.6$ bar,

P_6: $P_B = P_6 + P_A$, $2.6 = 3.4 + P_6$ $\therefore$ $P_6 = -0.8$ bar

The gauge readings show the pressure difference between the chambers connected and not absolute pressures. For example reading of $P_5 = -1.6$ bar. Such a vacuum is not possible.

Problem 2.3. *A container has h_w cm of water over which h_k cm of kerosene of specific gravity 0.9 floats. The gauge pressure at the base was 4 kN/m^2. If the ratio of $h_w/h_k = 1.25$,* ***determine the heights of the columns.***

Summing the pressures due to the two columns, (As $h_k = h_w/1.25$)

$$h_w \times 1000 \times 9.81 + [(h_w/1.25) \times 900 \times 9.81] = 4000$$

$\therefore$ $\mathbf{h_w} = 0.2371$ m, or **23.71 cm**

$\therefore$ $\mathbf{h_k} = 0.1896$ m, or **18.96 cm**

Problem 2.4. *A U-tube open to atmosphere is first filled to a sufficient height with mercury. On one side water of volume equal to 200 mm column over which kerosene of density 830 m^3/kg of volume equal to 250 mm column are added.* ***Determine the rise in the mercury column*** *in the other limb.*

Let the rise in mercury column be h, Then

$$h \times 13600 \times 9.81 = (0.2 \times 9.81 \times 1000) + (0.25 \times 9.81 \times 830)$$

Solving, $\mathbf{h = 0.02996\ m}$ or about **3 cm**.

Problem 2.5. *The pressure due to the atmosphere at the earths surface is 101.3 kPa.* ***Determine pressure at 10,000 m altitude****, assuming that the condition of air can be represented by the law $Pv^{1.4}$ = constant. Temperature at ground level is 27°C.*

The law can be written as $[P/(\rho g)^{1.4}]$ = const. or $P/\gamma^{1.4}$ = const.

Denoting the index as k, $P_1\ \gamma_1^{-k} = P_2\gamma_2^{-k} = P\ \gamma^{-k}$

Let the specific weight at altitude y be γ. Then $\gamma = (p/p_1)^{1/k}\gamma_1$

The hydrostatic equation is $dP/dy = -\gamma$ or $dP = -\gamma\ dy$ ... (*A*)

The equation Pv^k = constant can be rewritten as,

$$P/\gamma^k = P_o/\gamma_o^{\,k}$$

$\therefore$ $$\gamma^k = (P/P_o)\gamma_o^{\,k}$$

or $$\gamma = P^{1/k}\ P_o^{-1/k}\ \gamma_o = \gamma_o P_o^{-1/k}\ P^{1/k},$$

substituting in *A* and separating variables

$\therefore$ $dP/P^{1/k} = -P_o^{-1/k}\ \gamma_o\ dy$, integrating between limits y_o and y

$$[k/(k-1)]\ [P^{(k-1)/k} - P_o^{(k-1)/k}] = P_o^{-1/k}\ (gP_o/RT_o)\ (y - y_o)$$

$$= -P_o^{-1/k}\ (gP_o/RT_o)\ y$$

$$P^{(k-1)/k} = \{P_o^{(k-1)/k} - [((k-1)/k)]\ P_o^{(k-1)/k}\ (g/RT_o)\ y\}$$

$$P = P_o\ \{1 - [(k-1)/k]\ (g/RT_o)\ y\} \qquad \textbf{(P. 2.5.1)}$$

The values at various altitudes are calculated using the equation and compared with air table values. T_o = 300 K

Altitude, m	**1000**	**2000**	**4000**	**6000**	**8000**	**10000**	**20000**
Calculated P/P_o	**0.89**	**0.79**	**0.61**	**0.47**	**0.35**	**0.25**	**0.03**
From air tables	**0.8874**	**0.7848**	**0.6086**	**0.4657**	**0.3524**	**0.2622**	**0.0561**

Problem 2.6. *In a fresh water lake the specific weight of water γ is found to vary with depth y as $\gamma = K\,\gamma_o/(K + \gamma_o y)$ where K is the bulk modulus. At the surface γ_o = 9810 N/m^3 and P_o = 101.3 kPa. If the pressure measured at 1500 m was 14860 kPa,* ***determine the value of K.***

$$dP/dy = -\gamma = -K\,\gamma_o/(K + \gamma_o y)$$

$$\therefore \quad dP = \{-K\,\gamma_o/(K + \gamma_o y)\}\,dy = -K\,\gamma_o dy/(K + \gamma_o y)$$

Integrating between the surface and the depth

$$P - P_o = -(K.\gamma_o/\gamma_o)\ln[(K + \gamma_o y)/(K + \gamma_o \times 0)] \qquad \textbf{- (P .2.6)}$$

$$= -K \ln[(K + \gamma_o y)/K]$$

$$\therefore \quad P = P_o - K \ln[(K + \gamma_o y)/K]$$

Note : y is –ve as measured downwards, substituting the given values

$$14860 \times 10^3 = 101.3 \times 10^3 - K \ln\{[K - (9810 \times 1500)]/K\}$$

$$14758.5 \times 10^3 = -K \ln[K - (14.715 \times 10^6)/K]$$

Solving by trial (generally K is of the order of 10^9)

Assumed value of K	2×10^9	2.5×10^9	3×10^9
RHS	14769×10^3	14758.5×10^3	14751×10^3

2.5×10^9 gives the value nearest to LHS

$$\therefore \quad \mathbf{K = 2.5 \times 10^9\ N/m^2}$$

The specific weight at this location is

$$\gamma = (2.5 \times 10^9 \times 9810)/\{2.5 \times 10^9 + [9810 \times (-1500)]\}$$

$$= 9868.08\ \text{N/m}^2$$

The pressure at various depths are tabulated,

Depth, m	1000	2000	4000	6000
P, kPa	9930	19798	39652	59665

Problem 2.7 *A chemical reaction vessel of the shape given in figure is full of water with the top of the longer limb sealed and the top of the smaller limb open to atmosphere.* ***Determine the pressure at B, the top of the longer limb.*** *The density of water is 992 kg / m^3 at this condition. Using steam table indicate whether water will boil at this point if temperature is 30° C.*

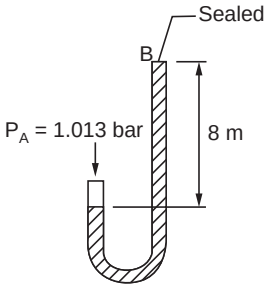


Figure P. 2.7

$\mathbf{P_B} = 101.3 \times 10^3 - (8 \times 992 \times 9.81) = 23447.8$ N/m^2 or **23.45 kPa**

At 30°C, the saturation pressure as read from steam table is 4.24 kPa, hence there will be no boiling at B. If the water just begins to boil, what should be the length of the limb?

$4.246 \times 10^3 = 101.3 \times 10^5 - h \times 992 \times 9.81$. Solving, $h = 9.973$ m

Problem 2.8. *A manometer of the shape shown in figure has limb A filled with water of specific gravity 1 and the other limb with oil of specific gravity 0.95. The area of the enlarged mouth portion is 50 times the area of the tube portion. If the pressure difference is 22 N/m^2,* ***calculate the height h***.

Let the water level when $P_1 = P_2$ be at x-x. Then the pressures at these points are equal as the same liquid fills the volumes below x-x. Let the height of oil on the left limb above x-x be H. The height of water in the other side will be (γ_o/γ_w) H or 0.95 H (in this case). Now let pressure P_2 act on the oil side limb and let the level of water below move down by distance h to the level yy. The pressure on both limbs at the level yy are equal. Now the liquid heights in each limb can be calculated.

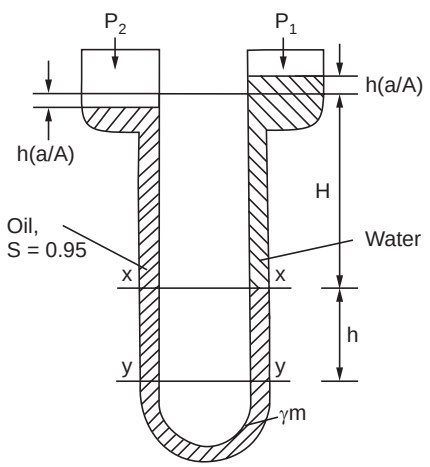


Figure P. 2.8

The rise of level in the water side will be $(a/A).h$ (As the filled volumes remain the same).

The fall of oil level in the other limit will be also $(a/A).h$

P_y is now calculated. Consider water side

$$P_y = P_1 + 9.81 \times 1000[0.95H + h + h(a/A)]$$

On the oil side, $P_y = P_2 + 9.81 \times 950[H + h - h(a/A)]$, As $P_2 - P_1 = 22$

$$22 = 9.81\ [-950H - 950h + (950 \times h/50) + 950H + 1000h + (1000 \times h/50)]$$

$$= 9.81\ [50h + 19h + 20h] = 9.81 \times 89h$$

$\therefore \quad 22 = 9.81 \times 89h, \quad \therefore \quad$ **h = 0.0252 m or 25.2 mm**

A manometer with constant limb area will give a reading of only 2.24 mm of water. Thus the sensitivity is improved appreciably by this arrangement.

Problem 2.9. *A U-tube manometer has both its limbs enlarged to 25 times the tube area. Initially the tube is filled to some level with oil of specific weight γ_m. Then both limbs are filled with fluid of specific weight γ_s to the same level, both limbs being exposed to the same pressure. When a pressure is applied to one of the limbs the manometric fluid rises by h m. Derive an expression for the pressure difference in the limbs. In both cases assume that the liquid level remains in the enlarged section.*

Consider stationary condition, when both pressures are equal. Let the fluid with specific weight γ_s be having a height H.

After pressures are applied, consider pressures at y as the reference.

Consider the left limb:

$$P_y = P_2 + \left(H + \frac{h}{2} - \frac{h}{2}\frac{a}{A}\right)\gamma_s$$

Consider the right limb:

$$P_y = P_1 + \frac{h}{2}\gamma_m + \left(H - \frac{h}{2} + \frac{h}{2}\frac{a}{A}\right)\gamma_s$$

Equating and solving

$$\therefore \quad P_2 - P_1 = \frac{h}{2}\gamma_m + \left(H - \frac{h}{2} + \frac{h}{2}\frac{a}{A}\right)\gamma_s - \left(H + \frac{h}{2} - \frac{h}{2}\frac{a}{A}\right)\gamma_s$$

$$= \frac{h}{2}\gamma_m + \gamma_s\left(h\frac{a}{A} - h\right)$$

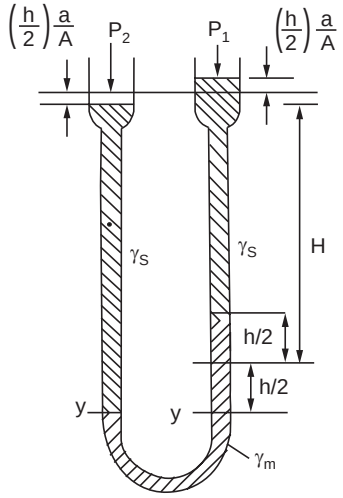


Figure P. 2.9

Let $P_2 - P_1 = 40$ N/m^2, $\gamma_m = 1000 \times 9.81$, $\gamma_s = 0.9 \times 1000 \times 9.81$

$$\therefore \qquad 40 = \frac{h}{2} \times 9810 + 0.9 \times 9810\left(\frac{h}{25} - h\right) = 1334.2\,h.$$

$$\therefore \qquad h = 0.02998 \text{ m or } 30 \text{ mm}.$$

If a U-tube with water was used the deflection will be of the order of 4 mm.

Problem 2.10. *A U-tube is filled first with a fluid of unknown density. Over this water is filled to depths as in figure. Lubricating oil of specific gravity 0.891 is filled over the water column on both limbs. The top of both limbs are open to atmosphere. Determine the density of the unknown fluid (dimensions in mm).*

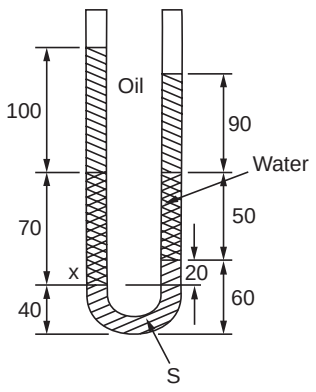


Figure P. 2.10

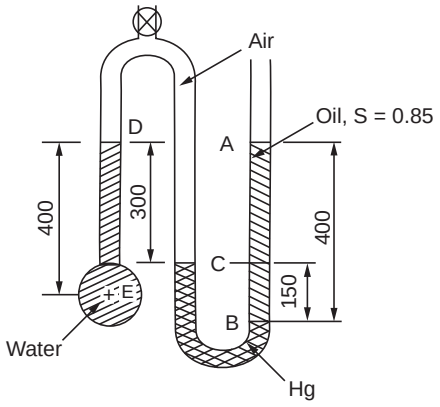


Figure P. 2.11

Consider level *x-x* in figure, on the left limb the pressure at this level is

$$P_{XL} = (70 \times 9.81 \times 1000/1000) + (100 \times 9.81 \times 891/1000)$$

On the right limb at this level,

$$P_{XR} = [(20/1000) \times 9.81 \times \rho] + [(50/1000) \times 9.81 \times 1000] + [(90/1000) \times 9.81 \times 891]$$

equating and solving, $\boldsymbol{\rho = 1445.5\ kg/m^3}$

Note: Division by 1000 is to obtain specific gravity.

Problem 2.11. *A compound manometer is used to measure the pressure in a pipe E carrying water. The dimensions are shown in Figure P.2.11. Determine the pressure in the pipe.*

Calculations can be started from the open limb where the pressure is known

$P_A = P_{atm} = 1.013$ bar $= 101300$ Pa

$P_B = P_A + \rho_o g h_o = 1.013 \times 10^5 + (850 \times 9.81 \times 0.4) = 104635.4$ Pa

$P_C = P_B - \rho_{Hg} g\, h_{Hg} = 104635.4 - (13600 \times 9.81 \times 0.15) = 84623$ Pa

$P_D = P_C = 84623$ Pa (300 mm air column does not contribute much)

$\mathbf{P_E} = P_D + \rho_w g h_w = 84623 + (1000 \times 9.81 \times 0.4) = 88547\ Pa$ or **88547 kPa**

Problem 2.12 *A U-tube with a distance of 120 mm between the limbs is filled with a liquid to mid level for use as a crude accelerometer fixed on a moving vehicle. When the vehicle is accelerated the difference in level between the limbs was measured as 32 mm.* ***Determine the acceleration.***

Let the angle connecting the liquid surfaces in the limbs be θ,

Then $\tan\theta = (h/2)/(L/2) = h/L$

using equation 2.6.7, $\tan\theta = a_x/(g + a_y)$

as $a_y = 0$, $\tan\theta = a_x/g$ or $\mathbf{a_x} = g(h/L) = 9.81 \times (0.032/0.12) = \mathbf{2.616\ m/s^2}$

Problem 2.13. *A container in the shape of a cube of 1 m side is filled to half its depth with water and placed on a plane inclined at 30° to the horizontal. The mass of the container is 50.97 kg. The coefficient of friction between the container and the plane is 0.30. Determine the angle made by the free surface with the horizontal when the container slides down. What will be the angle of the free surface if the container is hauled up with an acceleration of 3 m/s² along the plane.*

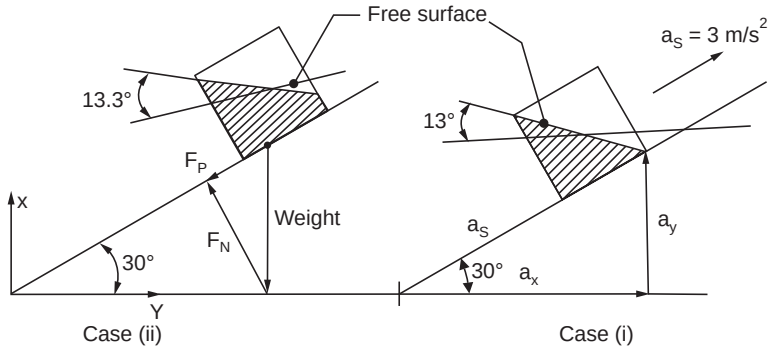


Figure P. 2.13

Irrespective of the inclination if the acceleration along and perpendicular to the horizontal are calculated, then the angle made by the free surface can be obtained using equation 2.6.7

$\tan \theta = - [a_x/(g + a_y)]$

The total mass $= (1 \times 1 \times 0.5 \times 1000) + 50.97 = 550.97$ kg

Case (*i*) Force along the plane

$F_x = 550.97 \times 9.81 \times \cos 60 = 2702.5$ N

Force normal to plane $F_y = 550.97 \times 9.81 \times \sin 60 = 4680.9$ N

The friction force acting against F_x is $F_y\ \mu = 4680.9 \times 0.3 = 1404.26$ N

Net downward force along the plane = $F_x - F_y.\mu = (2701.5 - 1404.26) = 1298.24$ N

Acceleration along the plane, $a_s = F/m = 1298.24/550.97 = 2.356$ m/s^2

The component along horizontal, $a_x = 2.356 \times \cos 30 = - 2.041$ m/s^2

The component along vertical, $a_y = - 1.178\ m/s^2$ (downwards)

$$\mathbf{\tan \theta} = (- 2.041)/(9.81 - 1.178) = + 0.2364$$

$\therefore$ $\theta = +$ **13.3°**

Case (ii) $a_x = 3 \cos 30 = 2.598$ m/s^2, $a_y = 3 \sin 30 = 1.5$ m/s^2

$\tan \theta = - [2.598/(9.81 + 1.5)]$

$\therefore$ **$\theta = -$ 12.94° with horizontal**

Problem 2.14, *A tank 0.4 m × 0.2 m size and of height 0.4 m is filled with water upto a depth of 0.2 m. The mass of the container is 10 kg. The container slides without friction downwards on a surface making 30° with the horizontal. Determine the angle the free surface makes with the horizontal. If the tank is moved up with the same acceleration* ***determine the slope of the free surface.***

Refer Fig. P.2.13

Total mass = $1000\ (0.4 \times 0.2 \times 0.2) + 10 = 26$ kg

Force along the surface = $26 \times 9.81 \times \cos 60 = 127.53$ N

Acceleration $a_s = 127.5/26 = 4.905$ m/s^2

Acceleration along x direction = $4.905 \times \cos 30 = - 4.2478$ m/s^2

Acceleration along y direction = $4.905 \times \sin 30 = - 2.4525$ m/s^2

$\tan \theta = - [- 4.2478/(9.81 - 2.4525)]$ $\therefore \theta = 30°$, same as the slope of the plane.

This is an interesting result. Try to generalise assuming other angles of inclination.

When moving up, with the same acceleration,

$a_x = 4.2478$, $a_y = 2.4525$,

$\tan \theta = - [4.2478/(9.81 + 2.4525)]$

$\therefore$ **$\theta = -19.1°$, slope = 0.3464**

Problem 2.15. *An aircraft hydraulic line pressure is indicated by a gauge in the cockpit which is 3 m from the line. When the aircraft was accelerating at 10 m/s^2 at level flight, the gauge indicated 980 kPa.* ***Determine the pressure at the oil line using equation 2.6.9.*** *Specific gravity of oil is 0.9.*

$$P_2 = P_1 - (\rho.a_x)\ (x_2 - x_1)$$

$\mathbf{P_2} = 980 \times 10^3 - [(900 \times 10) \times (- 3)] = 1007 \times 10^3$ Pa or **1007 kPa**

Problem 2.16. *A tank as in Fig.P.2.16 is filled with water. The left side is vented to atmosphere.* ***Determine the acceleration*** *along the right which will cause the pressure at A to be atmospheric.*

$$P_2 = P_1 - \rho.a_x\,(x_2 - x_1)$$

For the pressure at A to be atmospheric, there should be a reduction of 4 m of water column due to the acceleration.

Initial pressure all over the surface = P_{atm} + 4 m of water head

$$\tan\theta = 4/4 = a_x/g$$

$$\therefore \quad a_x = 9.81 \text{ m/s}^2$$

alternately, the general equation can be used, choosing B as origin,

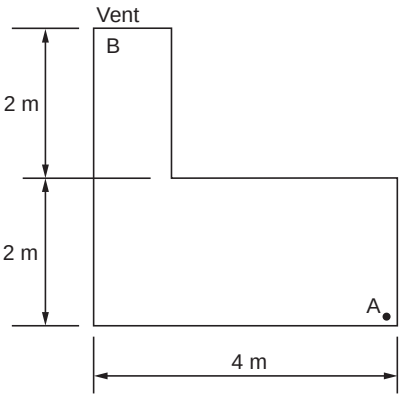


Figure P. 2.16

$$P = -[\gamma\,(a_x/g)x] - \gamma\,y$$

$$= -9810\,(a_x/g)\,x - 9810y$$

$$P = -1000\,a_x x - 9810\,y$$

In this case, $P = 0,\ x = 4,\ y = -4$

$$\therefore \quad 0 = -4000a_x + 4 \times 9810, \quad \therefore \quad \mathbf{a_x = 9.81\ m/s^2}$$

Problem 2.17. *A fully air conditioned car takes a curve of radius 250 m at 90 kmph. The air within the car can be taken to move as a solid. A child holds a balloon with a string and it is vertical along straight road.* ***Determine the direction of the string*** *measured from the vertical during the turn.*

The balloon will move opposite to the pressure gradient at the location,

$$\tan\theta = a_x/(g + a_y),$$

During the travel along the curve, $a_x = r.\omega^2$, $a_y = 0$

speed, $v = 90 \times 1000/3600 = 25$ m/s

$$\omega = (v/\pi D).2\pi = (25 \times 2\pi)/(\pi/500) = 1/10 \text{ rad/s}$$

$$\therefore \quad \tan\theta = 250 \times (1/10^2) \times (1/9.81) = 0.255, \quad \therefore \quad \mathbf{\theta = 14.3°}$$

As pressure increases outwards, the balloon will turn inwards by 14.30 to the vertical.

Problem 2.18. *A U-tube shown in figure is filled with water at 30°C and is sealed at A and is open to atmosphere at D.* ***Determine the rotational speed along AB*** *in rad/s is the pressure at the closed end A should not fall below the saturation pressure of water at this temperature.*

From steam tables at 30°C, saturation pressure is read as 0.04241 bar

using equation 2.7.1 as the situation is similar to forced vortex

$$(P_{r_2} - P_{r_1}) = \rho(\omega^2/2)\,(r_2^2 - r_1^2)$$

$$r_1 = 0,\ P_{r_2} = 1.013 \text{ bar},$$

$$P_{r_1} = 0.04241 \text{ bar},\ \rho = 1000 \text{ kg/m}^3,\ r_2 = 0.1 \text{ m}$$

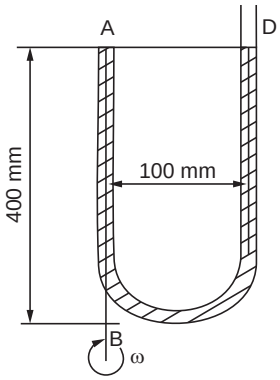


Figure P. 2.18

substituting, $(1.013 - 0.042421) \times 10^5 = (1000/2).\omega^2 \times 0.1^2$

$\therefore$ **ω = 139.33 rad/s**

Problem 2.19. *Gas centrifuges are used to produce enriched uranium. The maximum peripheral speed is limited to 300 m/s. Assuming gaseous uranium hexafluride at 325°C is used,* ***determine the ratio of pressures at the outer radius to the centre****. The molecular mass of the gas is 352. Universal gas constant = 8314 J/kgK.*

Equation 2.6.1 namely $dP/ds = -r \sin\theta + \rho a_s$

reduces when s is horizontal and in the case of rotation to

$dp/dr = \rho a_s = \rho r\omega^2$, For a gas $\rho = P/RT$, substituting

$dP/P = (\omega^2/RT)\, dr$, integrating between limits

$$\ln (P_2/P_1) = (\omega^2/2RT)r^2 = V^2/2RT \qquad \textbf{(P. 2.19)}$$

V = 300 m/s, R = 8314/352, T = 325 + 273 substituting

$$\ln (P_2/P_o) = 300^2 \times 352/[2 \times 8314 \times (325 + 273)] = 3.186$$

$\therefore$ **P_2/P_o = 24.19**

Problem 2.20. *A container filled with oil of density 800 kg/m³ is shown in figure. The small opening at A is exposed to atmosphere.* ***Determine the gauge pressures at B, C, D and E*** *when (i) a_x = 3.9 m/s² and a_y = 0 (ii) a_x = 2.45 m/s² and a_y = 4.902 m/s².*

Determine also the values of a_x and a_y if $P_A = P_B = P_C$

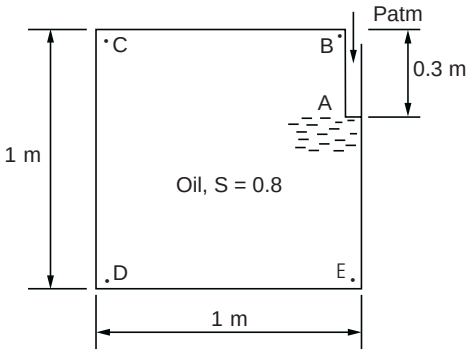


Figure P. 2.20

Case (i). Pressure at A is atmospheric in all cases. a_x = 3.9 m/s², a_y = 0, when accelerated along the x direction, the imaginary free surface angle θ is given by (as a_y = 0)

$$\tan\theta = -a_x/g = 3.9/9.81$$

$\therefore$ The slope is 3.9/9.81 = 0.39755 as the length is 1 m, C' will be above C by 0.0976 m head of fluid. As compared to A, B is at 0.3 liquid head above *i.e.*, P_B is lower

$\therefore$ **P_B** = – 0.3 × 9.81 × 800 = – **2354.4 Pa**

P_C = $P_{C'} - P_C$ = 9.81 × 800 [0.39755 – 0.3] = **765 Pa**

P_D = P_C + (1 × 9.81 × 800) = **8613 Pa**

P_E = 0.7 × 0.8 × 9.81 = **5493 Pa**

All the pressures are gauge pressures with atmospheric pressure as reference pressure

Case (ii). a_x = 2.45 m/s^2, a_y = 4.902 m/s^2

In this case $\tan\theta = a_x/(a_y + g)$ or the slope is 2.45/(9.81 + 4.902) = 0.16653

At B, the pressure is less than at A by a column of 0.3 m of liquid, but the weight is increased by the upward acceleration.

$\therefore$ **P_B** = 0.3 × 800(9.81 + 4.902) = **3531 Pa**

Now C' is at 0.16653 m above A. C is above C' by (0.3 – 0.1665) m

$$\mathbf{P_C} = -0.3 \times 800(9.81 + 4.902) \times (0.3 - 0.16653) = \mathbf{-1570\ Pa}$$

$$\mathbf{P_D} = P_C - [1 \times 0.3 \times 800(9.81 + 4.902)] = \mathbf{10200\ Pa}$$

$$\mathbf{P_E} = 0.7 \times 800(9.81 + 4.902) = \mathbf{8239\ Pa}$$

Case (iii). If $P_A = P_B$ then the weight of the liquid column should be zero due to the acceleration a_y,

$\therefore$ $\mathbf{a_y} = -g$ or $\mathbf{9.81\ m/s^2}$ upwards,

If $P_C = P_B$, automatically $B\ C$ should be constant pressure surface. So $\mathbf{a_x = 0}$

Problem 2.21. *At an instant an aircraft travelling along 40° to the horizontal at 180 m/s, decelerates at 4 m/s^2. Its path is along a concave upward circular curve of radius 2600 m.* ***Determine the position of the free surface of the fuel in the tank.***

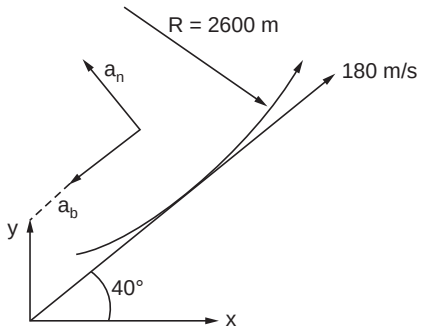


Figure P. 2.21

The path of the aircraft is shown in figure. The accelerations are indicated. The acceleration towards the centre of the curve is given by

$$a_x = V^2/R = 180^2/2600$$

$$= 12.5\ \text{m/s}^2, \text{ towards centre}$$

The acceleration along the tangent $a_t = -4\ \text{m/s}^2$

The components along x and y directions are

$$a_x = -4\cos 40 - 12.5 \sin 40 = -11.09\ \text{m/s}^2$$

$$a_y = -4 \sin 40 + 12.5 \cos 40 = 7.01\ \text{m/s}^2$$

$$\tan \theta = a_x / (a_y + g) = 11.09/(7.01 + 9.81) = 0.659,$$

$\therefore$ $\mathbf{\theta = 33.4°}$,

Slope of the free surface of fuel = 0.659

Problem 2.22. *A tanker lorry of cylindrical shape 6 m in length is filled completely with oil of density 830 kg/m^3. The lorry accelerates towards the right. If the pressure difference between the front and back at the centre line should not exceed 40 kPa, (gauge) what should be the maximum acceleration. Neglect the weight component.*

using equation 2.6.9, $\Delta p = \rho a_x (x_2 - x_1)$,

$$40000 = 830 \times a_x \times 6 \quad \therefore \mathbf{a_x = 8.03\ m/s^2}$$

Problem 2.23. *Air fills the gap between two circular plates held horizontal. The plates rotate without any air flowing out. If the radius is 60 mm and if the speed is 60 rpm, determine the pressure difference between the centre and the circumference.*

The air in the gap can be considered to rotate as a single body. As the level is the same the head difference between the centre and the outer radius is given by

$$\mathbf{h} = (\omega r)^2/2g_o = [(2\pi 60/60) \times 0.06]^2/(2 \times 9.81) = \mathbf{7.24 \times 10^{-3}\ m\ of\ air}$$

Considering density of air to be about 1.2,

$$\textbf{head of water} = (7.24 \times 10^{-3} \times 1.2)/1000 = \mathbf{8.69 \times 10^{-6}\ m}$$

REVIEW QUESTIONS

1. Define and explain the concept "pressure".
2. State and prove Pascal's law. Explain the consequences of the law.
3. Distinguish between gauge pressure, absolute pressure and vacuum pressure.
4. Derive the expression for the pressure variation in a static fluid under gravitational forces. Indicate the modifications where pressure varies along vertical and horizontal directions.
5. Derive an expression for the distribution of force in static fluid subjected to whole body acceleration in a general direction – *s*.
6. Derive the expression for the angle made by the free surface in a liquid that is subjected to both acceleration and gravitation.
7. Derive an expression for the pressure distribution in an incompressible fluid accelerated horizontally.
8. Explain what is meant by forced vortex and derive the expression for the radial pressure distribution in forced vortex.
9. Explain the basic principle involved in measuring pressure and pressure difference using manometers. Indicate when the use of manometers is advantageous.
10. Explain how small pressure difference reading can be amplified by using a micro manometer or inclined tube manometer.

OBJECTIVE QUESTIONS

O Q.2.1 Fill in the blanks:

1. Pressure is defined as _____
2. Pascals law states _____
3. On a free surface of a liquid the pressure is _____
4. When gravitational forces are zero, the pressure exterted by a column of fluid is _____
5. The pressure exerted by a column of fluid of height y m and specific weight γ is _____
6. At zero horizontal accelerating conditions on earths surface, the free surface will be _____
7. Manometers use the principle of _____
8. Manometers are suitable for _____ pressure measurement.
9. In a forced vortex the height of liquid at the periphery of a cylinder of Radius R above that at the centre will be _____
10. The shape of free surface in a forced vortex is _____

Answers

1. As a measure of force distribution over any surface associated with a fluid (dF/dA) 2. that the pressure at a point in a fluid at rest is equal in magnitude in all directions 3. is the same at all points 4. is zero 5. $y\ \gamma$. 6. horizontal 7. basic hydrostatic equation, ($\Delta P = \Delta y\gamma$) 8. low 9. ($R^2\ \omega^2/2\ g_o$) 10. Paraboloidal.

O Q.2.2 Fill in the blanks:

1. If the density varies linearly with height the pressure will vary _____ with height.
2. When a fluid is decelerated at a rate equal to g in the vertical direction the pressure on the base will be _____
3. When a fluid in a container is accelerated along the x direction at a m/s^2, the angle the free surface will occupy is given by _____
4. In micromanometer, the density difference between the filler fluid and the manometer fluid should be _____
5. The capillary effect can be _____ when both limbs of a manometer have equal areas.
6. The shape of a forced vortex in the absence of gravity will be _____
7. The pressure at a point in fluid at rest is _____ of direction.
8. The pressure exerted by a liquid column on the base depends on the _____ of the liquid.
9. The level rise in the forced vortex is _____ of the fluid.
10. Due to horizontal acceleration, the free surface of the fluid will be 45° when the acceleration equals _____

Answers

1. exponentially 2. zero 3. tan $\theta = -a/g$ 4. small 5. neglected/equal on both sides 6. cylindrical 7. independent 8. specific weight 9. independent 10. $a_x = g$

O Q.2.3 Fill in the blanks with increases, decreases, or remains constant.

1. The pressure in a fluid at rest _____ with depth.
2. Along the free surface in a liquid, the pressure _____
3. In a fluid at rest the pressure at a point _____
4. As specific weight increases, the head of liquid for a given pressure _____
5. As the density of manometric fluid decreases, the manometric deflection for the same pressure difference _____
6. As a container with liquid is accelerated the pressure on the base along the direction of acceleration _____
7. The forced vortex rise _____ as density of the liquid increases.
8. The forced vortex rise _____ with rotational speed.
9. In a micromanometer, the gauge deflection will increase if the area ratio _____
10. In inclined tube manometer, the gauge reading _____ when the angle is reduced.

Answers

Increases : 1, 5, 8, 9, 10 **Decreases** : 4, 6 **Remains constant** : 2, 3, 7

O Q.2.4 Indicate whether the statements are correct or incorrect.

1. In a fluid at rest, the pressure at a point varies with direction.
2. In a fluid at rest the pressure at a constant level will be equal at all locations.
3. The pressure on the base of a liquid column will depend upon the shape of the column.
4. The pressure over a free surface of a fluid at rest will vary with location.
5. For low pressure measurement a manometric fluid with low density will be better.
6. In a manometer, the fluid column will rise if the pressure measured is above the atmosphere.
7. In a manometer, the fluid column will fall if the pressure inside is less than atmospheric.
8. The vacuum gauge reading will increase as the absolute pressure decreases.

9. The absolute pressure is equal to the vacuum gauge reading.
10. Absolute pressure = atmospheric pressure – vacuum gauge reading.

Answers

Correct : 2, 5, 6, 7, 8, 10 Incorrect : 1, 3, 4, 9

O Q.2.5 Choose the correct answer:

1. The gravity at a location is 5 m/s^2. The density of fluid was 2000 kg/m^3. The pressure exerted by a column of 1 m of the fluid will be
   (*a*) 400 N/m^2 (*b*) 10,000 N/m^2
   (*c*) 2000 N/m^2 (*d*) 5 N/m^2
2. In a circular cylinder of 0.2 m dia and 0.4 m height a fluid of specific weight 1200 × 9.81 N/m^3 is filled to the brim and rotated about its axis at a speed when half the liquid spills out. The pressure at the centre is
   (*a*) 0.2 × 1200 × 9.81 N/m^2 (*b*) Zero
   (*c*) 0.4 × 1200 × 9.81 N/m^2 (*d*) 0.1 × 1200 × 9.81 N/m^2
3. In a forced vortex
   (*a*) the fluid velocity is inversely proportional to the radius
   (*b*) the fluid rotates without any relative velocity
   (*c*) the rise depends on the specific weight
   (*d*) the rise is proportional to the cube of angular velocity
4. In a forced vortex, the level at a radius of 0.6 m is 0.6 m above the centre. The angular velocity in radians is
   (*a*) 11.44 (*b*) 5.72
   (*c*) 32.7 (*d*) 130.8
5. The shape of forced vortex under gravitational conditions is
   (*a*) hyperboloid (*b*) spherical
   (*c*) paraboloid (*d*) cylindrical
6. In a manometer using mercury as manometric fluid and measuring the pressure of water in a conduit, the manometric rise is 0.2 m. The specific gravity of mercury is 13.55. The water pressure in m of water is
   (*a*) 14.55 × 0.2 (*b*) 13.55 × 0.2
   (*c*) 12.55 × 0.2 (*d*) none of the above
7. A horizontal cylinder half filled with fuel is having an acceleration of 10 m/s^2. The gravitational forces are negligible. The free surface of the liquid will be
   (*a*) horizontal (*b*) slopes in the direction of acceleration
   (*c*) vertical (*d*) slopes in the direction opposite of acceleration
8. In a static fluid, with y as the vertical direction, the pressure variation is given by
   (*a*) $\frac{dp}{dy} = \rho$ (*b*) $\frac{dp}{dy} = -\rho$
   (*c*) $\frac{dp}{dy} = \gamma$ (*d*) $\frac{dp}{dy} = -\gamma$

9. The specific weight of a fluid is 20,000 N/m^3. The pressure (above atmosphere) in a tank bottom containing the fluid to a height of 0.2 m is
   (*a*) 40,000 N/m^2 (*b*) 2000 N/m^2
   (*c*) 4000 N/m^2 (*d*) 20,000 N/m^2
10. In a differential manometer a head of 0.6 m of fluid *A* in limb 1 is found to balance a head of 0.3 m of fluid *B* in limb 2. The ratio of specific gravities of *A* to *B* is
   (*a*) 2 (*b*) 0.5
   (*c*) cannot be determined (*d*) 0.18

Answers

(1) *b* (2) *b* (3) *b* (4) *b* (5) *c* (6) *c* (7) *c* (8) *d* (9) *c* (10) *b*

O Q.2.6 Match the pairs:

(*a*)	Free surface in forced vortex	(1) Vertical
(*b*)	Free surface in static fluid	(2) Paraboloid
(*c*)	Free surface in forced vortex without gravity	(3) Negative slope
(*d*)	Free surface in a horizontally accelerating fluid	(4) Horizontal

Answers

a - 2, *b* - 4, *c* - 1, *d* - 3

EXERCISE PROBLEMS

E.2.1. A chamber is at a pressure of 100 kN/m^2. A gauge fixed into this chamber Fig. E.2.1 to read the outside pressure shows 1.2 kN/m^2. Determine the outside pressure.

[101.2 kN/m^2 absolute]

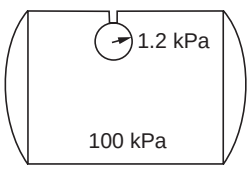


Figure E. 2.1

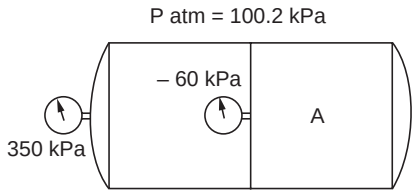


Figure E. 2.2

E.2.2. Determine the absolute and gauge pressures in chamber *A* as shown in Fig. E.2.2, the gauge pressure being referred to atmospheric pressure of 1.02 × 10^5 N/m^2. **[390.2, 290 kN/m^2]**

E.2.3. In an artificial atmosphere, the specific weight of air varies with the altitude *y* as $\gamma = c.y$, where γ is in N/m^3 and *y* is in m. The pressure at $y = 0$ is 5000 N/m^2. *c* is a dimensional constant having a unit of N/m^4. In this case *c* has a value of 1. Determine the expression for pressure variation with altitude. **[P = 5000 – (y^2/2)]**

E.2.4. Determine the pressure below 1000 m in the sea if the specific weight changes as $\gamma = K.\gamma_1/(K + \gamma_1.y)$ where *K* is the bulk modulus having a value of 2 × 10^9 N/m^2 and *y* is the depth in m. The surface pressure is 101.3 kN/m^2 and γ_1 = 9810 N/m^3. **[9935 kN/m^2]**

E.2.5. A vessel of the shape shown in Fig. E.2.5 is filled with a liquid of specific gravity 0.92. The pressure gauge at *A* reads 400 kN/m^2. Determine the pressure read by a gauge (Bourdon type) fixed at *B*. Neglect gauge height. **[454.15 kN/m^2]**

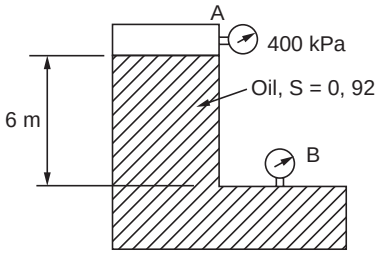


Figure E. 2.5

E.2.6. Determine the pressure above the atmosphere at point 3 for the manometer and dimensions shown in Fig. E.2.6. **[65 kN/m^2]**

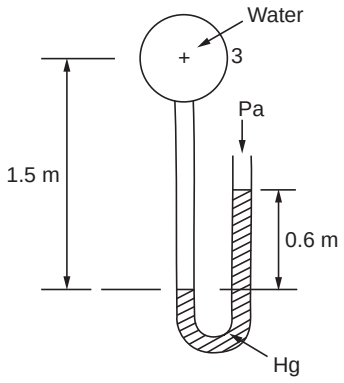


Figure E. 2.6

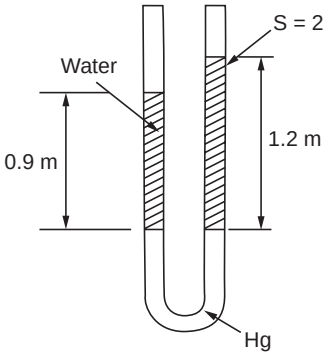


Figure E. 2.7

E.2.7. In a U-tube shown in Fig. E.2.7, open to atmosphere at both ends, a column of 0.9 m of water balances a column of 1.2 m of an unknown liquid. Determine the specific gravity of the unknown liquid. **[0.75]**

E.2.8. Determine the pressure at point *X* for the situation shown in Fig. E.2.8 **[8.04 kPa]**

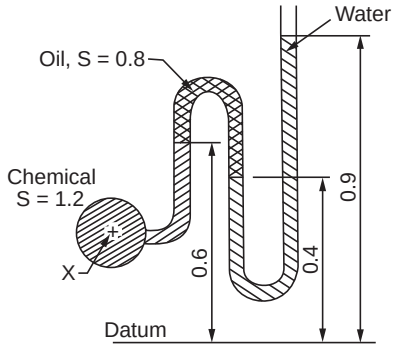


Figure E. 2.8

E.2.9. For the manometer shown in Figure E.2.9, determine the length *AB*. The pressure at point 1 and point 4 are 30 kPa and 120 kPa. **[68.3 cm]**

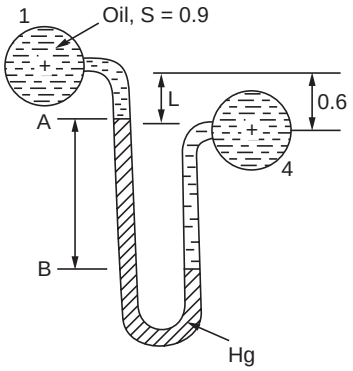


Figure E. 2.9

E.2.10. Determine the pressure at *A* above the atmosphere for the manometer set up shown in Fig. E.2.10. **[111.91 kPa]**

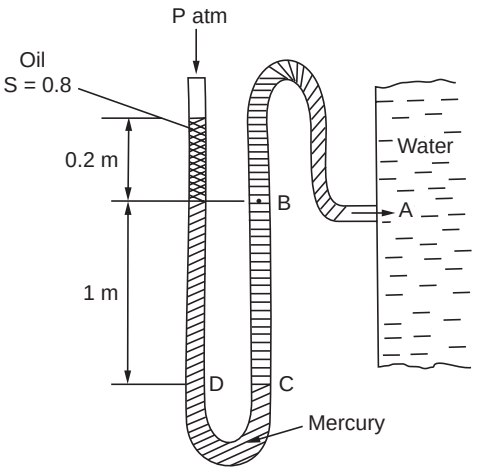


Figure E. 2.10

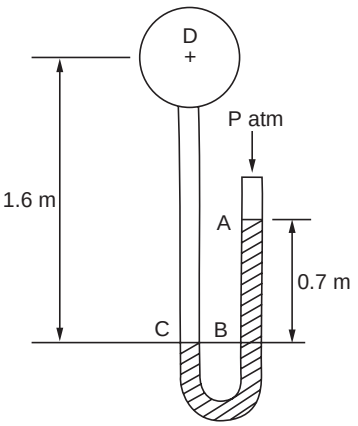


Figure E. 2.11

E.2.11. For the situation shown in Fig. E.2.11, determine the pressure at point *D*. The specific gravity of the oil is 0.9 and that of the manometer fluid is 0.7. **[94.04 kN/m^2]**

E.2.12. In a micromanometer the area of the well chamber is 12 times the area of the U tube section. The manometric fluid is having a specific gravity of 1.03 and the filler fluid is water. The flowing fluid in which the pressure is to be determined is air with a density of 1.2 kg/m^3 at the measuring condition. When pressures are equal, the level from the top to the filler fluid is

8 cm. The manometric fluid is 18 cm from top at filling. Under measuring condition the manometric fluid movement in one limb is 4 cm. Determine the pressure difference indicated.

[88.866 N/m^2]

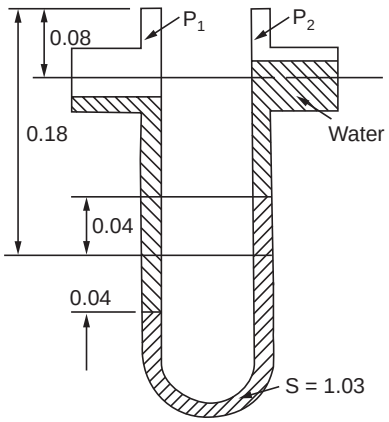


Figure E. 2.12

E.2.13. An inclined tube manometer with limb at 10° to horizontal shows a column length of 8 cm above the reservoir level. The specific weight of the fluid is 900 × 9.81 N/m^3. Determine pressure above atmospheric level. **[122.65 N/m^2]**

E.2.14. Determine the pressure difference between *A* and *B* shown in Fig. E.2.14.

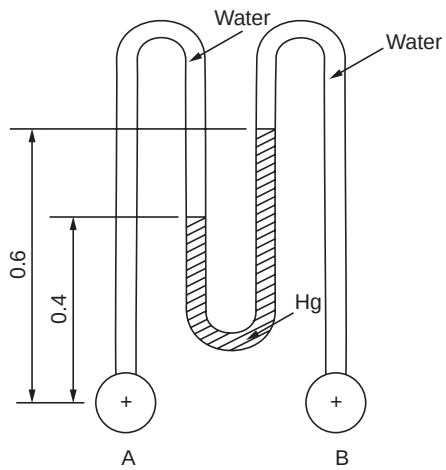


Figure E. 2.14

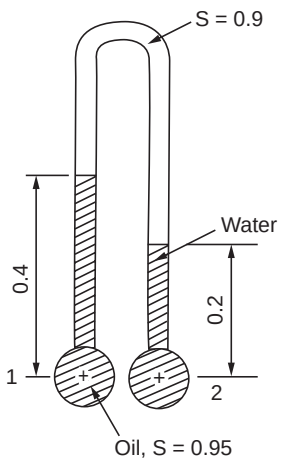


Figure E. 2.15

E.2.15. Determine the pressures at location 1 and 2 in Fig. E.2.15.

E.2.16. The atmospheric pressure at an elevation of 300 m was 100 kPa. when the temperature was 20°C. If the temperature varies at the rate of – 0.006° C/m, determine the pressure at height of 1500 m.

E.2.17. The pressure at sea level was 102 kPa and the temperature is constant with height at 5°C. Determine the pressure at 3000 m.

E.2.18. The bourden type pressure gauge in the oxygen cylinder of a deep sea diver when he is at a depth of 50 m reads 500 kPa. Determine the pressure of oxygen above atmospheric pressure. Assume sea water density is constant and is 1006 kg/m^3.

E.2.19. The density of a fluid at rest increases with depth as $1000 + 0.05h$ kg/m^3 where h is the depth in m from the surface. Determine the hydrostatic pressure at depth of 100 m.

E.2.20. A cylinder containing oil of specific gravity 0.92 as shown in Fig. E.2.20 is accelerated at 10 m/s^2 towards (*i*) right and (*ii*) left. The reading under accelerating conditions at the right end was 200 kN/m^2. The tube is 3 m long. Determine the pressure at the left end.

[227.6, 172.4 kN/m^2]

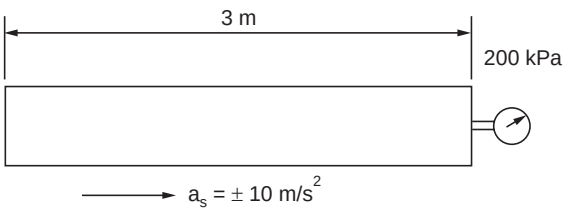


Figure E. 2.20

E.2.21. Using figure in Example 2.20, if the fluid density varies as $\rho = 0.3 + 8 \times 10^{-6} P$, where density is in kg/m^3 and P is in N/m^2 and if the pressure gauge at the right end reads 120 kN/m^2, determine the pressure at the left end, if the acceleration is to the right at 10 m/s^2.

[120.038 kN/m^2]

E.2.22. A rocket is accelerating horizontally to the right at 10 g. The pressure gauge is connected by a 0.6 m length tube to the left end of the fuel tank. If the pressure in the tank is 35 bar, and if fuel specific gravity is 0.8, determine the pressure gauge reading. **[35.471 × 10^5 N/m^2]**

E.2.23. A rectangular pan 0.3 m high, 0.6 m long and 0.3 m wide contains water to a depth of 0.15 m. Determine the acceleration which will cause water to spill. **[4.905 m/s^2]**

E.2.24. Determine the liquid level at the centre when a tall cylinder of 1.2 m dia filled upto a depth of 0.6 m is rotated at 77 rpm. **[0]**

E.2.25. A cylinder of radius 0.6 m filled partially with a fluid and axially rotated at 15 rad/s is empty upto 0.3 m radius. The pressure at the extreme edge at the bottom was 0.3 bar gauge. Determine the density of the fluid. **[987.65 kg/m^3]**

E.2.26. A tank containing liquid of specific gravity of 0.8 is accelerated uniformly along the horizontal direction at 20 m/s^2. Determine the decrease in pressure within the liquid per metre distance along the direction of motion.

E.2.27. The liquid in a tank when accelerated in the horizontal direction, assumes a free surface making 25° with the horizontal. Determine the acceleration.

E.2.28. A closed tank of cubical shape of 1 m side is accelerated at 3 m/s^2 along the horizontal direction and 6 m/s^2 in the vertical direction. Determine the pressure distribution on the base. Assume the base to be horizontal.

E.2.29. A closed cubical tank of 1.5 m side is filled to 2/3 of its height with water, the bottom face being horizontal. If the acceleration in the horizontal (along the right) and vertical directions are 5 m/s^2 and 7 m/s^2. Determine the pressures at the top and bottom corners.

E.2.30. A tube with closed ends filled with water is accelerated towards the right at 5 m/s^2. Determine the pressure at points 1, 2, 3 and 4. Calculate the acceleration that water will boil at point 4 at 40°C.

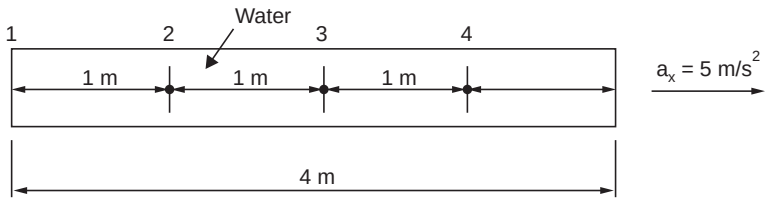


Figure E. 2.30

E.2.31. A cubical box of 1 m side is half filled with water and is placed in an inclined plane making 30° with the horizontal. If it is accelerated along the plane at 2 m/s^2 (*i*) upwards, (*ii*) downwards, determine the angle attained by the free surface.

E.2.32. A cylindrical vessel containing water is rotated as a whole. The pressure difference between radii 0.3 m and 0.6 m is 0.3 m of water. Calculate the rotational speed.

E.2.33. A small bore pipe 3 m long and one end closed is filled with water is inclined at 20° with the vertical and is rotated at 20 rpm with respect to a vertical axis passing through its mid point. The free surface is at the top of the pipe. Determine the pressure at the closed end.

E.2.34. The U-tube shown in Fig. E.2.34 is rotated at 120 rpm about the vertical axis along *A*-*A*. Determine the pressure at 1 and 2.

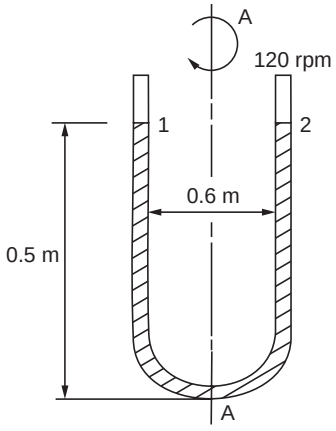


Figure E. 2.34

E.2.35. A hollow sphere of inside radius r is filled with water and is rotated about a vertical axis passing through the centre. Determine the circular line of maximum pressure.

E.2.36. A cylindrical vessel containing water is rotated about its axis at an angular speed ω (vertical). At the same time, the container is accelerated downwards with a value of v m/s^2. Derive an expression for the surface of constant pressure.

E.2.37. A box of cubical shape of 1.5 m side with base horizontal filled with water is accelerated upwards at 3 m/s^2. Determine the force on one of the faces.

3 Forces on Surfaces Immersed in Fluids

3.0 INTRODUCTION

In the previous chapter the pressure distribution in fluids in static and dynamic condition was discussed. **When a fluid is in contact with a surface is exerts a normal force on the surface.** The walls of reservoirs, sluice gates, flood gates, oil and water tanks and the hulls of ships are exposed to the forces exerted by fluids in contact with them. The fluids are generally under static condition. For the design of such structures **it is necessary to determine the total force** on them. It is **also necessary to determine the point of action of this force.** The point of action of the total force is known as **centre of pressure or pressure centre**. From the basic hydrodynamic equation it is known that the force depends on the pressure at the depth considered.

i.e., $P = \gamma h$. Force on an elemental area dA as a depth, h, will be

$$dF = \gamma h dA \tag{3.0.1}$$

The **total force is obtained by integrating** the basic equation over the area

$$F = \gamma \int_A h dA \tag{3.0.2}$$

From the definition of centre of gravity or centroid

$$\int_A h dA = \bar{h} A \tag{3.0.3}$$

where $\bar{h}$ is the depth of the centre of gravity of the area.

To determine the point of action of the total force, moment is taken of the elemental forces with reference to an axis and equated to the product of the total force and the distance of the centre of pressure from the axis namely h_{cp}

$$F. h_{cp} = \int_A h dF = \gamma \int_A h^2 dA \tag{3.0.4}$$

The integral over the area is nothing but the second moment or the moment of inertia of the area about the axis considered.

Thus there is a need to know the centre of gravity and the moment of inertia of areas.

3.1 CENTROID AND MOMENT OF INERTIA OF AREAS

In the process of obtaining the resultant force and centre of pressure, the determination of first and second moment of areas is found necessary and hence this discussion. The moment of the area with respect to the y axis can be obtained by summing up the moments of elementary areas all over the surface with respect to this axis as shown in Fig. 3.1.1.

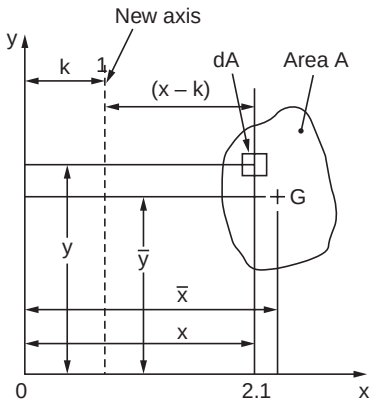


Figure. 3.1.1 First moment and second moment of an area

$$\text{Moment about } y \text{ axis} = \int_A x\, dA \tag{3.1.1}$$

$$\text{Moment about } x \text{ axis} = \int_A y\, dA \tag{3.1.2}$$

The integral has to be taken over the area. If moments are taken with respect to a parallel axis at a distance of k from the y axis equation 3.1.1. can be written as

$$\int_A (x - k)\, dA = \int_A x\, dA - k \int_A dA = \int_A x\, dA - k\, A \tag{3.1.3}$$

As k is a constant, it is possible to choose a value of $\bar{x} = k$, such that the moment about the axis is zero. The moment about the axis through the centre of gravity is always zero.

$$\int_A x\, dA - \bar{x}\, A = 0$$

Such an axis is called **centroidal** y axis. The value of $\bar{x}$ can be determined using

$$\bar{x} = (1/A) \int_A x\, dA \tag{3.1.4}$$

Similarly the centroidal x axis passing at $\bar{y}$ can be located using

$$\bar{y} = (1/A) \int_A y\, dA \tag{3.1.5}$$

The point of intersection of these centroidal axes is known as the **centroid** of the area.

It can be shown that the moment of the area about any line passing through the centroid to be zero.

With reference to the Fig. 3.1.1, the second moment of an area about the y axis. I_y is defined as

$$I_y = \int_A x^2\, dA \tag{3.1.6}$$

Considering an axis parallel to y axis through the centroid and taking the second moment of the area about the axis and calling it as I_G, where $\bar{x}$ is the distance from the axis and the centroid.

$$I_G = \int_A (x - \bar{x})^2 \, dA \tag{3.1.7}$$

$$I_G = \int_A x^2 \, dA - 2\bar{x} \int_A x \, dA + \int_A \bar{x}^2 \, dA$$

By definition $\int_A x^2 \, dA = I_y$, $\int_A x^2 \, dA = \bar{x} \, A$

As $\bar{x}^2$ is constant, $\int_A \bar{x}^2 \, dA = \bar{x}^2 \, A$. Therefore

$$I_G = I_y - 2\bar{x}^2 \, A + \bar{x}^2 \, A = I_y - \bar{x}^2 \, A \tag{3.1.8}$$

or
$$I_y = I_G + \bar{x}^2 A \tag{3.1.9}$$

Similarly
$$I_y = I_G + \bar{y}^2 \, A \tag{3.1.10}$$

The moment of inertia of an area about any axis is equal to the sum of the moment of inertia about a parallel axis through the centroid and the product of the area and the square of the distance between this axis and centroidal axis. These two equations are used in all the subsequent problems.

The second moment is used in the determination of the centre of pressure for plane areas immersed in fluids.

The product of inertia is defined as

$$I_{xy} = \int_A xy \, dA = I_{Gxy} + \bar{x} \, \bar{y} \, A \tag{3.1.11}$$

It can be shown that whenever any one of the axes is an axis of symmetry for the area, $I_{xy} = 0$.

The location of the centre of gravity, moment of inertia through the centroid I_G and moment of inertia about edge I_{edge} (specified) for some basic shapes are given in Table 3.1.

Table 3.1 Centre of Gravity and Moment of Inertia for some typical shapes

	Shape	CG	I_G	I_{base}
1.	Triangle, side b height h and base zero of x axis	$h/3$	$bh^3/36$	$bh^3/12$
2.	Triangle, side b height h and vertex zero of x axis	$2h/3$	$bh^3/36$	$bh^3/12$
3.	Rectangle of width b and depth D	$D/2$	$bD^3/12$	$bD^3/3$
4.	Circle	$D/2$	$\pi D^4/64$	–
5.	Semicircle with diameter horizontal and zero of x axis	$2D/3\pi$	–	$\pi D^4/128$
6.	Quadrant of a circle, one radius horizontal	$4R/3\pi$	–	$\pi R^4/16$
7.	Ellipse : area $\pi bh/4$ Major axis is b, horizontal and minor axis is h	$h/2$	$\pi bh^3/64$	–

8.	Semi ellipse with major axis as horizontal and $x = 0$	$2h/3\ \pi$	–	$\pi\ bh^3/128$
9.	Parabola (half) area $2bh/3$ (from vertex as zero)	$y_g = 3h/5$ $x_g = 3b/8$	–	$2bh^3/7$

3.2 FORCE ON AN ARBITRARILY SHAPED PLATE IMMERSED IN A LIQUID

Case 1 : Surface exposed to gas pressure : For plane surface, force = area × pressure The contribution due to the weight of the gas column is negligible. The resultant acts at the centroid of the area as the pressure at all depths are the same.

Case 2 : Horizontal surface at a depth y.

$P = -y \times \gamma$ and as y is – ve, force = $Ay\gamma$ in which y may also be expressed as head of the fluid. The resultant force acts vertically through the centroid of the area, Here also the pressure at all locations are the same.

Case : Plane inclined at angle θ with horizontal. Refer Fig. 3.2.1

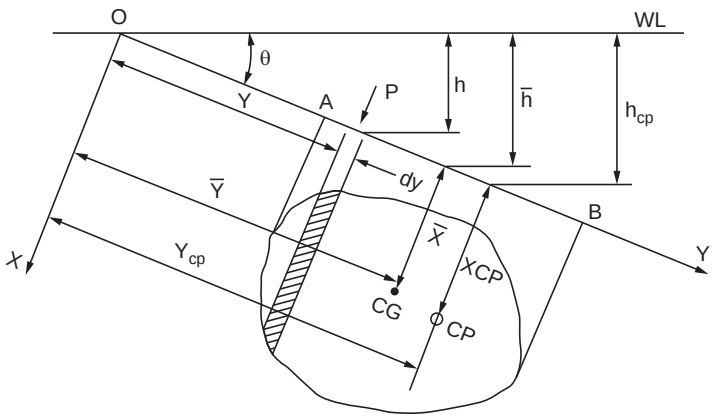


Figure. 3.2.1 Plane surface immersed in liquid at an angle

Consider the plane AB of the given shape immersed in the liquid at an angle θ to the horizontal (free surface). Let the trace of the plane (the end view of the line where it meets the horizontal plane) be "O". Consider this line as reference and set up the axes as shown in figure. Consider the elemental area dA. The force dF on the elemental area is given by

$$dF = P\ dA = \gamma h dA = \gamma\ y\ \sin\theta\ dA$$

The total force over the whole area is obtained by integration of this expression over the whole area.

$$F = \int_A \gamma\ y\ \sin\theta\ dA = \gamma \sin\theta \int_A y\ dA$$

From the definition of centroidal axis at $\bar{y} = \int_A y\ dA = \bar{y}\ A$. So

$$F = \gamma A \bar{y} \sin \theta \tag{3.2.1}$$

Calling the depth at $\bar{y}$ (distance of centroid from the surface) as $\bar{h}$,

As $$\bar{h} = \bar{y} \sin \theta, F = \gamma A \bar{h} \tag{3.2.2}$$

This equation is extensively used in the calculation of total force on a surface. Equations (3.1.4), (3.1.5) and those given in Table 3.1 are used to obtain the location of the centroid.

The following important conclusions can be drawn from this equation.

1. $\gamma\bar{h}$ equals the pressure at the centroid. The total force thus equals the product of area and the pressure at the centroid.

2. A special case of the situation is a vertical surface where $\theta = 90°$ and $\sin \theta = 1$. and so $\bar{h} = \bar{y}$ in this case.

3.3 CENTRE OF PRESSURE FOR AN IMMERSED INCLINED PLANE

The centre of pressure is determined by taking moments of the force on elementary areas with respect to an axis (say O in Fig. 3.2.1) and equating it to the product of the distance of the centre of pressure from this axis and the total force on the area (as calculated in section 3.2). For surfaces with an axis of symmetry, the centre of pressure will lie on that axis. In other cases x_{cp} and y_{cp} are calculated by the use of moments.

With reference to the Fig. 3.2.1, let CP (x_{cp}, y_{cp}) be the centre of pressure.

$$x_{cp} F = \int_A x \; P \, dA \quad \text{and} \quad y_{cp} F = \int_A y \; P \, dA$$

The area element considered here being *dxdy*. Referring to the Fig. 3.1.1

$$x_{cp} \gamma \bar{y} \; A \sin \theta = \int_A \gamma \; x \, y \sin \theta \, dA \qquad (P = \gamma y \sin \theta, F = \gamma \bar{y} \; A \sin \theta)$$

$$x_{cp} = (1/ \bar{y} \; A) \int_A x \; y \, dA, \qquad \text{As } \int_A x \; y \, dA = I_{xy}$$

From equations 3.1.9 and 3.1.10

$$x_{cp} = I_{xy} / \bar{y} \; A = (I_{Gxy} + \bar{x} \; A \; \bar{y})$$

$$= (I_{Gxy} / A \; \bar{y}) + \bar{x} \tag{3.3.1}$$

In case $x = \bar{x}$ or $y = \bar{y}$ is an axis of symmetry for the area, $I_{Gxy} = 0$ and the centre of pressure will lie on the axis of symmetry.

Along the y direction (more often the depth of centre of pressure is required)

$$y_{cp} = I_x / \bar{y} \; A, \text{ As } I_X = I_G + \bar{y}^2 \, A$$

$$y_{cp} = (I_G / \bar{y} A) + \bar{y} \tag{3.3.2}$$

I_G is the moment of inertia along the centroidal axis and $\bar{y}$ is the location of centroid along the y direction.

If the plane is vertical, then, $\bar{y} = \bar{h}$ the depth to the centroid.

In case the height h_{cp} is required instead of y_{cp}, (along the plane) then substituting

$$y_{cp} = h_{cp}/\sin\theta \quad \text{and} \quad \bar{y} = \bar{h} / \sin\theta \tag{3.3.2a}$$

Substituting in equation (3.3.2)

$$h_{cp} / \sin\theta = [I_G/\{(\bar{h} / \sin\theta) \times A\}] + \bar{h}/\sin\theta$$

$$\therefore \quad h_{cp} = [(I_G \sin^2\theta)/ \bar{h}\, A)] + \bar{h} \tag{3.3.3}$$

This is the general equation when the depth of the centre of pressure is required in the case of inclined planes. If $\theta = 90°$ (vertical surface), then $\sin^2\theta = 1$.

These equations are fairly simple, the main problem being the calculation of the moment of inertia for odd shapes.

Example 3.1. *The wall of a reservoir is inclined at 30° to the vertical. A sluice 1m long along the slope and 0.8 m wide is closed by a plate. The top of the opening is 8 m below the water level.* ***Determine the location of the centre of pressure*** *and the total force on the plate.*

The angle with the horizontal is 60°. The depth of centre of gravity,

$$\bar{h} = 8 + (0.5 \times \sin 60) / 2 = 8.433 \text{ m}$$

Total force $= \gamma\, \bar{h}\, A = 1000 \times 9.81 \times 8.433 \times 1 \times 0.8 =$ **66182 N**

$$\boldsymbol{h_{cp}} = (I_G \sin^2\theta / \bar{h}\, A) + \bar{h},\ I_G = (1/12)\, bd^3$$

$[(1/12)\ 0.8 \times 1^3 \times \sin^2 60/8.433 \times 0.8] + 8.433 =$ **8.44 m**

Distance along the wall surface, 8.44/ cos 30 = 9.746 m

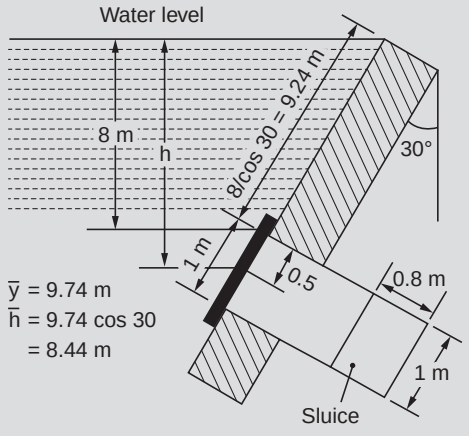


Figure Ex. 3.1 Problem model

Example 3.2. ***Determine the total force and its point of action*** *on an annular lamina of 1m ID and 3 m OD placed at an inclination of 30 degrees to the horizontal under water. The depth of centre of lamina from water surface is 8 m.*

Total force = $\gamma A \bar{h}$ = 1000 × 9.81 × π (3² – 1¹) × 8/4
= **493104.38 N** (depth is directly specified)

Depth of centre of pressure = $(I_G \sin^2 \theta / \bar{h} A) + \bar{h}$, $I_G = (\pi / 64) (D^4 - d^4)$
= $[(\pi/64) (3^4 - 1^4) \sin^2 30 - \{8 \times ((3^2 - 1^2) \pi/4)\}] + 8$
= **8.0195 m**

3.3.1 Centre of Pressure for Immersed Vertical Planes

Case 1: A rectangle of width b and depth d, the side of length b being horizontal.

Case 2: A circle of diameter d.

Case 3: A triangle of height h with base b, horizontal and nearer the free surface.

Assuming the depth of CG to be P m in all the cases. $\bar{h} = \bar{y}$ in the case

$$h_{cp} = (I_G / A \bar{y}) + \bar{y}$$

Case 1: $I_G = bd^2/12$, $\bar{y} = P$, $A = bd$

$$\mathbf{h_{cp}} = (bd^3/12\ bd\ P) + P = \mathbf{(d^2/12\ P) + P} \quad (3.3.4)$$

Case 2: $I_G = \pi d^4/64$, $\bar{y} = P$, $A = \pi d^2/4$

$$\mathbf{h_{cp}} = (\pi d^4 \times 4/64\ \pi d^2 P) + P = \mathbf{(d^2/16\ P) + P} \quad (3.3.5)$$

Case 3: $I_G = bh^3/36$, $\bar{y} = P$, $A = bh/2$

$$\mathbf{h_{cp}} = (2\ b\ h^3/\ 36\ b\ h\ P) + P = \mathbf{(h^2/18\ P) + P} \quad (3.3.6)$$

These equations can be used as a short cut under suitable situations.

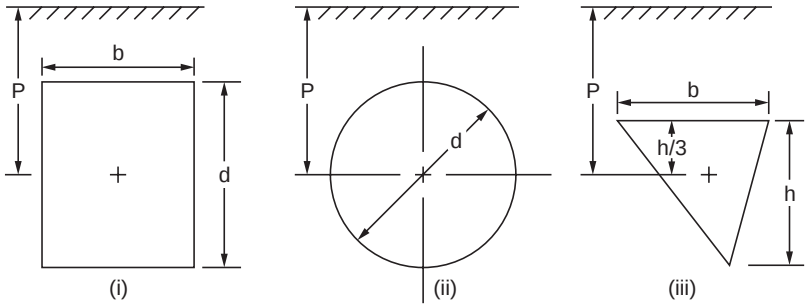


Figure 3.3.1 Vertical Surfaces

Example 3.3. *An oil tank is filled to a height of 7.5 m with an oil of specific gravity 0.9. It has a rectangular gate 1m wide and 1.5 m high provided at the bottom of a side face.* ***Determine the resultant force on the gate and also its point of action.***

Force on the gate from oil side = $\gamma A \bar{h}$ = (0.9 × 1000 × 9.81) (1 × 1.5) (6 + 0.75) = **89394 N**

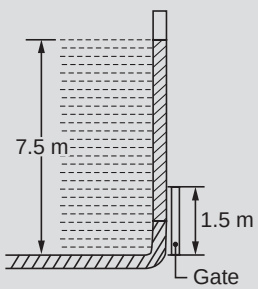


Figure Ex. 3.3

$$\mathbf{h_{cp}} = (I_G / \bar{y}\, A) + \bar{y};\ I_G = bd^3/12;\ \bar{y} = \mathbf{6.75\ m}$$

$$= ((1 \times 1.5^3/12)/6.75 \times 1 \times 1.5) + 6.75 = \mathbf{6.78\ m}$$

Check using eqn. 3.3.4, which is simpler,

$$\mathbf{h_{cp}} = (d^2/12P) + P = (1.5^2/12 \times 6.75) + 6.75 = \mathbf{6.78\ m}$$

The resultant force will act at a distance of 6.78 m from the surface of oil at the centre line of the gate.

Example 3.4. ***Determine the net force and its point of action*** *over an L shaped plate submerged vertically under water as shown in Fig. Ex. 3.4. The top surface of the plate is 1.5 m below water surface.*

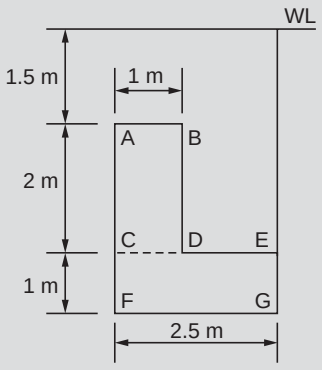


Figure Ex. 3.4

The plate can be considered as two rectangles (*i*) ABCD 1m wide and 2 m deep and (*ii*) CEFG 2.5 m wide and 1 m deep.

$$F_i = \gamma\, h_i A_i = (1000 \times 9.81)$$

$$(1.5 + 2.0/2)\ (1 \times 2) = 49050\ \text{N}$$

$$F_{ii} = \gamma\, h_{ii} A_{ii} = (1000 \times 9.81)$$

$$(1.5 + 2 + 1/2)\ (1 \times 2.5) = 98100\ \text{N}$$

Total force on the plate = **147150 N**

Considering *ABCD* $h_{cpi} = (I_{gi} / \bar{h}_i\, A_i) + \bar{h}_i$

$$= [(1 \times 2^3/12)/(1 \times 2.5 \times 2)] + 2.5 = 2.633\ \text{m}$$

Also by equation (3.5.1), $h_{cpi} = (d^2/12P) + P = (2^2/12 \times 2.5) = 2.633$ m

$$h_{cpii} = (I_{gii}/h_{ii}\, A_{ii}) + h_{ii} = ((2.5 \times 1^3/12)\ (1/4 \times 2.5)) + 4 = 4.021\ \text{m}$$

Also by equation 3.3.4, $h_{cpii} = (1^2/12 \times 4) + 4 = 4.021$ m

In order to locate the point of action of the resultant force, moment is taken with reference to the surface to determine the depth.

$$\mathbf{h_{cpy}} = (F_i\, h_{cpi} + F_{ii}\, h_{cpii})/(F_i + F_{ii})$$

$$= (49050 \times 2.633 + 98100 \times 4.021)/(49050 + 98100) = \mathbf{3.5583\ m}$$

Moment is taken about *AF* to determine the lateral location

$$\mathbf{h_{cpx}} = [(49050 \times 0.5) + (98100 \times 1.25)]/(49050 + 98100) = \mathbf{1.0\ m}$$

The resultant force acts at a depth of 3.5583 m and at a distance of 1.0 m from the edge *AF*.

3.4 COMPONENT OF FORCES ON IMMERSED INCLINED RECTANGLES

Consider a case of a rectangle of $a \times d$, with side d inclined at θ to the horizontal, immersed in a fluid with its centroid at a depth of $\bar{h}$ m. For this case it can be shown that (*i*) **The horizontal component of the resultant force equals the force on the vertical projection of the area and** (*ii*) **The vertical component equals the weight of the fluid column above this area.**

The net force acting perpendicular to the area is given by = $\gamma A \bar{h}$

The horizontal component equals = $\gamma A\, \bar{h} \sin\theta$

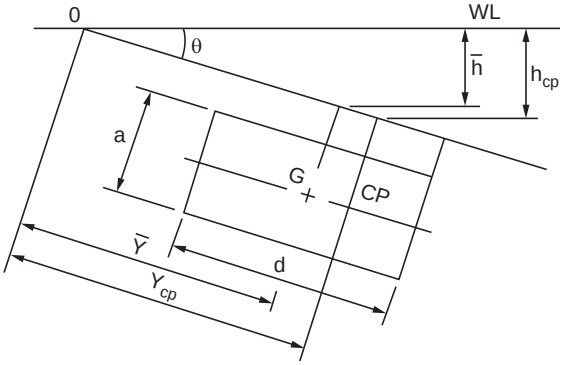


Figure 3.4.1

The vertical projection of the area = $A \sin \theta$

The centroid of this area will also be at $\bar{h}$

The force on the projected area = $\gamma A \bar{h} \sin \theta$.

Hence the horizontal component of the force equals the force on the vertical projection of the area.

The vertical component = $\gamma A \bar{h} \cos \theta$

The horizontal projection of the area = $A \cos \theta$

The volume of the fluid column above this surface = $A \cos \theta \bar{h}$

The weight of the fluid column = $\gamma A \bar{h} \cos \theta$

Hence the vertical component of the force equals the weight of the fluid column above the area.

It can also be shown that the location of the action of the horizontal component will be at the centre of pressure of the projected area and the line **of action of the vertical component will be along the centroid of the column of the liquid above the plane.**

Using equation (3.3.2), denoting the distance along the plane as y,

$$y_{cp} = (I_G / \bar{y} \, . A) + \bar{y}, \text{ and in case } \textbf{the edge is at the free surface, } \bar{y} = d/2$$

The equation reduces to $y_{cp} = (2/3)\, d$ or $(2/3)\, \bar{y} \sin \theta$ from free surface (3.4.1)

Example 3.5. *An automatic gate which will open beyond a certain head h is shown in Fig. Ex. 3.5.* ***Determine the ratio of h/L.*** *Neglect the weight of gate, friction etc.*

Consider 1 m width of the gate

The vertical force = $L\, \gamma\, h$ and acts at a distance $L/2$ from O.

The horizontal force on the gate = $\gamma\, h\, h/2$ and acts at $h/3$ distance from O.

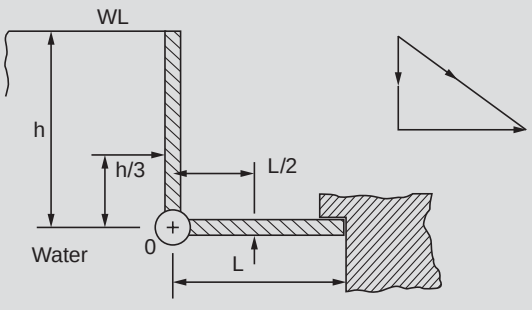


Figure Ex. 3.5

Note : When the water level is at the top of vertical plate, the centre of pressure will be at (2/3)h from top. See 3.4.1.

Taking moments about O, $L\,\gamma\,h\,(L/2) = \gamma\,h\,(h/2)\,(h/3)$

$$L^2 = h^2/3 \text{ (or) } \mathbf{L = h/(3)^{0.5} = 0.5774\ h}$$

For example if L = 2 m, then h = 3.4641 m.

Example 3.6. *A modified form of automatic gate is shown in Fig. Ex. 3.6. Determine the value of h in terms of L, W, and L_w where W is the weight for unit width of gate and L_w is the distance from O to the line of action of W.*

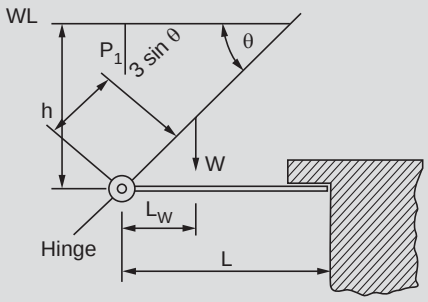


Figure Ex. 3.6

Considering unit width,

The vertical force on the horizontal side = $\gamma\,h\,L$ and acts upwards at $L/2$ from O.

Weight of the gate = W and acts downwards at L_w from O.

Total force normal to the plate $F = \gamma\,.\,A.\,\bar{h}$

For unit width, $A = h/\sin\theta$ and $\bar{h} = h/2$

Using equation (3.4.1.), the line of action along the inclined side can be obtained as $h/3 \sin\theta$ from bottom edge, or $\left(\frac{2}{3}\frac{h}{\sin\theta}\text{ from top}\right)$

Pressure force normal to plate = $\gamma\,(h/2)\,(h/\sin\theta)$ and acts at $(h/3 \sin\theta)$ from O.

Taking moments about O,

$$W.L_w + \gamma\,(h/2)\,(h/\sin\theta)\,(h/3\sin\theta) = \gamma\,h\,L\,L/2$$

$$\mathbf{(\gamma / 6\ sin^2\theta)\ h^3 - (\gamma L^2 / 2)\ h + W\ L_w = 0}$$

This is a cubic equation in h and can be solved by trial. With the calculators available presently cubic equations can be solved directly.

Example 3.7. *An automatic gate which opens beyond a particular head is as shown in Fig. Ex. 3.6. For the following data, determine the value of water head h to open the gate.* $\theta = 50°$, $L = 1m$, $W = 8000\ N$, $L_w = 0.265\ m$.

Using the equation derived in example 3.6,

$$(\gamma / 6 \sin^2\theta)\ h^3 - \gamma\,(L^2/2)\ h + W\,L_w = 0$$

$$(9810/ 6 \sin^2 50)\ h^3 - 9810(1^2 / 2)\ h + 8000 \times 0.265 = 0$$

or

$$2786\ h^3 - 4905\ h + 2120 = 0$$

Solving by trial, h = 1 m., **Hence the gate will open when water rises to 1m above *O***, (The other solution is 0.506 m).

3.5 FORCES ON CURVED SURFACES

***(i)* Vertical forces : The vertical force on a curved surface is given by the weight of the liquid enclosed by the surface and the horizontal free surface of the liquid.** The

force acts along the centre of gravity of the volume. In case there is gas pressure above the surface, the force due to gas pressure equals the product of horizontal projected area and the gas pressure and acts at the centroid of the projected area.

If the other side of the surface is exposed to the same gas pressure, force due to the gas pressure cancels out. This applies to doubly curved surfaces and inclined plane surfaces.

***(ii)* Horizontal forces: The horizontal force equals the force on the projected area of the curved surface and acts at the centre of pressure of the projected area.** The value can be calculated using the general equation.

$$F = \gamma A \bar{h},$$

where A is the projected area and $\bar{h}$ is the depth of the centroid of the area.

These two statements can be proved as indicated below. Refer to Fig. 3.5.1. The volume above the surface can be divided into smaller elements. At the base of each element, the vertical force equals the weight of the small element. Thus the total vertical force equals the sum of the weights of all the elements or the weight of the liquid enclosed between the area and the horizontal surface. Consider an imaginary vertical surface $A'B'$. The element between $A'B'$ and the surface AB is in equilibrium. $A'B'$ gives the projected vertical area. The horizontal force on this area due to liquid pressure should equal the horizontal force on the curved surface for the volume $A'B'AB$ to be in equilibrium. Hence the horizontal force equals the force on the projected area due to liquid pressure.

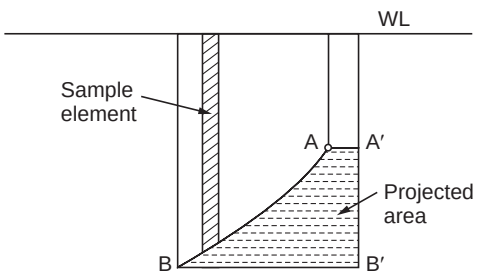


Figure 3.5.1

Example 3.8. *Determine the force exerted by sea water (sp. gravity = 1.025) on the curved portion AB of an oil tanker as shown in Fig. Ex. 3.8. Also determine the direction of action of the force.*

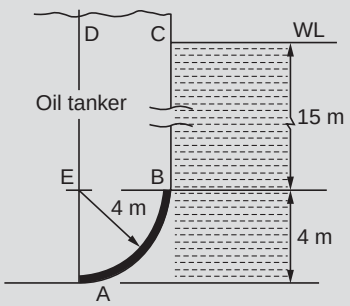


Figure Ex. 3.8

Consider 1m width perpendicular to paper,

The horizontal component of the **force** acting on the curved portion AB

$$= \gamma A \bar{h} = (1025 \times 9.81)(4 \times 1)(15 + 4/2)$$

$$= \mathbf{683757\ N}$$

Line of action of this horizontal force $= \bar{h} + (I_G/\bar{h}\,A)$

$$= 17 + [1 \times 4^3/12]\,[1/(17 \times 4 \times 1)] = 17.0784\ \mathbf{m}$$

from top and towards left

The vertical force is due to the volume of sea water displaced.

Vertical force = [volume *BCDE* + volume *ABE*] γ

= [(15 × 4 × 1) + (4^2 × π × 1/4)] [1025 × 9.81]

= **729673 N** (acts upwards).

To find the location of this force : Centre of gravity of the column *BCDE* is in the vertical plane 2 m from the edge.

Centre of gravity of the area *ABE* = (4 – 4*R*/3π) from the edge

= (4 – 4 × 4/3π) = 2.302 m from the edge.

Taking moments of the area about the edge, the line of action of vertical force is

= [(2.302 × 4^2 π/4) + 2 × (15 × 4)] / [(4^2 π/4) + (15 × 4)] = **2.0523 m from the edge.**

The resultant force = (683757^2 + $729673^{2)0.5}$ = **999973.16 N**

The direction of action to the vertical is,

tan θ = 683757 / 729673 = 0.937 ∴ θ = 43.14°

The answer can be checked by checking whether the resultant passes through the centre of the circle (as it should) by taking moments about the centre and equating them.

729673(4 – 2.0523) – 683757 × 2.078 = 337

Compared to the large values, the difference is small and so the moments are equal and the resultant can be taken as zero. Hence the resultant can be taken to pass through the centre of the cricle.

Example 3.9. *Determine the magnitude and direction of the resultant force due to water on a quadrant shaped cylindrical gate as shown in Fig. Ex. 3.9. Check whether the resultant passes through the centre.*

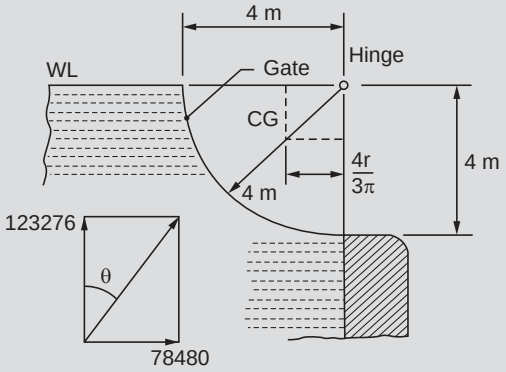


Figure Ex. 3.9

The horizontal force = $\gamma A \bar{h}$ where *A* is the projected area.

Considering unit width,

Horizontal force = 9810 × 2 × 4 × 1 = 78480 N, to the right

It acts at the centre of pressure of the projected area

i.e., at = 1.333 m from the bottom (*i.e.,* (1/3) × 4)

Vertical force = the weight to the liquid displaced

= π × 4^2 × 1 × 9810/4 = 123276 N, upwards.

It acts at $4r/3\pi$ = 1.698 m, from the hinge.

Resultant force = $(123276^2 + 78480^2)^{0.5}$ = 146137 N

Angle is determined by tan θ = 78480/123276 = 0.6366,

∴ θ = 32.48° where θ is the angle with vertical.

To check for the resultant to pass through the centre the sum of moment about *O* should be zero.

$$78480 \times (4 - 1.3333) - 123276 \times 1.698 = 42.$$

Compared to the values the difference is small and these can be assumed to the equal. Hence the resultant passes through the centre of the cricle.

3.6 HYDROSTATIC FORCES IN LAYERED FLUIDS

Two fluids may sometimes be held in a container one layer over the other. In such cases the total force will equal the some of the forces due to each fluid. The centre of pressure has to determined for each layer separately with reference to the centroid of each area. The location of the point of action of the total force can be determined taking moments about some convenient references.

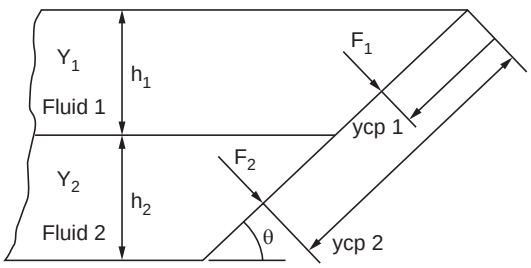


Figure 3.6.1

Total force, $F = F_1 + F_2 + ... = P_1A_1 + P_2A_2 + ...$

The depth of centre of pressure of fluid 1 is determined using the eqn. (4.3.2)

$$y_{cp1} = -\frac{\rho_1 g \sin\theta\, I_{xx}}{P_1 A_1} + \bar{y}_1 \qquad (3.8.1)$$

This distance is with respect to the centroid of the area.

Note : (P_1 / ρ_1) gives the head of the fluid as ρ is different for different fluids, this form is prferable.

Example 3.10. *A tank 20 m deep and 7 m wide is layered with 8 m of oil, 6 m of water and 4 m of mercury. Determine the total hydrostatic force and resultant centre of pressure on the side. Specific gravity of oil is 0.881 and that of mercury is 13.6.*

Pressures at the centroid of each layer is

$$P_{cg1} = 881 \times 4 \times 9.81 = 34570.44 \text{ N/m}^2$$

$$P_{cg2} = (881 \times 4 \times 9.81) + (1000 \times 3 \times 9.81) = 64000 \text{ N/m}^2$$

$$P_{cg3} = (881 \times 4 \times 9.81) + (1000 \times 3 \times 9.81) + (13600 \times 2 \times 9.81)$$

$$= 330832 \text{ N/m}^2$$

$$F_1 = P_{cg1} \times A_1 = 34570 \times 8 \times 7 = 1935944 \text{ N}$$
$$F_2 = P_{cg2} \times A_2 = 64000 \times 6 \times 7 = 2688018 \text{ N}$$
$$F_3 = P_{cg3} \times A_3 = 330832 \times 4 \times 7 = 9263308 \text{ N}$$
$$\text{Total force} = 13.887 \times 10^6 \text{ N/m}^2$$

$$y_{cp1} = -\frac{\rho_1 g \sin\theta\, I_{xx}}{P_1 A_1} = \frac{881 \times 9.81 \times 8^3 \times 7}{12 \times 34570} = 1.333 \text{ m as } \theta = 90°$$

$$y_{cp2} = \frac{1000 \times 9.81 \times 6^3 \times 7}{12 \times 2688018} = 0.46 \text{ m},$$

$$y_{cp3} = \frac{13600 \times 9.81 \times 4^3 \times 7}{12 \times 9263308} = 0.54 \text{ m}$$

The line of action of the total force is determined by taking moment about the surface.

$$y \times 13.887 \times 10^6 = (5.333 \times 1935944) + (11.46 \times 268808) + (16.54 \times 9263308)$$

Solving $y = 13.944$ m.

Example 3.11. *A tank contains water upto 3 m height over which oil of specific gravity 0.8 is filled to 2 m depth. Calculate the pressure at 1.5 m, 2 m and 2.5 m. Also calculate the total force on a 6 m wide wall.*

(*i*) At 1.5 m depth, $P_{1.5} = (0.8 \times 1000) \times 9.81 \times 1.5 = 11770 \text{ N/m}^2$

$P_{2.0} = (0.8 \times 1000) \times 9.81 \times 2 = 15700 \text{ N/m}^2$

$P_{2.5} = 15700 + (1000 \times 9.81 \times 0.5) = 20600 \text{ N/m}^2$

At the base, $P = 15700 + (1000 \times 3 \times 9.81) = 45130 \text{ N/m}^2$

Total force, $F = (15700 \times 2 \times 6/2) + \{(45130 + 15700) \times 3 \times 6/2\} = 641670$ N

SOLVED PROBLEMS

Problem 3.1. *The force due to water on a circular gate of 2m dia provided on the vertical surface of a water tank is 12376 N. Determine the level of water above the gate. Also* ***determine the depth of the centre of pressure from*** *the centre of the gate.*

Total force on the gate $= \gamma A \bar{h} = (1000 \times 9.81)(\pi \times 2^2/4) \times \bar{h} = 123276.09$ N

Solving $\bar{h} = 4$ m. So the depth of water above the centre = 4 m

Centre of action of this force

$$= (I_G / \bar{h} A) + \bar{h} = (\pi \times 2^4/64)/(4 \times \pi \times 2^2/4) + 4$$
$$= \mathbf{4.0625 \text{ m, from the water level}}$$

or 0.0625 m from the centre of the gate

Check using eqn. (3.4.2), $h_{cp} = (D^2/16\bar{h}) + \bar{h} = (2^2/(16 \times 4) + 4 = 4.0625$ m

Problem 3.2. *A circular plate of 3 m dia is vertically placed in water with its centre 5 m from the free surface. (i) Determine the force due to the fluid pressure on one side of the plate and also its point of application. (ii) Also find the diameter of a concentric circle dividing this area into two so that force on the inner circular area will equal the force on the annular area. (iii) Determine also the centre of pressures for these two areas separately. (iv) Show that the centres of pressure of the full area lies midway between these two centres of pressure.*

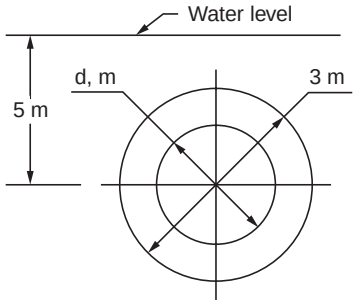


Figure P. 3.2

(*i*) The total force on the circular plate

$$\gamma A \bar{h} = 9810 \times (\pi \times 3^2/4) \times 5 = 346714 \text{ N}$$

Centre of pressure $= \bar{h} + (I_G/\bar{h}\, A) = 5 + (\pi \times 3^4 \times 4/64 \times 5 \times \pi \times 3^2)$

$= \mathbf{5.1125\ m.}$

(*ii*) The given condition is, force on the circular area = force on annular area, with outside diameter being 3 m. For the areas the centroid depth $\bar{h}$ is the same.

$$\gamma A_1 \bar{h} = \gamma A_2 \bar{h}$$

assuming that the diameter of the smaller circle to be d,

$$A_1 = \pi (3^2 - d^2)/4,\ A_2 = \pi d^2/4 \text{ equating}$$

$\therefore \quad 3^2 - d^2 = d^2$ or $2d^2 = 3^2$ or $d = 2.12132$ m.

The force on the inner circular area = $9810\ (\pi \times 3^2/2 \times 4) \times 5 =$ **173357 N**

(Checks as it is half of 346714 N)

Centre of pressure for the circle

$= 5 + [\pi \times 2.12132^4 \times 4/64 \times \pi \times 2.12132^2 \times 5] =$ **5.05625 m**

Centre of pressure for the annulus

$= 5 + [\{\pi\ (3^4 - 2.12132^4)/64\}/\{5 \times \pi\ (3^2 - 2.12132^2)/4\}]$

$=$ **5.16875 m**

The mid point of these two is = (5.16875 + 5.05625)/2 = 5.1125 m

(same as the centre of pressure of full area)

Problem 3.3. *A plane 3 × 4 m is vertically placed in water with the shorter side horizontal with the centroid at a depth of 5m. (i) Determine the total force on one side and the point of action of the force. (ii) Also determine the size of an inner rectangle with equal spacing on all sides, the total force on which will equal the force on the remaining area. (iii) Determine the centre of pressures for these areas and compare the moments of these forces about a horizontal axis passing through the centre of pressure for the whole area.*

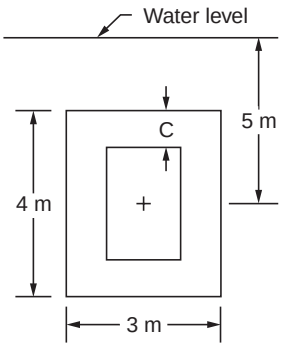


Figure P. 3.3

(*i*) The total force = $\gamma A \bar{h}$ = 1000 × 9.81 × 5 × 4 × 3 = 588600 *N*

Centre of pressure = 5 + (1/2) (3 × 4^3/5 × 12) = **5.2667 m**

(*ii*) Assume a spacing of *C* m on all sides to form the inner rectangle. The depth *CG* is the same for both areas

As $\gamma A_1 \bar{h} = \gamma A_2 \bar{h}$,

$A_1 = A_2$; (4 × 3) = 2(4 – 2*C*) (3 – 2*C*)

This reduces to $C^2 - 3.5\,C + 1.5 = 0$;

Solving *C* = 0.5 or 3.0. As *C* = 3 is not possible,

C = 0.5. The smaller rectangle is of 2 m × 3 m size,

(Check : 2 × 3 = 6, 4 × 3 = 12, half the area)

Pressure on the smaller rectangle = 1000 × 9.81 × 5 × 3 × 2 = **294300 N**

(*iii*) Centre of pressure for the inner area = 5 + (1/2) (2 × 3^3/2 × 3 × 5) = **5.15 m**

Centre of pressure for the outer area

= 5 + (1/2) (3 × 4^3 – 2 × 3^3) / [5 × (4 × 3 – 3 × 2)] = **5.3833 m**

(*iv*) Moment of the force on the inner area about the *CP* of the whole area

= 294300 (5.26667 – 5.15) = 34335 Nm (clockwise, acts above)

Moment of the force on the outer area

= 294300 (5.3833 – 5.26667)

= 34335 Nm (anti clockwise, acts below)

The moments are equal but are opposite in sign and the total is zero.

Problem 3.4. *A water tank has an opening gate in one of its vertical side of 10 m × 5m size with 5m side in the horizontal direction. If the water level is upto the top edge of the gate, locate three horizontal positions so that equal forces will act at these locations due to the water pressure.*

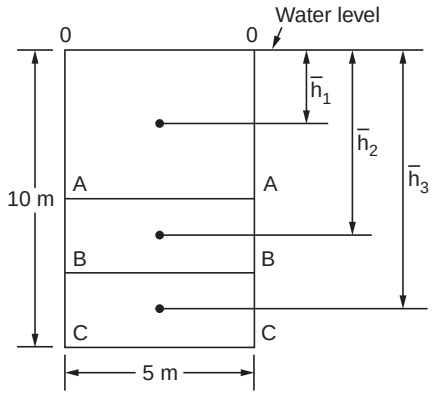


Figure P. 3.4

To solve this problem the area should be divided into three parts in each of which the force will be equal to 1/3 of the total force on the surface. Then the centre of pressure for each of the areas should be located to obtain the points of application of the forces.

Let $\bar{h}_1$ be the centroid of the top portion of the surface on which the force acting is equal to 1/3 of the total. Then

(*i*) $$\gamma\, \bar{h}_1\, A_1 = (1/3)\, \gamma A\, \bar{h}$$

$$\gamma \bar{h}_1 (2\, \bar{h}_1 \times 5) = (1/3) \times \gamma \times (10 \times 5) \times 5$$

$$\therefore \quad \bar{h}_1{}^2 = 25/3$$

$$\therefore \quad \bar{h}_1 = 2.887 \text{ m},$$

Therefore the depth of the top strip is $2 \times 2.887 = 5.774$ m

Centre of pressure for first strip $= [I_{G1}/\bar{h}_1\, A_1] + \bar{h}_1$

$= [5 \times 5.774^3/12 \times 2.887\, (5 \times 5.774)] + 2.887 =$ **3.8493 m**

(*ii*) For the second strip, let the centroid be $\bar{h}_2$.

The depth of this second strip $= 2(\bar{h}_2 - 2\, \bar{h}_1)$

Force on the second strip, $\gamma\, \bar{h}_2\, A_2 = (1/3)\, \gamma\, A\, \bar{h}$

$$\gamma\, \bar{h}_2 \times 5 \times 2\, (\bar{h}_2 - 2\, \bar{h}_1) = (1/3) \times \gamma \times 5 \times 10 \times 5$$

$$\bar{h}_2{}^2 - 2\, \bar{h}_1\, \bar{h}_2 - (25/3) = 0 \quad \text{or} \quad \bar{h}_2{}^2 - 2 \times 2.887\, \bar{h}_2 - (25/3) = 0$$

$$\bar{h}_2{}^2 - 5.774\, \bar{h}_2 - 8.33 = 0, \text{ solving } \bar{h}_2 = 6.97 \text{ m}$$

The depth for second strip is $= 2(\bar{h}_2 - 2\, \bar{h}_1)$

$= 2\, (6.97 - 2 \times 2.887) = 2.392$ m

Centre of pressure for this second strip $= [I_{G2}/\bar{h}_2\, A_2] + \bar{h}_2$

$= [5 \times 2.392^3/12 \times 6.97 \times (5 \times 2.392)] + 6.97 =$ **7.038 m**

(*iii*) The depth of the third and bottom strip is

$= 10 - 5.774 - 2.392 = 1.834$ m

$$\bar{h}_3 = 10 - (1.834/2) = 9.083 \text{ m}$$

Centre of pressure for third strip is $= [I_{G3}/\bar{h}_3\, A_3] + \bar{h}_3$

$= 5 \times 1.834^3/(12 \times 9.083 \times 5 \times 1.834) + 9.083 =$ **9.114 m**

To keep the gate closed supports at these locations will be optimum *i.e.*, at depths of 3.849, 7.038 and 9.114 m. The average of these values will equal the depth of centre of pressure for the whole gate *i.e.,* 6.667 m (check).

Problem 3.5. *A triangular surface is kept vertical in water with one of its edges horizontal and at the free surface. If the triangle is divided by a line drawn from one of the vertices at the free surface such that the total force is equally divided between the parts, determine the ratio by which the opposite side is divided.*

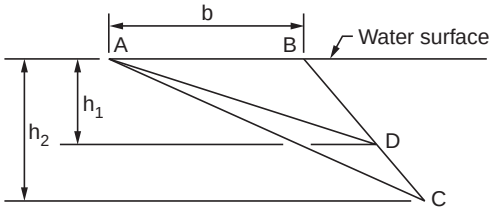


Figure P. 3.5

Consider the triangle *ABC*. Let the *AD* divided the triangle such that the total force on *ABD* equals the force on *ACD*.

Let h_1 be the height of triangle *ABD* and let h_2 be the height of triangle *ABC* along the depth.

The force on *ABC* = 2 × the force on *ABD*

The centroids will be at $h_1/3$ and $h_2/3$ for these triangles.

$$\gamma\,(bh_2/2)(h_2/3) = 2\,(\gamma\,bh_1/2)(h_1/3), \text{ rearranging}$$

$$h_2^{\,2} = 2\,h_1^{\,2} \quad \text{or} \quad (h_2/h_1) = (2)^{0.5}$$

But $BC/BD = (h_2/h_1) = (2)^{0.5}$

The opposite side is divided the ratio of $(2)^{0.5}$ *i.e.*, *BC/BD* = 1.4142

$\therefore$ **CD/BD = 0.4142**

Problem 3.6. *In a water reservoir, the vertical gate provided for opening is a semicircular plate of dia 3 m with diameter horizontal and at the water level.* ***Determine the total pressure and its point of action if water level*** *is up to the top edge of the gate.*

Total force on the gate = $\gamma A\bar{h}$

$\bar{h}$ = centre of gravity of the semicircular surface $2D/3\pi$

$= 2 \times 3/3 \times \pi = 0.6366$ m

Total force = $1000 \times 9.81 \times (\pi \times 3^2/4 \times 2) \times 0.6366 = 22072$ N = 22.072 kN

$I_{base} = \pi D^4/128$ (about the diameter)

Depth centre of pressure = $I_{base}/A\bar{h}$

$= (\pi \times 3^4/128)(2 \times 4/\pi \times 3^2)\,(1/0.6366)$

= **0.8836 m**

Problem 3.7. *A water tank is provided with a gate which has a shape of a quadrant of a circle of 3 m radius. The gate is positioned in such a way that one straight edge of it is horizontal. Determine the force acting on the gate due to water and its point of action if the tank is filled with water upto 2 m above the edge.*

Distance of centre of gravity of the gate from the top edge = $2D/3\,\pi = 4r/3\,\pi$

Total pressure on the gate = $\gamma\,\bar{h}\,A$

$= 1000 \times 9.81\,[2 + 4 \times 3/3\,\pi]\,[\pi \times 3^2/4]$ = **226976 N**

Moment of inertia for the gate with reference to the diameter = $\pi D^4/ 2 \times 128$

$$I_G = I - \bar{y}^2 A$$

Moment of inertia with reference to the centroid

$= [(1/2) \times \pi D^4/128] - (\pi D^2/4 \times 4)(2D / 3\pi)^2$

$= [(1/2) \times \pi D^4/128] - \pi D^4/36\pi^2$

$= \pi D^4 [(1/256) - (1/36\pi^2)]$, Simplyfying

$I_G = \pi D^4/ 916$

Depth of centre of pressure = $[I_G/\bar{h} A] + \bar{h}$

$\bar{h} = 2 + 4r/3\pi = 2 + 4/\pi = 3.2732$ m as $r = 3$,

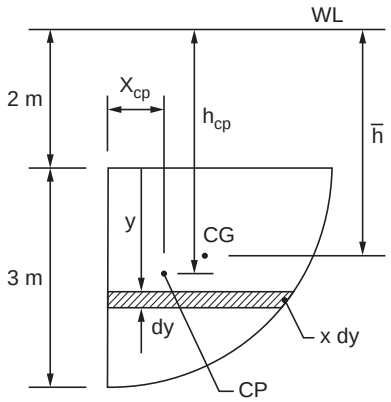


Figure P. 3.7

Depth of centre of pressure

$= [(\pi \times 6^4/916)/(3.2732 \times \pi 3^2/4)] + 3.2732$

$= 0.1921 + 3.2732 =$ **3.4653 m**

To determine the location of centre of pressure (as there is no line of symmetry with reference to the axes), moment of elementary forces of the elementary strips is taken with reference to the y axis and equated to the product of total force and the distance to the centre of pressure from x axis.

Circle equation is, $x^2 + y^2 = 9$ (taking centre as 0, 0)

Area of strip = $x.dy$, Force on the strip = $\gamma.h.x.dy$

Moment of force with respect to y axis $dM = \gamma h\, x\, dy\, x/2 = \dfrac{\gamma}{2}\, hx^2\, dy$,

[force acts at a distance of $x/2$ from y axis]

As $h = y + 2$ and $x^2 = (9 - y^2)$, $dM = (\gamma/2)(y + 2)(9 - y^2)\, dy$

Integrating the above expression from $y = 0$ to $y = 3$

$$M = (\gamma/2) \int_0^3 [-y^3 - 2y^2 + 9y + 18]\, dy$$

$$= \frac{\gamma}{2}\left[-\frac{y^4}{4} - \frac{2}{3}y^3 + \frac{9}{2}y^2 + 18y\right]_0^3$$

$= (\gamma/2)\,[56.25] = (9810/2) \times 56.25 = 275{,}906$ Nm

Equating the moment for the total force

$226{,}976 \times x_p = 275{,}906 \quad \therefore \quad x_p = 1.2156$ m from the left edge

The centre of pressure is located at 3.4655 m below the free surface and 1.2156 m from the vertical edge.

Problem 3.8. *A right angle triangle of 2m × 2m sides lies vertically in oil of specific gravity 0.9 with one edge horizontal and at a depth of 2m. Determine the net force on one side and its point of action.*

The centre of gravity lies at 1/3 height from base. The moment of inertia about the CG is $bh^3/36$ where b is the base and h is the height.

Total force = $\gamma A \bar{h}$ = 1000 × 9.81 × 0.9 × (2 × 2/2)(2 + 2/3) = **47088 N**

(if water F = 47088/0.9 N)

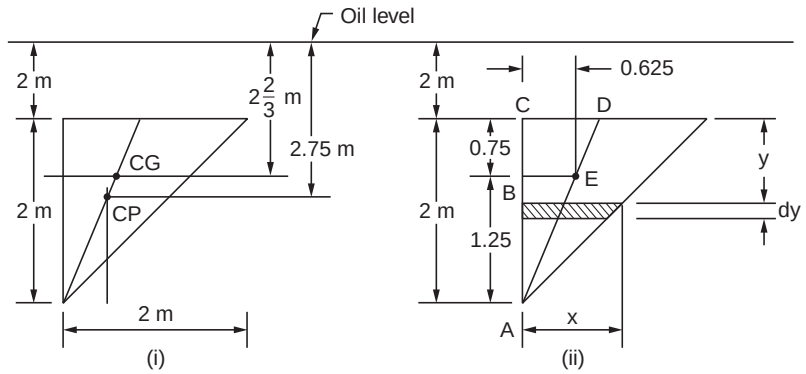


Figure P. 3.8

The depth of the centre of pressure = $h_{cp} = \bar{h} + (I_G/\bar{h}\ A)$

= (2 + 2/3) + [(2 × 3^3/36)/(2 + 2/3) (2 × 2/2)] = 2.75 m

In the x direction, the centre of pressure will be on the median line, which is the line of symmetry. Referring to the figure,

(BE/CD) = (AB/AC) = (1.25/2), CD = 1 so, BE = 0.625 m

The centre of pressure is **2.75 m from top and 0.625 m from the vertical side.**

Check:

A strip of width dy is considered. Force on the strip = $\gamma\, h\, A = \gamma\,(y + 2)\,(x\, dy)$

In this case, by similar triangle ($AF = FG = 2 - y$) $\quad \therefore \quad x = (2 - y)$

Force $dF = \gamma\,(y + 2)\,(2 - y)\,dy$ and Moment about the vertical edge = $dF\,(x/2)$

$dM = \gamma\,((2 - y)/\,2)\,(y + 2)\,(2 - y)\,dy$, Integrating from 0 to 2 for value of y,

$$M = (\gamma/2) \int_0^2 (y^3 - 2y^2 - 4y + 8)\,dy$$

$$= (9810 \times 0.9 / 2) \left[\frac{y^4}{4} - \frac{2}{3}y^3 - \frac{4}{2}y^2 + 8y \right]_0^2 = 29430 \text{ Nm}$$

Taking moments of the total force,

x_{cp} = 29430/47088 = 0.625 m (checks)

Whenever there is a line of symmetry for the axis, the centre or pressure will be on it.

Problem 3.9. *Determine the centre of pressure and the total force for the combined area as shown in Fig. P.3.9. Assume water is the liquid.*

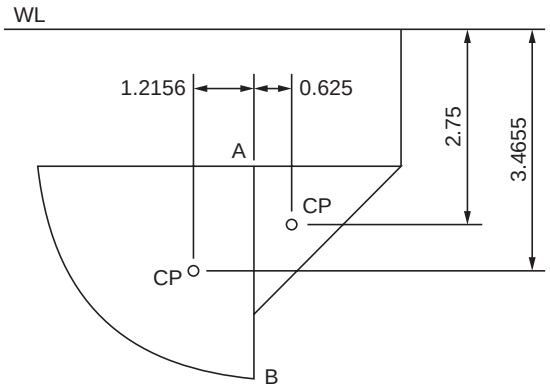


Figure P. 3.9

The shape is a combination of the shapes of problems P.3.7 and P.3.8.

The available values from these problems are

(*i*) For the Quadrant of circle the centre of pressure is at depth = 3.4655 m and distance from side = 1.2156 m

(*ii*) For the triangle, the centre of pressure is at depth = 2.75 m and distance from the side = 0.625 m

(note *CP* is independent of density as long as density is constant)

The forces are : C from the problems P.3.7 and P.3.8)

For triangle = 47088 × (1/0.9) = 52320 N

For quadrant = 226976 N

To locate the depth moment is taken about the surface.

Taking moments about *AB*, depth

$$y = [(226976 \times 3.4655) + (47088 \times 2.75/0.9)]/(226976 + 52320)$$

$$= \mathbf{3.3313\ m}$$

Taking moments about the common edge horizontal location is,

$$x = [(1.2156 \times 226976) - (0.625 \times 52320)]/(226976 + 52320)$$

= **0.8708 m** to the left of the common edge

The combined centre of pressure lies at a depth of 3.3313 m and 0.8708 m to the left from the vertical common edge.

Problem 3.10. *An oil tank has an opening of 2 m square with diagonal horizontal in one of its vertical wall as shown in Fig. P. 3.10. Determine the total force and torque required to close the opening by a hinged gate exactly if the oil (sp. gravity 0.90) level is 5m above the centreline of the gate.*

The centre of gravity for the plate is on its diagonal.

Moment of inertia = Moment of inertia of the top triangle + Moment of inertia of bottom triangle = $bh^3/12 + bh^3/12 = 2\ bh^3/12$

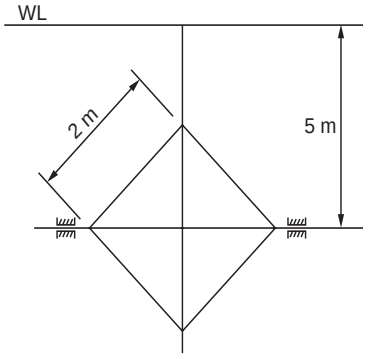


Figure P. 3.10

$$h = \sqrt{2^2 + 2^2}/2 = \sqrt{8}/2,\ b = \sqrt{8}$$

Moment of Inertia $= 2(\sqrt{8})\,(\sqrt{8}/2)^3/12 = 1.3333\ \text{m}^4,\ h = 5,\ A = 4$

Depth of centre of pressure $= (1.3333/5 \times 4) + 5 =$ **5.0667 m**

The centre of pressure lies on the vertical diagonal at a depth of 5.0667 m

Total force on the gate $= \gamma A\bar{h} = (1000 \times 9.81 \times 0.9) \times 4 \times 5 =$ **176.580 kN**

Torque required to close the gate $= (5.0667 - 5)\ 176580 =$ **11,778 Nm**

Problem 3.11. *A hinged gate is held in position by a counter weight W as shown in Fig. P. 3.11. The gate is L m long along the slope and b m wide. The counter weight, W acts perpendicular to the gate which is inclined at angle θ. Determine the height of water for the movement of the gate outwards. Neglect the weight of the gate.*

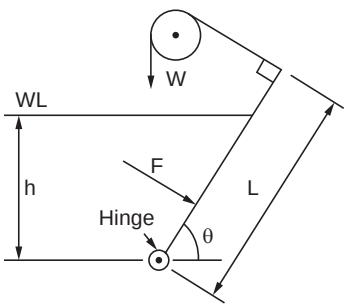


Figure P. 3.11

Let h m of water cause the gate to just start to move out,

Force on the gate $= \gamma A\,\bar{h} = \gamma\,(hb/\sin\theta)\,h/2$. The force acts at $(h/3.\sin\theta)$ from the hinge (position of centre of pressure). The tension of the rope will equal W. Taking moments about the hinge,

$$W.L = \gamma\,(hb/\sin\theta)\,(h/2)\,(h/3\sin\theta)$$

$$\therefore \qquad \mathbf{h^3 = 6\ W\ L\ \sin^2\theta\gamma b}$$

Problem 3.12. *A square shaped vertical closing for an opening in a water tank is pivotted along the middle, and held in place by a torque. Show that the torque required remains constant irrespective of the height of the fluid above the opening as long as it submerges the opening completely.*

The torque required = Net force × distance from centroid to centre of pressure

$$= \gamma A\,\bar{h}\,(h_{cp} - \bar{h}) = \gamma A\,\bar{h}\,(I_G/A\,\bar{h}) = \gamma I_G$$

As I_G depends only on the plate dimension, the torque is independent of the height of fluid subject to the condition that it submerges the plate fully. As force increases the distance of centre of pressure from *CG* decreases and hence this result.

Problem 3.13. *A square gate of side, a closing an opening is to be hinged along a horizontal axis so that the gate will open automatically when the water level reaches a certain height above the centroid of the gate. Determine the distance of this axis from the centroid if the height h is specified.*

The gate will begin to open if the hinge is at the level of the centre of pressure and the water level just begins to rises, in which case the centre of pressure will move upwards causing the opening.

$(h_{cp} - \bar{h})$ gives the distance of the hinge from the centroid

$$(h_{cp} - \bar{h}) = (I_G/A\bar{h}),\ \bar{h} = h/2,\ I_G = a^4/12 \text{ (square plate of side } a)$$

$$\therefore \quad (h_{cp} - \bar{h}) = (a^4\ 2/12\ a^2\ h) = a^2/6h$$

The location of the hinge should be at ($a^2/6h$) below the centroid.

If for example h = 6 m and a = 2 m, then

$$(h_{cp} - \bar{h}) = 4/(6 \times 6) = 1/9\text{m, below the centroidal axis.}$$

Problem 3.14. *A gate of rectangular shape hinged at A divides the upstream and downstream sides of a canal 5 m wide as shown in figure. Determine the angle θ in terms of the head h_1 and h_2 for equilibrium. Neglect the weight of the gate. The angle between the plates is 90°.*

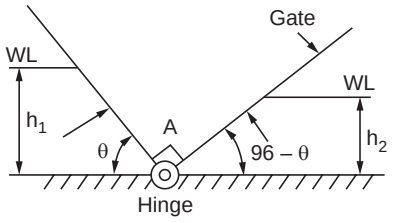


Figure P. 3.14

On the left side, the force is given by (per unit width) $\gamma\,(h_1/2)\,(h_1/\sin\theta)$ and it acts at $(1/3)\,(h_1/\sin\theta)$, from hinge. On the right side (as $\sin(90° - \theta) = \cos\theta$) the force is $\gamma(h_2/2)\,(h_2/\cos\theta)$ and acts at $(1/3)\,(h_2/\cos\theta)$, from hinge in both cases perpendicular to the plate.

Taking moments about the hinge,

$$\gamma\,(h_1/2)\,(h_1/\sin\theta)\,(h_1/2)\,(h_1/3\sin\theta) = \gamma\,(h_2/2)\,(h_2/\cos\theta)\,(h_2/3\cos\theta)$$

$$\therefore \quad \mathbf{(h_1/h_2)^3 = \tan^2\theta};\ \text{If } h_1 = h_2,\ \theta = 45° \text{ as } \tan^2\theta = 1\ \tan\theta = 1$$

This checks the expression. If h_1 = 2 m and h_2 = 3 m, then

$$(2/3)^3 = \tan^2\theta,\ \theta = 28.56° \text{ or the gate is tilted towards left.}$$

Problem 3.15. *Determine the total force and location of the centre of pressure on a rectangular plate 11 m long and 6 m wide with a triangular opening immersed in water at an angle of 40° to the horizontal as shown in figure. The top edge of the inclined plate is 6 m from the free surface.*

(*i*) Considering the whole plate without the opening

$$\bar{h} = 6 + (11/2)\sin 40 = 9.54 \text{ m}$$

$$\text{Total force} = \gamma\,\bar{h}\,A = 9810 \times 9.54 \times 11 \times 6/1000 = 6176.77 \text{ kN}$$

Depth of centre of pressure = $\bar{h} + (I_G \sin^2 \theta / A\ \bar{h})$

$= 9.54 + [6 \times 11^3/12]\ [\sin^2 40/(11 \times 6)\ 9.54] = 9.977$ m

(*ii*) Considering the triangular hole portion only

$$\bar{h} = 6 + (4 + 5/3) \sin 40 = 9.6425 \text{ m}$$

Total force = $\gamma\ \bar{h}\ A = 9810 \times 9.6425 \times (4 \times 5/2)/1000 = 945.33$ kN

Depth of centre of pressure

$= 9.6425 + [4 \times 5^3/36]\ [\sin^2 40/9.6425\ (4 \times 5/2)] = 9.702$ m

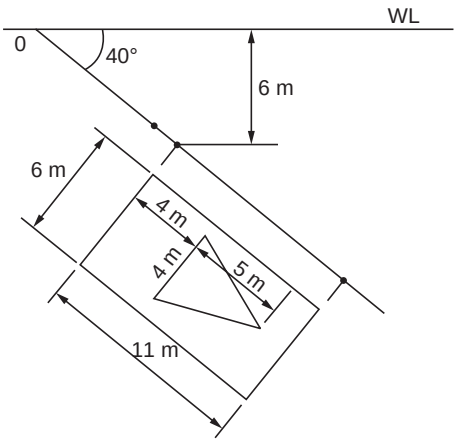


Figure P. 3.15

To determine the line of action of the resultant, moment is taken about *O*, at the surface

$[9.977 \times 6176770/\sin 40] - [9.702 \times 945330/\sin 40]$

$= [h/\sin 40]\ [6176770 - 945330]$

where h is the centre of pressure of the composite area.

Solving, depth of centre of pressure of the composite area of the gate = **10.027 m**

Calculate the depth of the *CG* of the area and check whether it is lower than 10.027 m.

CG is at 9.52 m depth. Net force on the composite area of the gate

$= (6176.77 - 945.33) = 5231.44$ kN

Problem 3.16. *Determine the resultant force and the direction of its action on the segmental gate shown in Fig. P. 3.16.*

Height, $h = 4 \times 2 \sin 30 = 4$ m. Considering 1 m width, the horizontal force = $9810 \times 2 \times 4 = 78480$ N. The line of action (centre of pressure) is 1.333 m from the bottom.

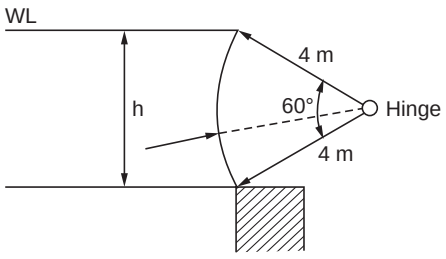


Figure P. 3.16

Vertical force (upwards) equals the weight of displaced. Volume of segment of circle

$$= [(\pi R^2 \times 60/360) - (2R \sin 30 \times R \cos 30/2)] \times 1 = 1.4494\ m^3$$

Weight = 1.449 × 9810 = **14218 N** upwards direction.

$\tan \theta = 78480/14218$, $\theta = 79.73°$ from vertical

The net force is $(78480^2 + 14218^2)^{0.5}$ = **79757.5 N**

The centre of gravity of a segment of a circle from centre is given by

$(2/3)\ R \sin^3 \theta/(\text{Rad}\ \theta - \sin\theta \cos\theta)$ where θ is half of the segment angle.

Substituting the values

$$h_{CG} = (2/3) \times 4 \times \sin^3 30/[(\pi/6) - \sin 30 \cos 30] = 3.67974 \text{ m}$$

taking moments about the centre,

$$(14218 \times 3.67974) - (2 - 1.33) \times 78477 = 0.$$

The quantities are equal. So the resultant passes through the centre and as the resulting moment about the centre is zero. **The line of action will pass through *O* and its direction will be (90 – 79.73)° with horizontal, as shown in figure.**

Problem 3.17. *A roller gate as shown in Fig. P. 3.17 has a span of 5 m. Determine the magnitude and direction of the resultant force on the cylinder when water just begins to overflow. Neglect weight of the gate.*

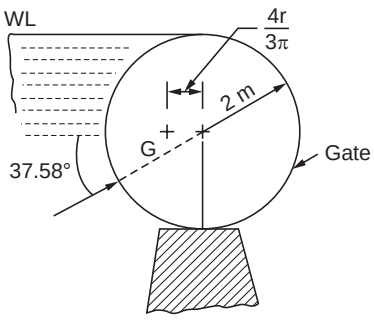


Figure P. 3.17

The horizontal force equals the force on the projected area,

$$\gamma A \bar{h} = 9810 \times 4 \times 5 \times 2 = 392400 \text{ N}$$

This acts at a distance of 1.3333 m from bottom twoards the right.

The vertical upward force is equal to the weight of the water displaced

$$= 5 \times 9810 \times \pi \times 2^2/2 = 308190 \text{ N}$$

It acts at a distance of $(4r/3\pi) = 0.84883$ m left of centre and upwards

$$\text{Resultant} = (392400^2 + 308190^2)^{0.5} = 498958 \text{ N}$$

Taking moments about the centre,

$$392400 \times (2 - 1.3333) - 0.84883 \times 308189 = 0.$$

As the net moment about the centre is zero the resultant passes through the centre. The direction with horizontal is given by tan θ = 308190 / 392400, θ = 37.58°.

Problem 3.18. *Determine the magnitude and direction of the force on the elliptical tank portion AB as shown in Fig P. 3.18.*

The horizontal force on the elliptical portion equals the force on the projected area, considering 1 m width, $\bar{h}$ = 5m, $A = 4 \times 1$

$$\gamma A \bar{h} = 9810 \times 4 \times 5 = 196200 \text{ N acts at 1.333 m from bottom}$$

Vertical force on the elliptical portion equals the weight of water above this area. Area of ellipse = $\pi bh/4$ where b and h are minor and major axis. $b = 6$, $h = 8$. $A = 12\pi$. The area here is 1/4 th of the ellipse. Hence, $A = 3\pi\ m^2$ (ellipse portion)

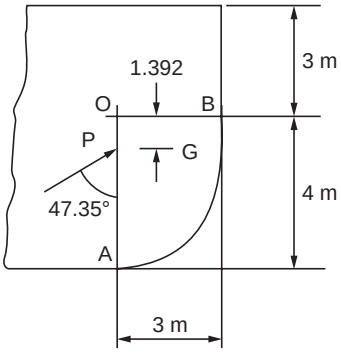


Figure P. 3.18

Rectangular portion above

$= 3 \times 3 = 9\ m^2$. Volume $= 1 \times (3\pi + 9)\ m^3$

Weight $= 9810\ (3\pi + 9)$

$= 180747$ N = Total vertical force

The centre of gravity of the quarter of elliptical portion

$= (4b/3\pi)$,

$= 4 \times 3/3\pi = 4/\pi\ m$ from the major axis (as $b = 3$ m)

Centre of gravity of the rectangle = 1.5 m from the wall

Taking moments and solving, the location x of vertical force

$= [9 \times 1.5 + 3\pi\ (3 - 4/\pi)] / [9 + 3\pi] =$ **1.616 m** from wall

Resultant force $= (196200^2 + 180747^2)^{0.5} = 266766$ N

To determine the line of action, let this line cut OA at a distance of h below O at P.

Then, taking moments about P, $(2.666 - h)\ 192600 = (3 - 1.616)\ 180747$

$\therefore$ **h = 1.392 m, the line of action passes through a point P, 1.392 m below O at an angle of 47.35° from vertical.**

Problem 3.19. *Determine the resultant force on the wall of a tank ABC as shown in Fig. P 3.19.*

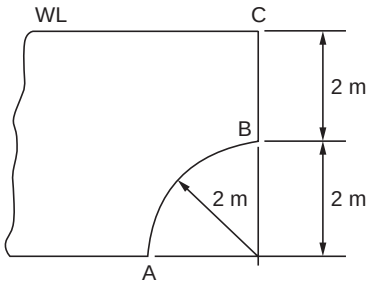


Figure P. 3.19

Considering unit width,

Horizontal force equals the force on the projected area = $\gamma A\ \bar{h}$

$= 9810 \times 4 \times 2 =$ **78480 N**

This force acts at 2.6667 m from the top

The vertical force equals the weight of the volume above the surface (unit width)

Vertical force $= 9810 \times (4 \times 2 - \pi\ 2^2/4) \times 1 =$ **47661 N** (downward)

Resultant $= (78480^2 + 47661^2)^{0.5} =$ **91818 N**

Angle with vertical θ : $\tan^{-1}\ (78480/47661) =$ **58.73°**

To fix the line of action, the line of action of the vertical force should be determined.

Problem 3.20. *The shape described as $x = 0.2y^2$ forms the wall of a gate. Derive expressions for the horizontal force, vertical force and the moment on the gate with respect to O, as shown in Fig. P. 3.20. Calculate the values for y = 3m.*

Consider unit width,

Horizontal force $= \gamma A\ \bar{h} = \gamma \times y \times (y/2) = \gamma y^2/2$

This force acts at $y/3$ from the bottom. Vertical force: weight of volume above the surface. Assuming unit width, the volume = area × width

$$A = \int_0^y x.dy = 0.2 \int_0^y y^2\, dy$$

$$= 0.2\, y^3/3 = xy/3$$

Vertical force = $(xy/3) \times \gamma \times$ width

$$= (0.2\, y^3/3) \times \gamma \times \text{width}$$

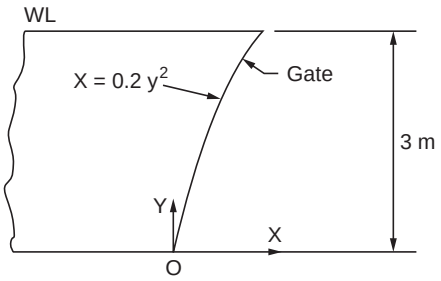


Figure P. 3.20

The position of line of action can be determined taking moment about the y axis. Let it be $\bar{x}$ from y axis.

$$\bar{x} \,.\, xy/3 = \int_0^y x \,.\, dy \,.\, x/2 = \frac{0.04}{2} \int_0^y y^4\, dy = \frac{0.04\, y^5}{10} = \frac{x^2\, y}{10},$$

$\therefore$ $\bar{x} = (3x/10) = 0.06\, y^2$

Clockwise moment about

$$O = (\gamma\, y^2/2) \times (y/3) + (0.2\, y^3/3)\, \gamma \times 0.06\, y^2$$

For y = 3 m and unit width

Horizontal force = $9810\ (3 \times 3/2) = 44145$ N

Vertical force = $0.2\, y^3\, \gamma/3 = 0.2 \times 27 \times 9810/3 = 17658$ N

Clockwise moment = $(\gamma \times 27/6)\ (1 + 0.024 \times 9) = 53680$ N m

The direction of the force with the vertical can be found using

$$\tan^{-1}\ (44145/17658) = 68.2°$$

Resultant = $(44145^2 + 17658^2)^{0.5}$ = **47546 N**

To locate the actual line of action of the force, perpendicular distance from O × force = moment

$\therefore$ Distance = 53680/47456 = 1.129 m

It cuts the vertical from O at 1.129/sin 68.2 = **1.216 m**

Problem 3.21. *A hemispherical bulge of 3m diameter inwards is as shown in Fig. P. 3.21. Determine for the given dimensions, the magnitude and direction of the resultant force on the wall of the bulge (i) when water is full (ii) water level comes to the top of the bulge and (iii) water level upto the centre of the bulge.*

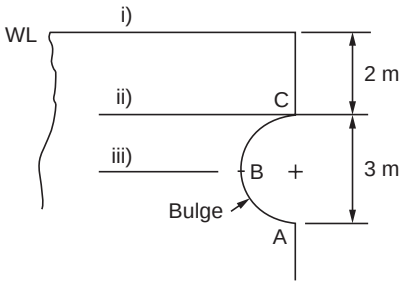


Figure P. 3.21

Force on the surface ABC is required. The projected area = $\pi\, D^2/4$

(*i*) Horizontal force = $9810 \times 3.5 \times \pi \times 3^2/4 = 242700$ N

Depth of centre of pressure

$$= 3.5 + (\pi/64)\ (3^4 \times 4/3.5 \times \pi \times 3^2) = 3.66071 \text{ m}$$

Vertical force = weight of the liquid displaced

$\gamma\,(1/2)\,(\,4\,\pi\,R^3/3) = 69343$ N and acts upwards

For the hemisphere, centre of action from surface = $3\,R/8 = 0.5625$ m from wall.

Note : The vertical force on the surface *AB* is due to the liquid column above it and acts upwards. The vertical force on the surface *BC* is due to the liquid column above it and acts downwards. **So the net force is due to the weight of the volume of liquid displaced and acts upwards.**

The resultant is given by $[242700^2 + 69343^2]^{0.5} = 252412$ N

The direction is given by (angle with vertical) θ,

$$\theta = \tan^{-1}(24700/69343) = 74.05°.$$

The angle with the horizontal will be 15.95°

Check whether the resultant passes through the centre by taking moment. It does.

$\{0.5625 \times 69343 - (0.16071 \times 242700)\} \cong 0$

(*ii*) When water level comes up to the edge, horizontal force

$$= \gamma\,h\,A = 9810 \times 1.5 \times (\pi \times 3^2/4) = 104014 \text{ N}$$

Horizontal force acts at $1.5 + (\pi\,3^4/64)\,(1/1.5)\,(4/\pi \times 3^2) = 1.875$ m

The vertical force remains the same.

Resultant = $[104014^2 + 69343^2]^{0.5} = 125009$ N

Does the resultant pass through the centre? Check.

Line of action, angle with vertical = $\tan^{-1}(104014/69343) = 56.31°$

(*iii*) When water comes to the centre, horizontal force

$$= 9810 \times 0.75 \times (\pi \times 3^2/8) = 26004 \text{ N}$$

$$I_G = I_b - A\,(\bar{h}^2),\ \bar{h} = 2D/3\,\pi,\ I_b = \pi\,D^4/128$$

$$\therefore \qquad I_G = 0.55565$$

Centre of pressure = $0.75 + [0.55565/(0.6366 \times \pi \times 1.5^2/2)]$

$$= \mathbf{0.9973\ m,\ down\ from\ B.}$$

Vertical force = $\gamma\,(1/4)\,(4\,\pi\,R^3/3) = 34672$ N (upwards)

Resultant = 43340N, $\theta = 36.87°$ with vertical.

Problem 3.22. *An oil tank of elliptical section of major axis 3 m and minor axis 2 m is completely filled with oil of specific gravity 0.9. The tank is 6 m long and has flat vertical ends. Determine the forces and their direction of action on the two sides and the ends.*

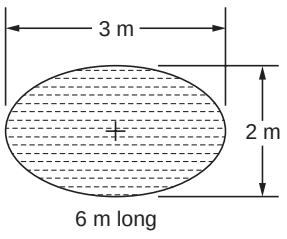


Figure P. 3.22

Considering the surface of the left half of the tank, horizontal force = $\gamma\,\bar{h}\,A$

$$= 9810 \times 0.9 \times 1 \times 2 \times 6 = 105948 \text{ N to the left}$$

Line of action = $1 + (6 \times 2^3/12)\,(1/1)\,(1/6 \times 2) = 1.333$ m from top. Similar force acts on the right half of the tank to the right, at the same level.

Vertical force on the left half = Weight of displaced liquid

$= 9810 \times 0.9 \times (\pi \times 3 \times 2/4 \times 2) = 20803$ N

downward and the location is $4h/3\pi = 4 \times 1.5/3\pi = 0.63662$ m, from centre line

Resultant = $(105948^2 + 20803^2)^{0.5}$ = **107971 N**

Direction (with vertical) = $\tan^{-1}(105948/20803)$ = **78.89°**. Similar force acts on the other half.

Ends: Elliptical surfaces : $F = \gamma \bar{h} A = 9810 \times 0.9 \times 1 \times \pi \times 3 \times 2/4$ = **41606 N**

Line of action $= 1 + (\pi \times 3 \times 2^3/64)(1/1)(4/\pi \times 3 \times 2)$

= **1.25 m from top.**

Problem 3.23. *A square section tank of 3 m side and 2 m length as shown in Fig. P. 3.23 has the top of one side wall in the shape of a cylinder as indicated. The tank is filled with water as indicated. Determine the horizontal and vertical forces on the curved surface. Also locate the line of action of the resultant force. The water is under a gauge pressure of 20,000 N/m^2.*

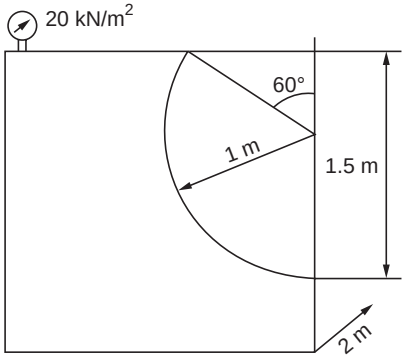


Figure P. 3.23

The horizontal force is the force due to water pressure on the projected area. It can be split up into two components (*i*) due to the water column and (*ii*) due to the pressure on the fluid

The horizontal force $= \gamma \bar{h} A + P A$

$= 9810 \times 0.75 \times (1.5 \times 2) + 20{,}000 \times (1.5 \times 2) = 22072.5 + 60{,}000$

$= 82072.5$ N (to the right)

The first component acts at the centre of pressure and the second at the centre of gravity.

Centre of pressure due to fluid pressure

$= 0.75 + (1/12)(2 \times 1.5^3 / 0.75 \times 2 \times 1.5) = 1.0$ m (from top).

Location of the net force is determined by taking moments about the top.

$= \{(22072.5 \times 1) + (60000 \times 0.75)\} / (22072.5 + 60{,}000)$

$= 0.8172$ m from top.

The vertical force can also be considered as the result of two action (*i*) the weight of displaced volume and (*ii*) the pressure on the projected area.

$= 9810 [(120 \times \pi \times 1^2 \times 2/360) + (\cos 60 \times \sin 60 \times 2/2)]$

$+ \sin 60 \times 2 \times 20000$

$= 24794 + 34641 = 59435$ N (upwards)

The resultant = $(82072.5^2 + 59435^2)^{0.5} = 101333$ N

The resultant acts at an angle (with vertical), $\tan^{-1}(82072.5 / 59434.9) = 54.09°$

Note : The problem can also be solved by considering an additional head of fluid equal to the gauge pressure.

Problem 3.24. *Determine the vertical and horizontal forces on the cylinder shown in Fig. P. 3.24. The cylinder is in equilibrium.*

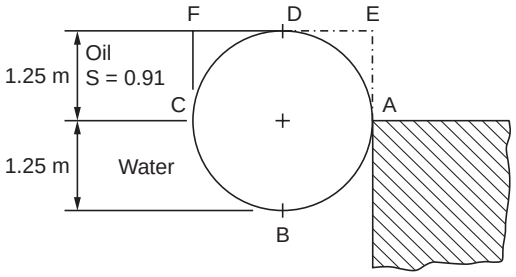


Figure P. 3.24

The horizontal force can be calculated as the sum of forces due to the oil and due to the water on projected area. Consider 1 m length. The horizontal force on *AB* and *BC* are equal and opposite.

The other horizontal force due to oil on *CD* is

$$= 9810 \times 0.91 \times (1.25/2) \times 1.25 \times 1$$

$$= \mathbf{6974.3\ N} \text{ and this force acts at}$$

$[(1.25/2) + 1 \times 1 \times 1.25^3 \times 2/12 \times 1.25 \times 1.25 \times 1) = 0.833$ m from the top surface

The vertical upward force equals the weight of water displaced + weight of oil displaced (*AEDG* + *CDG*)

$$= [9810 \times (\pi \times 1.25^2/2)] + [(1.25 \times 1.25) + \pi \times 1.25^2/4) \times 0.91 \times 9810]$$

$$= (24077 + 24904)\ \text{N}$$

Total upward force = **48981 N**

Workout the resultant as an exercise.

Problem 3.25. *A bridge is in the form of an elliptical arch and wter flows just touching the bottom. The major axis is 8 m and the minor axis is 3 m. Determine the upward force due to water pressure. The bridge is 4m wide.*

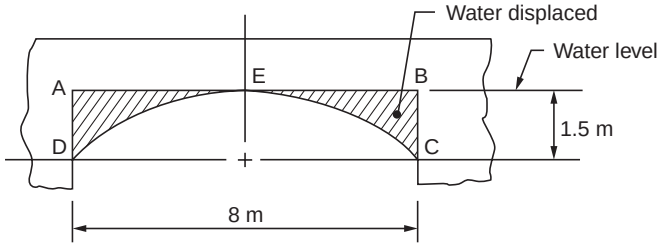


Figure P. 3.25

The upward force is due to the weight of water displaced as shown in figure.

Area of half ellipse = $\pi bh/8 = \pi \times 8 \times 3/8 = 3\pi \text{m}^2$

Area of rectangle = $8 \times 1.5 = 12\ \text{m}^2$

Weight of water displaced = $(12 - 3\pi) \times 4 \times 9810 = 101052$ N

Problem 3.26. *A channel is closed by two swinging lock gates each of 4 m wide and 6 m height and when closed the angle between them is 120°. On the upstream side the water level is 5.5 m and in the downstream it is 2m. Determine (i) normal force on each gate and (ii) the reaction between the gates. If the gates are hinged at 0.5 m and 5.5 m from the base, determine the reaction at each hinge.*

The normal force on each gate on the upstream side

$$= 9810 \times (5.5/2) \times 5.5 \times 4 = 593505\ N$$

This force acts at 5.5 / 3 = 1.8333 m from the bottom.

The normal force on the downstream side on each gate

$$= 9810 \times (2/2) \times 2 \times 4 = 78480 \text{ N}.$$

This force acts at 2/3 = 0.667 m from bottom.

Net normal force = 593505 – 78480 = 515025 N

To determine the reaction R considering the equilibrium and taking moments about the point A.

$515025 \times 2 = R \times 4 \sin 30$

R = 515025 N. This acts perpendicular to the contact as shown. The total reaction at the hinges should also equal this value. To determine the reaction at each hinge, moments can be taken with reference to the other hinge.

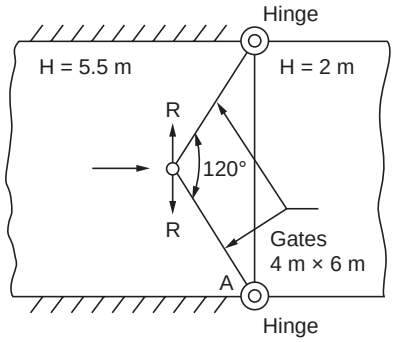


Figure P. 3.26

Taking moments from the top hinge at 5.5 m from base, (as half the force only is causing the reaction at the hinge, and as the reaction is at 30° to the plane of gate)

$(593505/2)\,(5.5 - 1.83333) - (78480/2)\,(5.5 - 0.6667)$

$$= R_1\,(5.5 - 0.5) \sin 30$$

R_1 = 359373 N and by similar calculation

R_2 = 155652 N, (check total as 515025)

Problem 3.27. *A dam section is 6 m wide and 20 m high. The average specific gravity of the material is 2.8. Determine the height of water which may just cause overturning of the dam wall.*

When the moment with downstream corner A of the structures weight equals the moment of the pressure force on the structure, the structure will tilt. *i.e*. Weight $\times AB$ = Pressure force $\times AC$

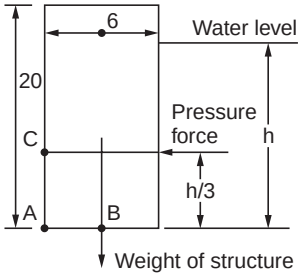


Figure P. 3.27

Considering 1m length, (CP is $h/3$ from bottom), taking moments

$$20 \times 6 \times 2.8 \times 9810 \times 3 = 9810 \times (h/2) \times h \times (h/3)$$

Solving, **h = 18.22 m.**

Problem 3.28. *Determine the resultant vertical force on the curved structure AB and also the line of action. At any section the height, x = 0.5 Z^2 where z is the width. Consider a width of 1m. Water stands upto x = 2m. (note the constant 0.5 should be dimensional, 1/m).*

The area and centre of gravity are to be determined by integration. Considering a strip at x and width dx,

$$A = \int_0^x Z\,dx \text{ where } Z = (x/0.5)^{0.5}$$

$$\therefore \qquad A = 0.9428\,x^{1.5}$$

To obtain the line of centre of gravity from $z = 0$ line

$$A\,\bar{Z} = \int_0^x Z\,dx\; Z/2 = \int_0^x x\,dx = x^2/2$$

$$\therefore \qquad \bar{Z} = (x^2/2)\,(1/0.9428\,x^{1.5}) = x^{0.5}/1.8856$$

The vertical force due to gas pressure = vertical projected area × pressure

Top width, $Z = (3/0.5)^{0.5} = 2.45$ m. Considering 1 m width.

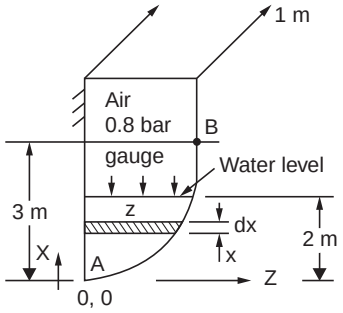


Figure P. 3.28

$$P = 0.8 \text{ bar} = 0.8 \times 10^5\ N/m^2$$

Pressure force = $2.45 \times 1 \times 0.8 \times 10^5$ N = 195959 N. This acts along 1.225 m from $Z = 0$ position. The vertical force due to the weight of water which stands upto $x = 2$ m over the surface.

Weight = volume × sp. weight = area × depth × sp. weight

$A = 0.9428\,x^{1.5}$, $x = 2$, $A = 2.6666$ m^2

Force due to water = $2.6666 \times 9810 = 26160$ N. This force acts at

$\bar{Z} = x^{0.5}/1.8856 = 0.75$ m from the vertical at $Z = 0$

Total force = 195959 + 26160 = 222119 N, To determine the line of action:

$195959 \times 1.225 + 26160 \times 0.75 = Z\,(222119)$,

$z = 1.169$ m.

REVIEW QUESTIONS

1. Explain the importance of the study of fluid forces on surfaces and submerged bodies.
2. Explain the concept of centroid of an area or centre of gravity. What will be the value of first moment of area about the centroid.
3. Explain the concept of Moment of Inertia of a surface and the application of the same in the study of forces due to fluid pressure on surfaces.
4. Derive an expression for the force on a thin plate of given arbitrary shape immersed in a liquid at an angle θ to the free surface.
5. Obtain simplified expressions for the centre of pressure of vertical planes. (*i*) plate (*ii*) circle (*iii*) triangle.
6. Show that in the case of a rectangle inclined to the horizontal, immersed in a fluid with its centroid at a depth, h (*i*) the horizontal component of the resultant force equals the force on the vertical projection of the area and (*ii*) the vertical component equals the weight of the fluid column above this area.
7. Explain how force on curved surfaces due to fluid pressure is determined.

OBJECTIVE QUESTIONS

O Q.3.1. Fill in the blanks

1. The point of action of resultant fluid forces is called _____.
2. The moment of area about any axis through the centre of gravity will be _____.
3. The second moment of area about an axis through the centre of gravity will _____ compared to any other axis.
4. The centre of pressure will generally be _____ the centroid.
5. The hydrostatic force on a submerged plane surface depends on the _____ of the centroid.
6. The force due to liquid pressure acts _____ to the surface.
7. The law for calculating hydrostatic pressure is _____.
8. The second moment about any axis differs from the second moment through a parallel axis through the centroid by _____.
9. The distance of centre of pressure from its centroid for a vertical area immersed in liquid is given by _____.
10. The vertical distance between the centriod and centre of pressure over a plane area immersed at an angle θ to the free surface is given by _____.
11. The pressure at the same horizontal level in a static liquid is _____.

Answers

(1) Centre of pressure, (2) zero, (3) lower, (4) below, (5) depth, (6) normal, (7) $(dp/dy) = -\gamma$, (8) $A\,x^2$, x-distance between the axes, 9. $I_G / A\,\bar{h}$, 10. $I_G \sin^2\theta / A\,\bar{h}$ 11. the centre.

O Q.3.2 Fill in the blanks:

1. The horizontal force on a curved surface immersed in a liquid is equal to the force on _____.
2. The vertical force on a curved surface equals the _____.
3. The line of action of horizontal force on a curved surface immersed in a liquid is_____.
4. The line of action of vertical force on a curved surface immersed in a liquid is _____.
5. The force due to gas pressure on curved surface in any direction _____.
6. The resultant force on cylindrical or spherical surfaces immersed in a fluid passes through _____.

Answers

(1) the vertical projected area, (2) the weight of column of liquid above the surface, (3) the centre of pressure of the vertical projected area, (4) the centriod of the liquid column above the surface, (5) equals the product of gas pressure and projected area in that direction, (6) the centre.

O Q.3.3 Fill in the blanks using increases, decreases or remains constant :

1. The force due to liquid pressure _____ with depth of immersion.
2. The distance between the centriod and the center of pressure _____ with depth of immersion.
3. When a plane is tilted with respect to any centriodal axis the normal force on the plane due to liquid pressure _____.
4. The location of centre of pressure of a plane immersed in a liquid _____ with change in density of the liquid.

Answers

Increases : 1, Decreases : 2, Remains Constant : 3, 4.

O Q.3.4 Indicate whether the statements are correct or incorrect :

1. The centre of pressure on a plane will be at a lower level with respect to the centroid.
2. In a plane immersed in a liquid the centre of perssure will be above the centroid.
3. The resultant force due to gas pressure will act at the centroid.
4. The vertical force on an immersed curved surface will be equal to the column of liquid above the surface.
5. The normal force on an immersed plane will not change as long as the depth of the centroid is not altered.
6. When a plane is tilted along its centroidal axis so that its angle with horizontal increases, the normal force on the plane will increase.

Answers

(1) Correct : 1, 3, 5 (2) Incorrect : 2, 4, 6.

O Q.3.5 Choose the correct answer :

1. The pressure at a depth 'd' in a liquid, (above the surface pressure) is given by
   (a) ρg (b) γd
   (c) $-\gamma d$ (d) $(\rho/g)d$ (usual notations)
2. The density of a liquid is 1000 kg/m^3. At location where g = 5 m/s^2, the specific weight of the liquid will be
   (a) 200 N/m^3 (b) 5000 N/m^3
   (c) 5000 × 9.81 / 5 N/m^3 (d) 5000 × 5 / 9.81 N/m^3
3. The centre of pressure of a rectangular plane with height of liquid h m from base
   (a) h/2 m from bottom (b) h/3 m from top
   (c) h/3 m from bottom
   (d) can be determined only if liquid specific weight is known.
4. The horizontal force on a curved surface immersed in a liquid equals
   (a) the weight of the column of liquid above the surface
   (b) the pressure at the centroid multiplied by the area
   (c) the force on the vertical projection of the surface
   (d) the pressure multiplied by the average height of the area.
5. The location of the centre of pressure over a surface immersed in a liquid is
   (a) always above the centroid
   (b) will be at the centroid
   (c) will be below the centroid
   (d) for higher densities it will be above the centroid and for lower densities it will be below the centroid.
6. The pressure at a point y m below a surface in a liquid of specific weight γ as compared to the surface pressure, P will be equal to
   (a) $P + (y/\gamma)$ (b) $P + y\gamma$
   (c) $P - (y\,g/\gamma)$ (d) $P + (y\,.\,g/\gamma)$.

7. When the depth of immersion of a plane surface is increased, the centre of pressure will
   (*a*) come closer to the centroid
   (*b*) move farther away from centroid
   (*c*) will be at the same distance from centroid
   (*d*) depend on the specific weight of the liquid.
8. A sphere of R m radius is immersed in a fluid with its centre at a depth h m The vertical force on the sphere will be
   (*a*) $\gamma\,(4/3)\pi\,R^3$ (*b*) $\gamma\pi\,R^2\,h$
   (*c*) $\gamma\,(\pi\,R^2\,h + 8\,\pi\,R^2/3)$ (*d*) $\gamma\,(\pi\,R^2\,h - 8\,\pi\,R^2/3)$.

Answers

(1) *b* (2) *b* (3) *c* (4) *c* (5) *c* (6) *b* (7) *a* (8) *a*.

O Q.3.6 Match the sets A and B :

(I)

	A	B
1.	Specific weight	(*a*) m^3
2.	Density	(*b*) m^4
3.	Second moment of area	(*c*) N/m^2
4.	First moment of area	(*d*) kg/m^3
5.	Pressure	(*e*) N/m^3

Answers

1 – *e*, 2 – *d*, 3 – *b*, 4 – *a*, 5 – *c*.

(II)

	A	B
1.	Centroid	(*a*) always positive
2.	Centre of pressure	(*b*) area moment zero
3.	Free surface	(*c*) resultant force
4.	Second moment of area	(*d*) constant pressure

Answers

1 – *b*, 2 – *c*, 3 – *d*, 4 – *a*.

(III) Moment of inertia of various shapes :

1.	Circle about centroidal axis	(*a*) $B\,h^3 / 36$
2.	Rectangle about centroidal axis	(*b*) $D^4 / 64$
3.	Triangle about centroidal axis	(*c*) $D^4 / 128$
4.	Semicircle about base	(*d*) $B\,h^3 / 12$

Answers

1 – *b*, 2 – *d*, 3 – *a*, 4 – *c*.

EXERCISE PROBLEMS

E.3.1. Determine the centroid of the following shapes shown in Fig E. 3.1 from the given reference lines.

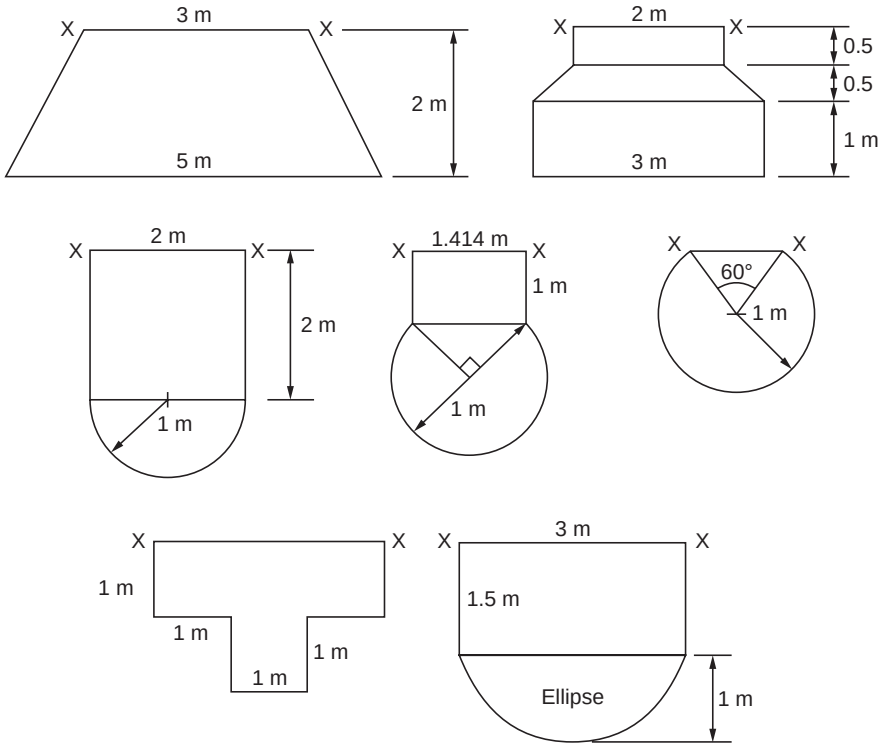


Figure E. 3.1

E.3.2. For the shapes in Fig E. 3.1, determine the moment of inertia of the surfaces about the axis *xx* and also about the centroid.

E.3.3. From basics (by integration) determine the forces acting on one side of a surface kept vertical in water as shown in Fig. E. 3.3.

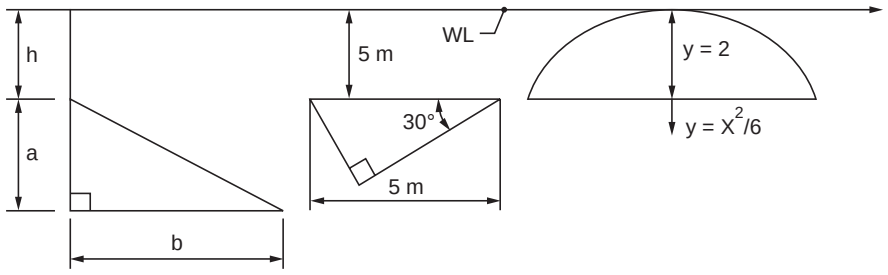


Figure E. 3.3

E.3.4. Determine the magnitude and location of the hydrostatic force on one side of annular surface of 2 m *ID* and 4 m *OD* kept vertical in water.

E.3.5. Determine the moment required to hold a circular gate of 4 m dia, in the vertical wall of a reservoir, if the gate is hinged at (*i*) the mid diameter (*ii*) at the top. The top of the gate is 8 m from the water surface.

E.3.6. Determine the compressive force on each of the two struts supporting the gate, 4 m wide, shown in Fig. E. 3.6.

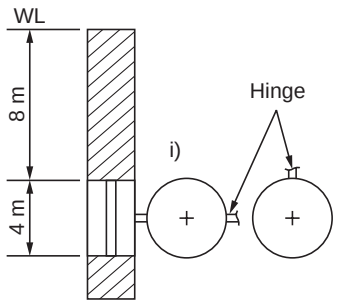


Figure E. 3.5

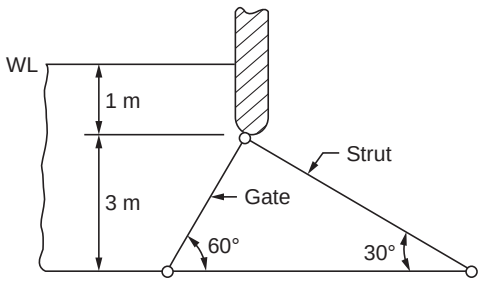


Figure E. 3.6

E.3.7. An annular plate of 4 m *OD* and 2 m *ID* is kept in water at an angle of 30° with the horizontal, the centre being at 4 m depth. Determine the hydrostatic force on one side of the plane. Also locate the centre of pressure.

E.3.8. A tank contains mercury upto a height of 0.3 m over which water stands to a depth of 1 m and oil of specific gravity 0.8 stands to a depth of 0.5 m over water. For a width of 1 m determine the total pressure and also the point of action of the same.

E.3.9. A trapezoidal gate of parallel sides 8 m and 4 m with a width of 3 m is at an angle of 60° to the horizontal as shown in Fig. E. 3.9 with 8 m length on the base level. Determine the net force on the gate due to the water. Also find the height above the base at which the resultant force acts.

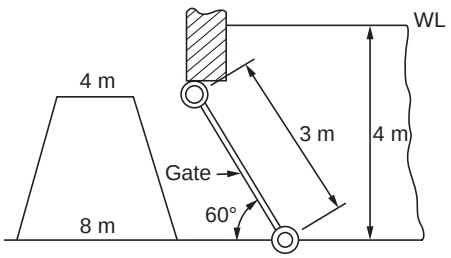


Figure E. 3.9

E.3.10. Show that the resultant force on a submerged plane remains unchanged if the area is rotated about an axis through the centroid.

E.3.11. A gate as shown in Fig. E. 3.11 weighing 9000 N with the centre of gravity 0.5 m to the right of the vertical face holds 3 m of water. What should be the value of counter weight W to hold the gate in the position shown.

E.3.12. A rectangular gate of 2 m height and 1 m width is to be supported on hinges such that it will tilt open when the water level is 5 m above the top. Determine the location of hinge from the base.

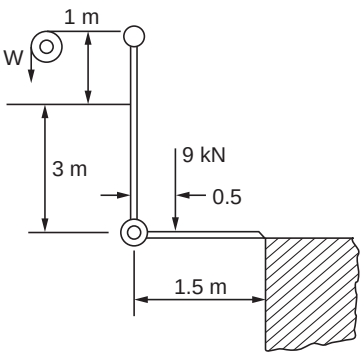


Figure E. 3.11

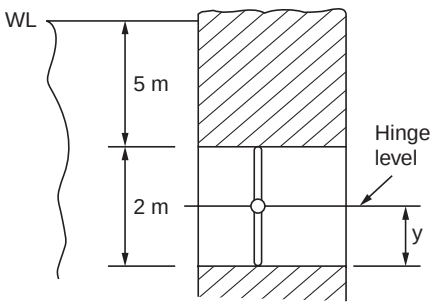


Figure E. 3.12

E.3.13. Show that as the depth of immersion increases, the centre of pressure approaches the centroid.

E.3.14. Determine the magnitude and line of action of the hydrostatic force on the gate shown in Fig. E. 3.14. Also determine the force at the edge required to lift the gate. The mass of the gate is 2500 kg and its section is uniform. The gate is 1 m wide.

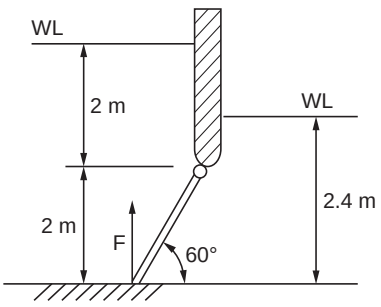


Figure E. 3.14

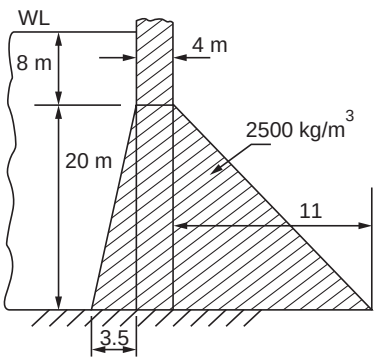


Figure E. 3.15

E.3.15. A dam section is shown in figure. Determine the location where the resultant hydrostatic force crosses the base. Also calculate the maximum and minimum compressive stress on the base.

E.3.16. An automatic flood gate 1.5 m high and 1 m wide is installed in a drainage channel as shown in Fig. E. 3.16. The gate weighs 6 kN. Determine height of water backing up which can lift the gate.

E.3.17. Compressed air is used to keep the gate shown in Fig. E. 3.17 closed. Determine the air pressure required.

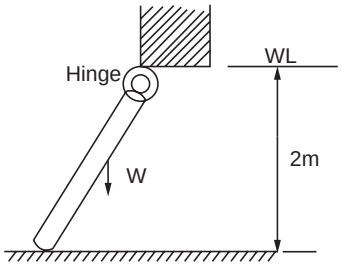


Figure E. 3.16

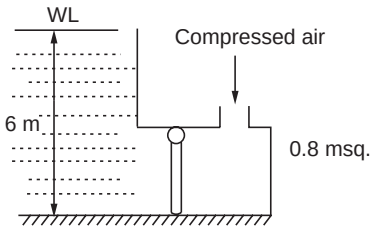


Figure E. 3.17

E.3.18. A spherical container of 6 m diameter is filled with oil of specific gravity 0.73. Determine the resultant force on one half of the sphere divided along the vertical plane. Also determine the direction of action of the force.

E.3.19. An inverted frustum of a cone of base dia 1 m and top dia 6 m and height 5 m is filled with water. Determine the force on one half of the wall. Also determine the line of action.

E.3.20. A conical stopper is used in a tank as shown in Fig. E. 3.20. Determine the force required to open the stopper.

E.3.21. Determine the total weight/m length of a gate made of a cylindrical drum and a plate as shown in Fig. E. 3.21, if it is in equilibrium when water level is at the top of the cylinder.

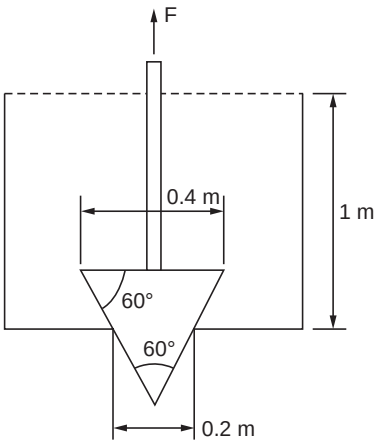


Figure E. 3.20

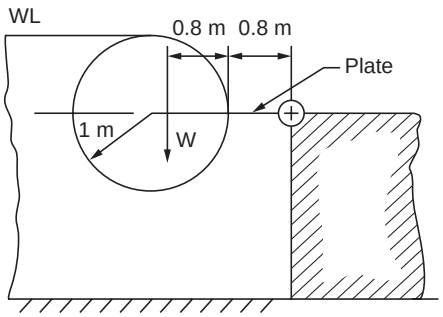


Figure E. 3.21

E.3.22. A gate 12 m long by 3 m wide is vertical and closes an opening in a water tank. The 3 m side is along horizontal. The water level is up to the top of the gate. Locate three horizontal positions so that equal forces acting at these locations will balance the water pressure.

[*3.4641 m, 8.4452 m 10.9362 m*]

4 Buoyancy Forces and Stability of Floating Bodies

4.0 ARCHIMEDES PRINCIPLE

In the previous chapter the forces due to fluid on surfaces was discussed. In this chapter the forces due to fluid on floating and submerged bodies is discussed. It is applicable in the design of boats, ships, balloons and submersibles and also hydrometers. In addition to the discussion **of forces** the **stability of floating bodies** due to small disturbances is also discussed.

If an object is immersed in or floated on the surface of fluid under static conditions a force acts on it due to the fluid pressure. This force is called **buoyant force**. The calculation of this force is based on Archimedes principle.

Archimedes principle can be stated as (*i*) **a body immersed in a fluid is buoyed up by a force equal to the weight of the fluid displaced and (*ii*) a floating body displaces its own weight of the liquid in which it floats.**

Other possible statements are: The resultant pressure force acting on the surface of a volume partially or completely surrounded by one or more fluids under non flow conditions is defined as **buoyant force and acts vertically on the volume.** The buoyant force is equal to the weight of the displaced fluid and **acts upwards through the centre of gravity of the displaced fluid.** This point is **called the centre of buoyancy** for the body.

This principle directly follows from the general hydrostatic equation, $F = \gamma Ah$ and is applied in the design of ships, boats, balloons and other such similar systems. The stability of such bodies against tilting over due to small disturbance can be also checked using this principle.

4.1 BUOYANCY FORCE

Consider the immersed or floating body shown in Fig. 4.1.1. The total force on the body can be calculated by considering the body to consist of a large number of cylindrical or prismatic elements and calculating the sum of forces on the top and bottom area of each element.

(*i*) **Immersed body.** Consider a prismatic element :

Let the sectional area be dA, Force on the top $dF_1 = dA\ \gamma\ h_1$ and

Force on the base $dF_2 = dA\ \gamma\ h_2$ (cancelling P_{atm}, common for both terms)

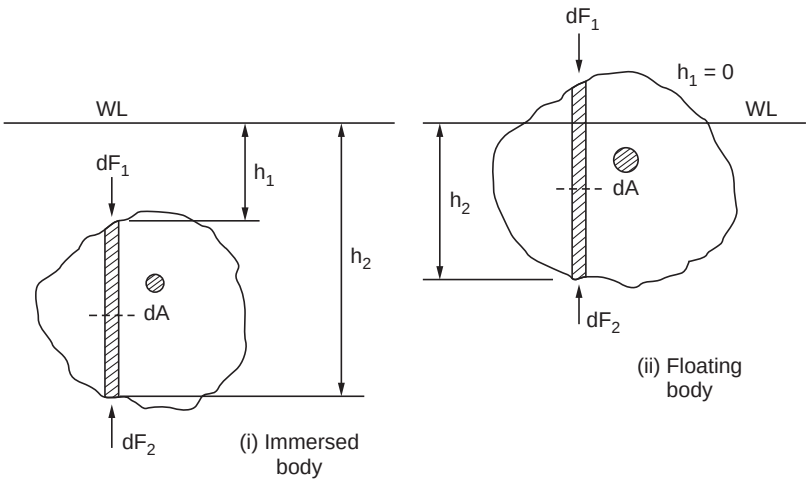


Figure 4.1.1 Proof for Archimedes principle

Net force on the element $(dF_2 - dF_1) = \gamma\, dA\, (h_2 - h_1) = \gamma\, dV$.

where dV is the volume of the element. This force acts upwards. as $h_2 > h_1$

Summing up over the volume, $F = \gamma\, V$ (or) the weight of the volume of liquid displaced.

(*ii*) **Floating body.** Considering an element of volume dV, Force on the top of the element $dF_1 = dA$. P_a and Force on the base of the element $dF_2 = dA\, (\gamma\, h_2 + P_a)$

$$dF_2 - dF_1 = \gamma\, dA\, h_2 = \gamma\, dV$$

where dV is the volume of the fluid element displaced. Summing up over the area,

$F = \gamma\, V$, the weight of volume displaced.

It is seen that the equation holds good in both cases – immersed or floating.

Example. 4.1 *A cylinder of diameter 0.3 m and height 0.6 m stays afloat vertically in water at a depth of 1 m from the free surface to the top surface of the cylinder.*

Determine the buoyant force on the cylinder. *Check the value from basics*

Buoyant force = Weight of water displaced

$= (\pi \times 0.3^2/4)\ 0.6 \times 1000 \times 9.81 = 416.06$ N

This acts upward at the centre of gravity G

Check: Bottom is at 1.6 m depth. Top is at 0.6 m depth

Buoyant force = Force on the bottom face – Force on top face

$= (\pi \times 0.3^2/4)\ (1000 \times 9.81 \times 1.6 - 1000 \times 9.81 \times 1.0)$

= 416.16 N

Example. 4.2 ***Determine the maximum weight that may be supported*** *by a hot air balloon of 10 m diameter at a location where the air temperature is 20° C while the hot air temperature is 80° C. The pressure at the location is 0.8 bar. R = 287 J/kg. K.*

The forces that act on the balloon are its weight downward and the buoyant force upwards. The buoyant force equals the weight of the cold surrounding air displaced. The difference between these two gives the maximum weight that may be carried by the balloon. The volume of cold air displaced equals the volume of the balloon. The pressure is assumed to be the same both inside and outside of the balloon.

Volume of balloon = $(4 \times \pi \times 5^3/3) = 523.6\ m^3$

Mass of hot air = $(PV/RT) = 0.8 \times 10^5 \times 523.6/287 \times (273 + 80)$

= 413.46 kg.

Weight of hot air = m.g = 413.46 × 9.81 = 4056 N

Weight of cold air = $[0.8 \times 10^5 \times 523.6 \times 9.81] /[287 \times (273 + 20)]$

= 4886.6 N

Weight that can be carried by the balloon = 4886.6 – 4056

= 830.6 N (*i.e.*, about 84.67 kg mass under earths gravity)

This should include the weight of the balloon material and fittings.

4.2 STABILITY OF SUBMERGED AND FLOATING BODIES

There are three possible situations for a body when immersed in a fluid.

(*i*) If the **weight of the body is greater** than the weight of the liquid of equal volume then the **body will sink into** the liquid (To keep it floating additional upward force is required).

(*ii*) If the **weight of the body equals** the weight of equal volume of liquid, then the body will submerge and may **stay at any location** below the surface.

(*iii*) If the **weight of the body is less** than the weight of equal volume of liquid, then the body will be partly submerged and **will float in** the liquid.

Comparison of densities cannot be used directly to determine whether the body will float or sink unless the body is solid over the full volume like a lump of iron. However the apparent density calculated by the ratio of weight to total volume can be used to check whether a body will float or sink. If apparent density is higher than that of the liquid, the body will sink. If these are equal, the body will stay afloat at any location. If it is less, the body will float with part above the surface.

A submarine or ship though made of denser material floats because, the weight/volume of the ship will be less than the density of water. In the case of submarine its weight should equal the weight of water displaced for it to lay submerged.

Stability of a body: A ship or a boat should not overturn due to small disturbances but should be stable and return, to its original position. Equilibrium of a body exists when there is no resultant force or moment on the body. A body can stay in three states of equilibrium.

(*i*) **Stable equilibrium: Small disturbances will create a correcting couple** and the body will go back to its original position prior to the disturbance.

(*ii*) **Neutral equilibrium Small disturbances do not create any additional force** and so the body remains in the disturbed position. No further change in position occurs in this case.

(*iii*) **Unstable equilibrium:** A small **disturbance creates a couple which acts to increase the disturbance** and the body may tilt over completely.

Under equilibrium conditions, two forces of equal magnitude acting along the same line of action, but in the opposite directions exist on a floating/submerged body. These are the gravitational force on the body (weight) acting downward along the centroid of the body and buoyant force acting upward along the centroid of the displaced liquid. Whether floating or submerged, under equilibrium conditions these two forces are equal and opposite and act along the same line.

When the position of the body is disturbed or rocked by external forces (like wind on a ship), the position of the centre of gravity of the body **(with respect to the body)** remains at the same position. But the shape of the displaced volume of liquid changes and so its centre of gravity shifts to a new location. Now these two forces constitute a couple which may correct the original tilt or add to the original tilt. If the couple opposes the movement, then the body will regain or go back to the original position. If the couple acts to increase the tilt then the body becomes unstable. These conditions are illustrated in Fig 4.2.

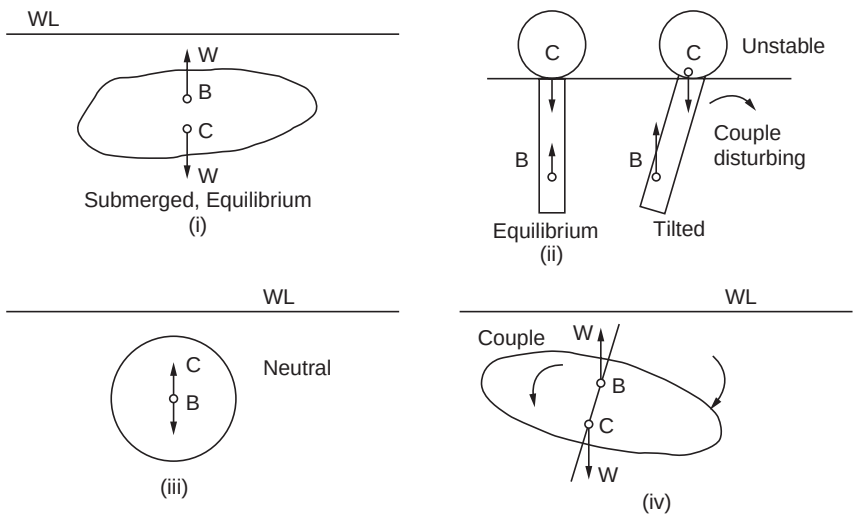


Figure 4.2.1 Stability of floating and submerged bodies

Figure 4.2.1 (*i*) and 4.2.1 (*ii*) shows bodies under equilibrium condition. Point C is the centre of gravity. Point B is the centre of buoyancy. It can be seen that the gravity and buoyant forces are equal and act along the same line but in the opposite directions.

Figure 4.2 (*iii*) shows the body under neutral equilibrium. The centre of gravity and the centre of buoyancy conicide.

Figures 4.2.1 (*iv*) and 4.2.1 (*v*) shows the objects in Figures 4.2.1 (*i*) and 4.2.1 (*ii*) in a slightly disturbed condition. Under such a condition a couple is found to form by the two forces, because the point of application of these forces are moved to new positions. In the case of Figure 4.2.1 (*iv*) the couple formed is opposed to the direction of disturbance and tends to return the body to the original position. This body is in a state of stable equilibrium. The couple is called righting couple. In the case of Figure 4.2.1 (*ii*) the couple formed is in the same direction as the disturbance and hence tends to increase the disturbance. This body is in unstable equilibrium. In the case of figure 4.2.1 (*iii*) no couple is formed due to disturbance as both forces act at the same point. Hence the body will remain in the disturbed position.

In the case of top heavy body (Figure 4.2 (*ii*)) the couple created by a small disturbance tends to further increase the tilt and so the body is unstable.

It is essential that the stability of ships and boats are well established. The equations and calculations are more involved for the actual shapes. Equations will be derived for simple shapes and for small disturbances. (**Note:** For practical cases, the calculations will be elaborate and cannot be attempted at this level.)

4.3 CONDITIONS FOR THE STABILITY OF FLOATING BODIES

(*i*) **When the centre of buoyancy is above the centre of gravity of the floating body, the body is always stable** under all conditions of disturbance. A righting couple is always created to bring the body back to the stable condition.

(*ii*) **When the centre of buoyancy coincides with the centre of gravity,** the two forces act at the same point. A disturbance does not create any couple and so **the body just remains in the disturbed position.** There is no tendency to tilt further or to correct the tilt.

(*iii*) **When the centre of buoyancy is below the centre of gravity** as in the case of ships, **additional analysis is required to establish stable conditions** of floating.

This involves the **concept of metacentre and metacentric height.** When the body is disturbed the centre of gravity still remains on the centroidal line of the body. The shape of the displaced volume changes and the centre of buoyancy moves from its previous position.

The location *M* at which the line of action of buoyant force meets the centroidal axis of the body, when disturbed, is defined as **metacentre.** The distance of this point from the centroid of the body is called metacentric height. This is illustrated in Figure 4.3.1.

If the metacentre is above the centroid of the body, the floating body will be stable. If it is at the centroid, the floating body will be in neutral equilibrium. If it is below the centroid, the floating body will be unstable.

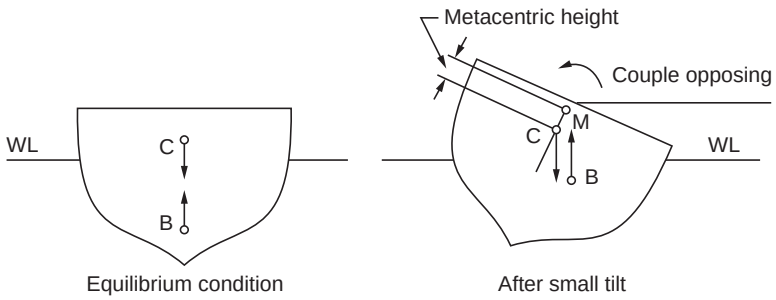


Figure 4.3.1 Metacentric height, stable condition

When a small disturbance occurs, say clockwise, then the centre of gravity moves to the right of the original centre line. The shape of the liquid displaced also changes and the centre of buoyancy also generally moves to the right. If the distance moved by the centre of buoyancy is larger than the distance moved by the centre of gravity, the resulting couple will act anticlockwise, correcting the disturbance. If the distance moved by the centre of gravity is larger, the couple will be clockwise and it will tend to increase the disturbance or tilting.

The distance between the metacentre and the centre of gravity is known is **metacentric height**. The magnitude of the righting couple is directly proportional to the metacentric height. Larger the metacentric height, better will be the stability. Referring to Fig 4.4.1, the centre of gravity *G* is above the centre of buoyancy *B*. After a small clockwise tilt, the centre of buoyancy has moved to *B′*. The line of action of this force is upward and it meets the body centre line at the metacentre *M* which is above *G*. In this case metacentric height is positive and the body is stable. It may also be noted that the couple is anticlockwise. If *M* falls below *G*, then the couple will be clockwise and the body will be unstable.

4.4 METACENTRIC HEIGHT

A floating object is shown in Figure 4.4.1 in section and plan view (part). In the tilted position, the submerged section is *FGHE*. Originally the submerged portion is *AFGHD*. Uniform section is assumed at the water line, as the angle of tilt is small. The original centre of buoyancy *B* was along the centre line. The new location *B′* can be determined by a moment balance. Let it move through a distance *R*. Let the weight of the wedge portion be *P*.

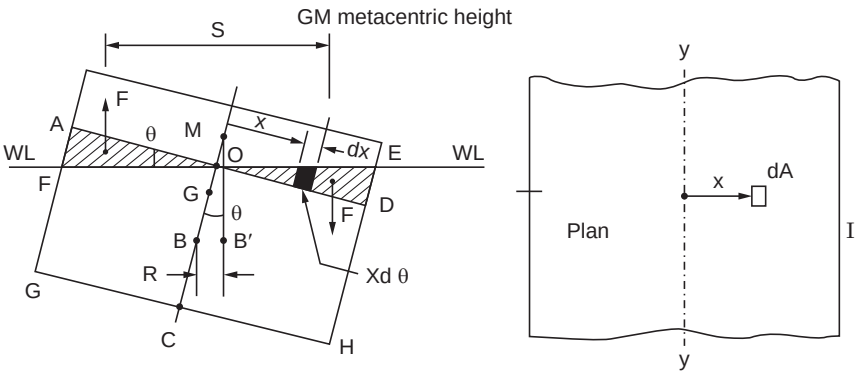


Figure 4.4.1 Metacentric height – derivation

The force system consists of the original buoyant force acting at *B* and the forces due to the wedges and the resultant is at *B′* due to the new location of the buoyant force.

Taking moments about *B*, $P \times S = W \times R$

The moment $P \times S$ can be determined by taking moments of elements displaced about *O*, the intersection of water surface and centre line.

Consider a small element at *x* with area *dA*

The height of the element = $\theta \times x$ (as θ is small, expressed in radians)

The mass of the element $\gamma \times \theta\, dA$ (γ – specific weight). The moment distance is *x*.

$$\therefore \qquad P \times S = \gamma\, \theta \int_A x^2\, dA = \gamma\, I\, \theta \qquad (4.4.1)$$

where $I = \int_A x^2\, dA$, moment of inertia about the axis $y - y$

$W \times R = \gamma\, \theta\, I$, but $W = V\gamma$ where *V* is the displaced volume

$$\therefore \qquad \gamma\, \theta\, I = V\gamma\, R \qquad (4.4.2)$$

From the triangle MBB′, $R = MB \sin \theta$ or $R = MB\, \theta$

$$\therefore \qquad MB = R/\theta = I/V \qquad (4.4.3)$$

Both *I* and *V* are known. As $V = W/\gamma$, the metacentric height is given by,

$$MG = MB \pm GB \qquad (4.4.4)$$

GB is originally specified. So the metacentric height can be determined. If *G* is above *B* –ve sign is used. If *G* is below *B* +ve sign is to be used.

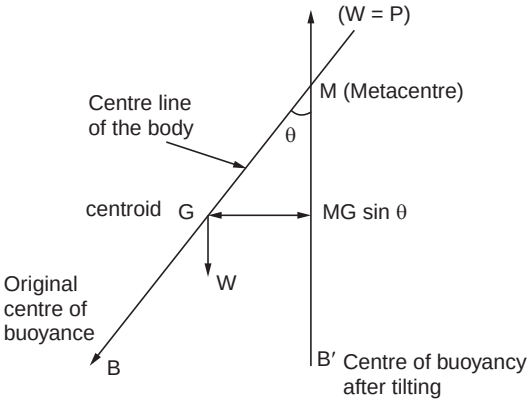


Figure 4.4.2

B = W acts vertically along $B'M$. in the upward direction W acts vertically downwards at G. The distance between the couple formed is $MG \sin\theta$. Hence the righting couple

$$= \gamma V \overline{MG}\ \theta = W \overline{MG} \sin\theta.$$

Example. 4.3 *A ship's plan view is in the form of an ellipse with a major axis of 36 m and minor axis of 12 m. The mass of the ship is 1000 tons. The centre of buoyancy is 1.8 m below the water level and the centre of gravity is 0.3 m below the water level.* ***Determine the metacentric height for rolling*** *(y – y axis) and* ***pitching*** (x – x axis).

$$MG = (I/V) \pm GB,\ GB = 1.8 - 0.3 = 1.5 \text{ m}$$

For rolling:

$$\mathbf{I} = x\,(bh^3/64) = \boldsymbol{\pi \times 36 \times 12^3/64} = 3053.63 \text{ m}^4,\ V = W/\gamma = \text{m}/\rho$$

Considering sea water of density 1030 kg/m^3 and V as the liquid volume displaced.

$$V = 1000{,}000/1030 = 970.87 \text{ m}^3,\ MB = I/V = 3053.63/970.87$$

$$MG = MB - GB = (3053.62/970.87) - 1.5 = 3.15 - 1.5 = \mathbf{1.65\ m}$$

(–ve) sign as B is below G). This is positive and so the ship is stable about rolling by small angles. For **pitching**

$$\mathbf{MG} = (\pi \times 12 \times 36^3/64 \times 970.87) - 1.5 = \mathbf{28.3\ m}$$

Highly stable in this direction. This situation is for small angles and uniform section at the water line.

4.4.1 Experimental Method for the Determination of Metacentric Height

The weight of the ship should be specified, say W. A known weight W_1 is located at a distance of X from the centre line. A plumb bob or pendulum is used to mark the vartical. The weight is now moved by $2X$ m so that it is at a distance of X m on the otherside of the centre line. The angle of tilt of the pendulum or plumb bob is measured. Then the disturbing moment is $W_1\,2X$. This equals the restoring couple $W\,MG \sin\theta$. For small angles $\sin\theta \simeq \theta$

$$W_1\,2X = W\,\overline{MG} \sin\theta = W\,\overline{MG}\ \theta\ (\theta \text{ in radians})$$

Metacentric height $MG = 2\,W_1\,X/W\,\theta$ (X is half the distance or distance of weight from centre and θ is the angle in radians). The angle can be measured by noting the length of the pendulum and the distance moved by the plumb bob weight.

Example. 4.4 *A ship displacing 4000 tons has an angle of tilt of 5.5° caused by the movement of a weight of 200 tons through 2m from one side of centre line to the other.* ***Determine the value of metacentric height.***

$$\overline{\mathbf{MG}} = (2W_1\,X/W\,\theta)$$

$$= 2 \times 200 \times 1000 \times 9.81 \times 1/4000 \times 1000 \times 9.81\ (.5.5 \times \pi/180)$$

$$= \mathbf{1.042\ m}$$

(here X is half the distance moved) check using the degree of tilt and $MG = 2MX/W \sin\theta$.

SOLVED PROBLEMS

Problem 4.1 *Determine* ***the diameter of a hydrogen filled*** *balloon to support a total of 1 kg at a location where the density of air is 0.8 kg/m^3 and that of the hydrogen in the balloon is 0.08 kg/m^3.*

The weight that can be supported equals the difference in weights of air and hydrogen.

$$9.81 \times 1 = (4/3) \times \pi \times r^3 \times (0.8 - 0.08) \times 9.81, \text{ Solving } r = 0.692 \text{ m}$$

$$\mathbf{D = 2r = 1.384\ m}$$

Problem 4.2 *Ship weighing 4000 tons and having an area of 465 m^2 at water line submerging to depth of 4.5 m in sea water with a density of 1024 kg/m^3 moves to fresh water.* ***Determine the depth of submergence in fresh water.*** *Assume that sides are vertical at the water line.*

Originally the weight of the ship equals the weight of sea water displaced. (omitting 9.81 in both numerator and denominator)

Volume of sea water displaced = $4000 \times 1000/1024 = 3906.25\ m^3$

To support the same weight, the volume of fresh water displaced

$$= 4000 \times 1000/1000 = 4000\ m^3$$

$$\text{Extra volume} = 4000 - 3906.25 = 93.75\ m^3$$

Area at this level = 465 m^2, Equivalent Depth = 93.75/465 = 0.2 m

$\therefore$ The depth of submergence in fresh water = 4.5 + 0.2 = **4.7 m**

Problem 4.3 *A bathy sphere of mass 6800 kg (empty) and having a diameter of 1.8 m is to be used in an ocean exploration. It is supported by a cylindrical tank of 3 m dia and 6 m lenght of mass 4500 kg when empty and filled with oil of specific gravity 0.7.* ***Determine the maximum mass of equipment*** *that can be supported in the bathy sphere. Assume density of sea water as 1024 kg/m^3. Neglect metal thickness. The important thing to note is that the limiting condition is when the supporting cylinder just submerges.*

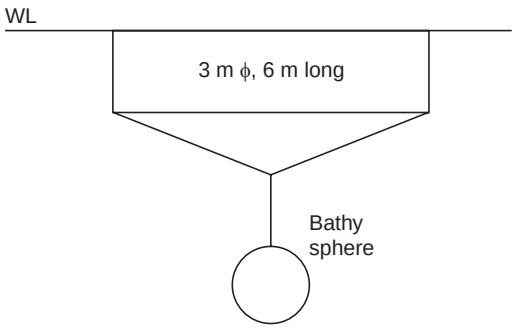


Figure P. 4.3

$\therefore$ The total volume displaced = volume of cylinder + volume of sphere

$$= (\pi \times 3^3 \times 6/4) + (4 \times \pi \times 0.9^3/3) = 45.465\ m^3$$

Weight of water displaced = $45.465 \times 1024 \times 9.81$ = **456717 N**

The weight of cylinder and oil = $9.81\ [4500 + (\pi \times 3^2 \times 6 \times 700/4)]$ = **335384.8 N**

The weight of empty sphere = $6800 \times 9.81 = 66708$ N

Total weight = 335384.8 + 66708 = **402092.8 N**

The additional weight that can be supported = 456717 – 402092.8

= 54624.5 N (about 5568 kg of mass)

Problem 4.4 *A cubical block with a density of 2500 kg/m^3 fully submerged in water is used to hold down a box 0.3 × 0.6 × 0.9 m size just submerged in water. The box has a mass of 110 kg.* ***Determine the weight of the block.***

The upward forces are due to buoyancy on the block and buoyancy on the box. The downward forces are due to the weight of the box and the block.

Let the side of the block be h m.

Total upward force $= (0.3 \times 0.6 \times 0.9 \times 9810) + (h^3 \times 9810)$

Downward force $= 9.81\ [110 + h^3 \times 2500]$

Equating, $1589.22 + 9810\ h^3 = 1079.1 + 24525\ h^3$

Solving $h = 0.326065$ m

Weight of the block $= 2500 \times 9.81 \times 0.326065^3 =$ **850.2 N (86.7 kg)**

Problem 4.5 ***Determine the volume and specific weight*** *of an object which weighs 22 N in water and 30 N in oil of specific gravity 0.80.*

Obviously this object is immersed in the fluid completely during weighment. If it is just floating, its weight is balanced by the buoyant force and so the apparent weight will be zero. Let its volume be V m^3. When in water, it displaces V m^3 and so also when in oil. Let buoyant force when in water be W_w and when in oil W_o

Let the real weight of the object be W kg

$$W - W_w = 22\ N\ ;\ W - W_O = 30\ N$$

Subtracting $W_w - W_O = 8$ N

$V(9810 - 0.80 \times 9810) = 8$; $\therefore$ **$V = 4.0775 \times 10^{-3}$ m^3** or 4.0775 litre

$\therefore \quad W_w = 4.0775 \times 9.81 = 40$ N

Substituting for W_w in equaion 1

$W = 22 + W_w =$ 62 N ; **specific weight** $= W/V =$ **15205.5 N/m^3**

Problem 4.6 *A hydrometer (to measure specific gravity of a liquid) is in the form of a sphere of 25 mm dia attached to a cylindrical stem of 8 mm dia and 250 mm length. The total mass of the unit is 14 grams.* ***Determine the depth of immersion*** *of the stem in liquids of specific gravity of 0.75, 0.85, 0.95. 1.05 and 1.15. Cheak whether the intervals are uniform.*

The volume of liquid displaced × sepecific weight = Weight of the hydrometer

The volume is made up of the sphere and the cylindrical of length h.

$$[(4/3)\ \pi\ (0.025/2)^3 + \pi\ (0.004)^2\ h] \times \text{sp. gravity} \times 9810 = 14 \times 9.81 \times 1/1000$$

$$[0.8181 \times 10^{-6} + 5.026 \times 10^{-5}\ h] \times \text{sp. gravity} = 1.4 \times 10^{-5}$$

This reduces to $h = (0.2786/\text{sp. gravity}) - 0.1628$

The only unknown is h and is tabulated below

Sp. gravity	0.75	0.85	1.00	1.05	1.15
h, mm	208.6	164.9	115.8	102.5	79.4

The intervals in mm are : 43.7, 34.5, 27.9, 23.1 and hence not uniform. This is due to the combined spherical and cylindrical shape.

Only the sphere will be immersed when the sp. gravity of liquid equals 1.711. Usually the major portion of the weight is placed in the spherical portion. So the buoyant force creates a righting couple and the instrument is stable till the spherical portion alone is immersed.

Problem 4.7 *The specific weight of a liquid varies as $\gamma = 9810\ (1 + y)$ where y is measured in m, downward from the surface. A block 1 m × 2 m area and 2 m deep weighing 19620 N floats in the liquid with the 2 m side vertical.* ***Determine the depth of immersion.***

The weight of liquid displaced = weight of the body

To determine the weight of the liquid displaced, consider a small thickness by at distance y. (as $\gamma = 9810\ (1 + y)$)

The weight of the element $dW = 1 \times 2dy \times \gamma = 2\ (1 + y)\ 9810\ dy$

Let the depth of immersion be D. Then integrating the expression and equating it to the weight of the solid.

$$2 \times 9810 \int_0^D (1 + y)\, dy = 19620\ ; \quad \therefore \quad \left[y + y^2/2\right]_0^D = 1$$

or $D^2 + 2D - 2 = 0$, Solving **D = 0.732 m** (the other root is negative). This is less than the depth of the body. So the assumption that the body floats is valid. Check whether it will be stable.

Problem 4.8 *An iceberg floats in sea water with 1/7 of the volume outside water.* ***Determine the density of ice.*** *The density of sea water is 1025 kg/m^3.*

The weight of any floating body equals the weight of liquid displaced by it. The iceberg displaces 6/7 of its volume of sea water. Let the vulume of iceberg be V m^3.

Then $1 \times \gamma \times V = (6/7)V \times 1025 \times 9.81; \quad \therefore \quad \gamma = 8618.79$ N/m^3

$\therefore$ density of **ice = γ/g = 878.57 kg/m^3**

Problem 4.9 *Two spheres, one heavier and weighing 12000 N and of diameter 1.2 m and the other lighter and weighing 4000 N, are tied with a rope and placed in water. It was found that the spheres floated vertically with the lighter sphere just submerging.*

Determine the diameter of the lighter sphere and the tension in the rope.

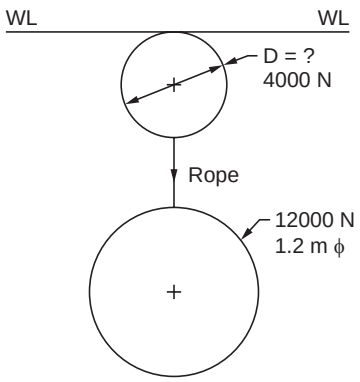


Figure P. 4.9

The buoyant force on the heavier sphere equals the weight of water displaced. Buoyant force on the heavier sphere = $(4/3) \times \pi \times 0.6^3 \times 9810 = 8875.9$ N

The weight of the sphere = 12000 N.

The difference between these two is the tension in the rope.

The **tenstion in the rope** = 12000 – 8875.9 = **3124.1 N**

The weight of the lighter sphere and the rope tension together should balance the buoyant force on the smaller sphere of diameter D.

$$(4/3) \times \pi \times R^3 \times 9810 = 4000 + 3124.1\ ; \quad \therefore \quad \mathbf{D = 1.1152\ m}$$

Problem 4.10 *A mass of volume 0.1 m^3 attached to a balloon of 0.3 m^3 at a pressure of 1.4 bar (abs) weigh totally 2000 N. The unit is released in the sea.* ***Determine the level to which the unit will sink****. Assume the specific weight of sea water as 10000 N/m^3 and the air temperature in the balloon remains constant.*

When the unit sinks to a level such that the weight equals the buoyant force, it will stop sinking further. The buoyant force equals the weight of water displaced. As it goes down in the water the volume of the balloon shrinks due to the increase in surrounding pressure.

Let the volume of the balloon at this level be V m^3.

$(0.1 + V)\ 10000 = 2000; \quad \therefore \quad V = 0.1$ m^3. The original volume was 0.3 m^3.

As $(P_2/P_1) = (V_1/V_2)$; $P_2 = 1.4 \times (0.3/0.1) = 4.2$ bar (abs)

The depth at which this pressure reached is given by

$$10000 \times y = (4.2 - 1.013) \times 10^5; \text{ Solving } y = 31.87 \text{ m.}$$

Hence the unit will **sink to a depth of 31.87 m**

Problem 4.11 *A helium balloon is floating (tied to a rope) at a location where the specific weight of air is 11.2 N/m³ and that of helium is 1.5 N/m³. The empty balloon weighs 1000 N.* ***Determine the diameter of the balloon*** *if the tension in the rope was 3500 N.*

The buoyant force on the balloon = Rope tension + weight of balloon

Volume × (sq weight of surrounding fluid air – sp. weight of helium)

= Rope tension + weight of balloon

$$V \times (11.2 - 1.5) = 3500 + 1000;$$

$\therefore \quad V = 463.92 \text{ m}^3$; $(4\pi R^3/3) = 463.92 \text{ m}^3$;

$\therefore$ **D = 9.6 m**

Problem 4.12 ***Determine the diameter of the sphere to open a cylindrical gate*** *hinged at the top and connected to the sphere as shown in Fig. P.4.12 when the water level reaches 6 m above the centre of the gate. The gate weighs 4500 N and its centroid coincides with the centroid of the semicircle. The sphere weighs 1500 N/m³. The width of the gate is 1.5 m.*

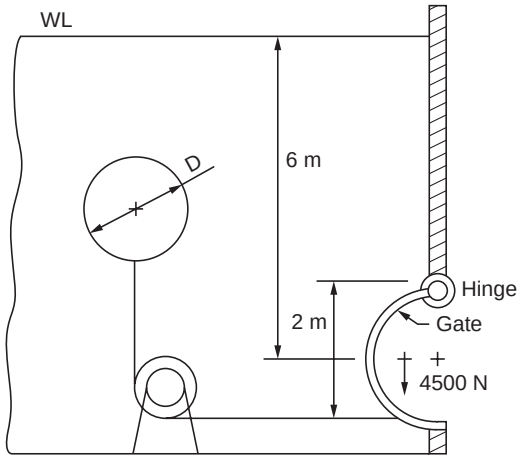


Figure P. 4.12

The forces on the gate are (*i*) tension in the rope equal to the buoyant force on the sphere minus the weight of the sphere. (*ii*) horizontal force due to pressure and (*iii*) vertical force due to pressure and weight.

Buoyant force = $(4/3)\ \pi R^3$ [9810 – 1500] N acts at 2 m from hinge.

as $h = 6$ m, horizontal force due to water = $9810 \times 6 \times 2 \times 1.5$ N = 176580 N

Centre of pressure = $6 + (1/12)\ (1.5 \times 2^3/6 \times 2 \times 1.5) = 6.0556$ m, acts from the hinge at: 1.0556 m.

Vertical force = Weight of gate + Pressure force (Pressure force equals the weight of the volume displaced, acts upwards), $R = 1$, $L = 1.5$ m,

Vertical upward force = $-4500 + \pi\ (R^2/2)\ L \times 1000 \times 9.81 = 18614$ N.

It acts at the centroid of the semicircle, $(4R/3\pi) = 0.4244$ m from hinge.

Taking moments about the hinge,

$$176580 \times 1.0555 = 18614 \times 0.4244 + (4/3)\,\pi R^3\,(9810 - 1500)$$

Solving, **R = 1.7244 m or D = 3.4488 m**

Problem 4.13 *An empty storage tank of square section 6 m side and 1.2 m high of mass 2250 kg is buried under loose soil at a depth of 1 m. The density of the soil is 480 kg/m^3. A spring causes water to seep below the tank.* ***Determine the height of water*** *that may cause the tank to break free and start to rise.*

At the point when the tank begins to break free due to water seeping all around, it can be considered that the tank floats with a weight equal to its own and the weight of soil above it. The total weight should equal the weight of water displaced.

If h is the height upto which the water rises, then

$$(2250 + 6 \times 6 \times 1 \times 480)\,9.81 = 9.81 \times 1000 \times 6 \times 6 \times h \quad \therefore \quad \mathbf{h = 0.5425\ m}$$

When water rises to about 0.5425 m from battom the tank will begin to break free.

Problem 4.14 *A wooden pole of 45 cm^2 section and 3 m length is hinged at 1.2 m above the water surface and floats in water at an angle of θ with vertical* ***Determine the value of the angle.*** *The pole weighs 90 N.*

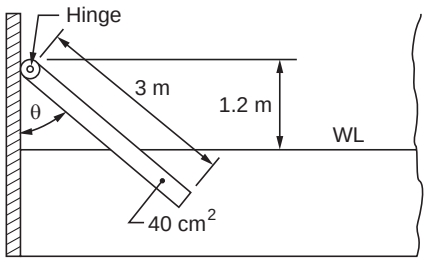


Figure P. 4.14

The problem is solved by taking moment of the weight at the hinge and equating it to the moment of the buoyancy force at the hinge. Taking moment **along the pole.**

Moment of the weight = (3/2) 90 = 135 Nm.

Length of the submerged portion = 3 – (1.2/cos θ),

Weight of the displaced water or buoyancy force

$$= (45/10^4) \times 9810\,[3 - (1.2/\cos\theta)].$$

The distance along the pole it acts = (1.2/cos θ) +(1/2) [3 – (1.2/cos θ)]

$$= (1/2)\,[3 + (1.2/\cos\theta)]$$

Moment of buoyant force = $(45/10^4)\,(9810/2)\,[3^2 - (1.2^2/\cos^2\theta)]$

Equating to 135 Nm and solving, **θ = 45°**.

Check = $(45/10^4)\,(9810/2)\,[3^2 - (1.2^2/\cos^2 45)] = 135.08$ Nm. *checks*

This can be extended to analyse the water level control valve in tanks.

Problem 4.15 *A wooden pole of 0.16 m square section of length 3m and weighing 425 N and of dimensions as shown in Fig. P. 4.15 floats in oil of specific gravity 0.815. The depth of oil above the hinge (friction negligible) is 2 m.* ***Determine the angle of inclination of the pole*** *with horizontal. Also determine the oil level for the pole to float vertically.*

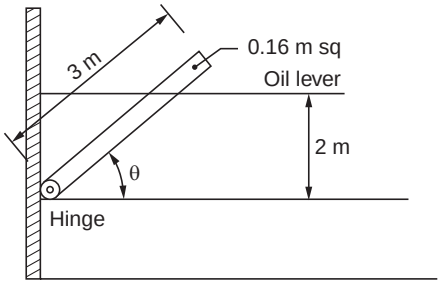


Figure P. 4.15

The problem is solved by taking moment about the hinge for the weight and the buoyant force and equating them. Let the angle of inclination be θ with horizontal. The moment for the weight about the hinge (along × direction) = 425 × (3/2) cos θ. The weight of oil displaced (buoyant force)

$$= (2/\sin\theta) \times 0.16^2 \times 9810 \times 0.815 \text{ N}$$

Moment arm = (2/sin θ) (1/2) (cos θ). Equating the moments,

$$425\ (3/2)\cos\theta = (2/\sin\theta) \times 0.16^2 \times 9810 \times 0.815\ (2/\sin\theta)(1/2)\cos\theta$$

Solving, **θ = 53.26°.**

When the pole begins to float vertically, the weight equals the buoyant force.

$h \times 0.16^2 \times 9810 \times 0.815 = 425$; **h = 2.0765 m.**

If the level rises above this value, a vertical force will act on the hinge.

Problem 4.16 *In order to keep a weight of 160 N just submerged in a liquid of specific gravity of 0.8, a force of 100 N acting upward is required. The same mass requires a downward force of 100 N to keep it submerged in another liquid.* ***Determine the specific weight of the second liquid.***

The volume of fluid displaced in both cases are equal as the weight is submerged.

Case (i) The buoyant force in this case = (160 – 100) N = 60 N

The volume of the fluid displaced = $60/(0.8 \times 1000 \times 9.81)\ m^3 = 7.65 \times 10^{-3}\ m^3$

Case (ii) The buoyant force = 160 + 100 = 260 N

Specific weight of the other liquid = $260/7.65 \times 10^{-3}$

= 34008 N/m³

(**Note** : Buoyant force = volume displaced × sp. weight of liquid)

density will be = (34008/9.81) = **3466.7 kg/m³**[c]

Problem 4.17 *A cylindrical container of 0.4 m dia. and 0.9 m height weighing 60 N contains oil of specific weight 8600 N/m³ to a depth of 0.3 m.* ***Determine the depth upto which it will float in water.*** *Also calculate the depth oil in the container so that the depth of oil and depth of immersion are equal.*

Case (*i*) Total weight of the container = $60 + \pi \times 0.2^2 \times 0.3 \times 8600$ N = 384.21 N

Volume of water displaced = $384.21/(1000 \times 9.81)\ m^3 = 0.0392\ m^3$

∴ Depth of immersion = $0.0392/(\pi \times 0.2^2)$ = **0.3117 m**

Case (ii) Let h be the depth of immersion. The depth of oil will also be h,

$60 + (\pi \times 0.2^2 \times h \times 8600) = \pi \times 0.02^2 \times h \times 9810$, Solving **h = 0.3946 m**

Problem 4.18 *A cylindrical hydrometer weighing 0.04 N has a stem diameter of 6 mm.* ***Determine the distance between the markings*** *for 0.8, 1.0 and 1.1 specific gravity values. Indicate the direction of the markings as up or down also.*

The volume displaced in each case equals the weight of the hydrometer.

(*i*) Relative density = 0.8. Let the volume displaced be V_1

$$V_1 \times 9810 \times 0.8 = 0.04; \quad \therefore \quad V_1 = 5.097 \times 10^{-6} \text{ m}^3$$

(*ii*) Relative density = 1. Let the volume displaced be V_2

$$V_2 \times 9810 = 0.04 ; \quad \therefore \quad V_2 = 4.0775 \times 10^{-6} \text{ m}^3.$$

$$(V_1 - V_2) = \pi (D^2/4)\, l_1;$$

Solving $\boldsymbol{l_1 = 0.036052}$ **m or 3.6 cm.** This is downwards as V_2 is less than V_1.

(*iii*) Relative density = 1.1. Let the volume displaced be V_3.

$$V_3 = 0.04/(9810 \times 1.1) = 3.707 \times 10^{-6} \text{ m}^3$$

$(V_3 - V_3)\ \pi (D^2/4) \times l_1$; $\boldsymbol{l_2 = 0.01311}$ **m or 1.31 cm**

This is also downwards as V_3 is less than V_2. As density increases, the depth of immersion decreases and is non linear.

Problem 4.19 *A wooden cylinder having a specific gravity of 0.6 has a concrete cylinder of the same diameter and 0.2 m length attached to it at one end. The specific gravity of the concrete is 2.5.* ***Determine the length of the wooden cylinder*** *for the composite block to float vertically.*

The limiting condition is for the composite block to float with top surface at water level. Let "h" be the length of the wooden cylinder. The weight of the composite block

$$= (\pi \times R^2 \times 0.2 \times 9810 \times 2.5) + (\pi \times R^2 \times h \times 9810 \times 0.6)$$

This equals the weight of water displaced when the block just floats.

The weight of water displaced = $\pi \times R^2 \times 9810\ (0.2 + h)$. Equating and solving,

$$0.5 + 0.6\, h = 0.2 + h \quad \therefore \quad h = 0.75 \text{ m}$$

The wooden cylinder **should be atleast 0.75 m long for the composite cylinder to float vertically.**

Work the problem for 1 m long cylinder and find the length above the water line.

Problem 4.20 *A right circular cylinder of diameter D m and height h m with a relative density of (S < 1) is to float in water in a stable vertical condition.* ***Determine the limit of the ratio D/h*** *for the required situation.*

For stability, the limiting condition is that the metacentre approach the centre of gravity. Using equation (4.4.3) and (4.4.4), (V – volume displaced), $MB = I/V$

$$MG = (I/V) \pm GB. \text{ Here } MG = 0 \text{ for the limiting condition.}$$

$(I/V) = GB \quad I = \pi D^4/64,\ V = \pi D^2\, h\ \text{S}/4, \quad \therefore \quad (I/V) = D^2/16hS$

Also from basics

$$GB = (h/2) - (h\,S/2) = h\,(1 - S)/2; \text{ equating, } (D^2/16hs) = [h\,(1 - S)]/2$$

$$\therefore \qquad (D/h) = 2\,[2\,S\,(1 - S)]^{0.5} \qquad \ldots (1)$$

For example if $S = 0.8, \dfrac{D}{h} = 1.1314$

$$\therefore \qquad D > h.$$

The diameter should be larger than the length. This is the reason why long rods float with length along horizontal. The same expression can be solved for limiting density for a given D/h ratio. Using equation 1

$$(D/h)^2 = 8\,S\,(1 - S) \quad \text{or} \quad 8\,S^2 - 8\,S + (D/h)^2 = 0$$

$$S = \{1 \pm [1 - (4/8)\,(D/h)^2]^{0.5}\}/2,$$

say **if (D/h) = 1.2,** then **S = 0.7646 or 0.2354**

Problem 4.21 *A right circular cylinder of 0.3 m dia and 0.6 m length with a specific weight of 7500 N/m^3 is to float vertically in kerosene of specific weight of 8900 N/m^3.* ***Determine the stability of the cylinder.***

$$MG = (I/V) - GB;\ I = \pi \times 0.3^4/64$$

Volume displaced, $V = (\pi \times 0.3^2 \times 0.6 \times 7500)/(4 \times 8900)$

Location of G = 0.3 m from bottom

Location of $B = 0.3 \times (7500/8900)$ m from the bottom.

$\therefore$ $GB = 0.3 - 0 - 3 \times (7500/8900)$

$MG = [(\pi \times 0.3^4 \times 4 \times 8900)/\ 64 \times \pi \times 0.3^2 \times 0.6 \times 7500)]$

$$- [0.3 - (0.3 \times 7500/8900)] = - 0.036 \text{ m.}$$

Hence, the cylinder is unstable.

Check : (use the eqn. 1 in problem 4.20)

$$D = 2h\ [2S\ (1 - S)]^{0.5}, \text{ here } S = 7500/8900$$

Substituting, $D = 2 \times 0.6\ [2 \times (7500/8900) \times (1 - 7500/8900)]^{0.5}$

$= 0.61788$ m. $D > h$.

This is the value of D which is required for stability. The given cylinder is of lower diameter and hence unstable.

Problem 4.22 ***Derive the expression for (D/h) for a hollow right circular cylinder*** *of outer diameter D and inner diameter kD and height h, to float vertically in a liquid with relative density S.*

The limiting condition for stability is $MG = 0$ or $(I/V) = GB$

With usual notations (refer P 4.20)

$$I = D^4\,(1 - k^4)/64;\ V = \pi\,D^2\,(1 - k^3)\,h\,S/4$$

where V is the volume of the liquid displaced. G is located at $h/2$ from base and B is located at $h\,S/2$ from base. $GB = h\,(1 - S)/2$;

Equating, $[D^4\,(1 - k^4)/64]\ [4/(D^2\,(1 - k^2)\,hS] = h\,(1 - S)/2$. Solving

$$\mathbf{(D/h) = 2[2S\ (1 - S)/(1 + k^2)]^{0.5}} \qquad ...(1)$$

For example if $k = 0$ this becomes a solid cylinder and the expression reduces to $(D/h) = 2\ [2S\ (\ 1 - S)]^{0.5}$ as in problem Problem 4.20.

Consider a thin cylinder, where $k = 0.9$ and $S = 0.8$ then, $(D/h) = 0.841$ (compare with Problem 4.20).

Problem 4.23 Check the stability *of a hollow cylinder with D = 1.2 m and h = 1.8 m with a specific gravity 0f 0.33333 to float in water. The ID is 0.5D.*

Refer Problem 4.20, equation 1.

Here, $k = 0.5$, $h = 1.8$ m. For stability, the minimum

Value of D is given by $D = 2h\ [2S\ (1 - S)/(1 + k^2)]^{0.5}$

Substituting the values, for stability

$$\mathbf{D} = 2 \times 1.8\ [2 \times 0.33333\ (1 - 0.33333)/(1 + 0.5^2)]^{0.5}$$
$$= \mathbf{2.147\ m > 1.8}$$

The specified diameter is only 1.2 m. **So it is not stable.** Considering the calculated value,

$$D = 2.147 \text{ m}, I = \pi \times 2.147^4\ (1 - 0.5^4)/64.$$
$$V = \pi \times 2.147^2\ (1 - 0.5^2)\ 1.8 \times 0.333333/4;\ (I/V) = 0.60002 \text{ and}$$
$$GB = 1.8\ (1 - 0.33333)/2 = 0.6. \text{ Hence, } MG = 0. \text{ So, } \textit{checks.}$$

Problem 4.24 *Determine the* ***metacentric height of a torus of mean diameter D*** *with a section diameter d and specific gravity 0.5 when it floats in water with its axis vertical.*

The specific gravity is 0.5. So it floats such that half its volume will be displaced.

$$MG = (I/V) - GB\ ;\ I = \pi\ [(D + d)^4 - (D - d)^4]/64$$

as the section along the free surface is annular with $OD = D + d$ and $ID = D - d$.

$$V = (1/2)\ (\pi d^2/4)\pi D.$$

Centre of gravity is on the water surface. Centre of buoyancy will be at the CG of displaced volume equals $2d/3\pi$.

$$GB = (2d/3\pi)$$
$$\therefore \quad MG = \{\pi\ [(D + d)^4 - (D - d)^4] \times 8/[64 \times \pi^2 \times D \times d^2]\ \} - (2d/3\pi)$$

Simplifying $\quad MG = \{[(D + d)^4 - (D - d)^4]/[8 \times \pi \times d \times d^2]\} - (2d/3\pi)$

Problem 4.25 Determine the metacentric height of a torus of D = *1.8 m and d = 0.6 m with specific gravity 0.5 when floating in water with axis vertical.*

Refer Problem 4.24 $\mathbf{MG} = \{[(D + d)^4 - (D - d)^4]/[8 \times \pi \times D \times d^2]\} - \{2d/3\pi\}$

$$= ([2.4^4 - 1.2^4]/[8 \times \pi \times 1.8 \times 0.6^2]\} - \{2 \times 0.6/3\pi\} = \mathbf{1.7826\ m}$$

[**Note :** If the relative density is different from 0.5, the determination of the value of GB is more involved as the determination of the position of CG is difficult]

Problem 4.26 *Derive on expression for the ratio of length, h to side, a of a square log to float stably in a vertical direction. The relative density of the log is S.*

The limiting condition for floating in a stable position is that metacentre and centre of gravity coincide. or $(I/V) = GB$.

$I = (a^4/12)$ where a is the side of square. The volume displaced $V = a^2 h S$ where h is the immersion height. Position of $G = h/2$ and position of $B = h\ S/2$, $GB = h\ (1 - S)]/2$

$$I/V = [(a^4/12)(1/a^2 hS)] = [h\ (1 - S)/2];$$
$$\therefore \quad (a/h) = [6S\ (1 - S)]^{0.5}$$

Consider $S = 0.8$, then $(a/h) = 0.98$, $\quad S = 0.5$, then, $(a/h) = 1.23$

as S decreases (a/h) increases.

The sides should be longer than the height. The is the reason why long logs float with length along horizontal. The expression can be generalised for a rectangular section with sides a and k.a (where k is a fraction). Then the stability is poorer along the shorter length ka.

$$I = ak^3\ a^3/12;\ V = h\ Sa^2\ k\ ;\ GB = h(1 - S)/2$$
$$(k^3 a^4/12)\ (1/a^2 khS) = h(1 - S)/2;\ (a/h) = [65\ (1 - S)]^{0.5}/k \qquad \text{...A}$$

Here the side has to be still larger or the height shorter. This expression can be used also to determine the limiting density for a given (side/height) ratio to float stably in a vertical position.

Consider the general eqn. A, reordering,

$$(k^2 \times a^2/h^2) = 6S - 6S^2 \text{ or } S^2 - S + (k^2a^2/6h^2) = 0$$

$$\therefore \quad S = \{ 1 \pm [1 - (4k^2a^2)/(6h^2)]^{0.5} \}/2$$

if $k = 1$, then $S = [1 \pm \{1 - (4a^2/6h^2)\}^{0.5}]/2$

Problem 4.27 *Derive the expression for the ratio of base diameter to the height of a cone to float in a fluid in a stable condition given the relative density between the solid and the fluid as S.*

This case is different from the cylinder due to variation of area along the height (Refer Problem 4.20). The situation is shown in the Fig. P. 4.27. The cone displaces liquid upto a depth h where the diameter is d.

The limiting conditions is that $MG = 0$ or $(I/V) = GB$. In this case the volume displaced and the relationship between D and d and H and h are to be established.

$$(1/3)(D^2/4)\, HS = (1/3)\,(d^2/4)\, h \; ; \; D^2\, HS = d^2h; \; S = \frac{d^2}{D^2} \cdot \frac{h}{H}$$

Also as $(h/H) = (d/D)$

$$h/H = (d/D) = S^{1/3}$$

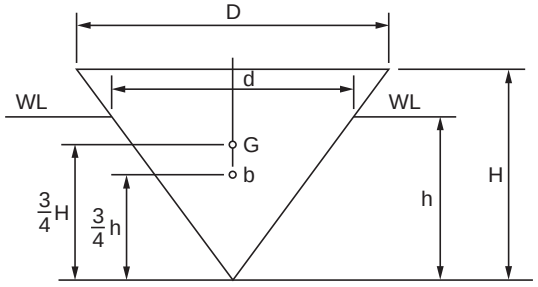


Figure P. 4.27

Volume displaced $= (1/3)(\pi d^2/4)\, h = (1/3)\,(\pi D^2/4)\, HS$

$$I = \pi d^4/64 = \pi D^4\, S^{4/3}/64 \; (d^4 = D^4 \,.\, S^{4/3})$$

$$\therefore \quad I/V = (D^4\, S^{(4/3)}/64)/(D^2\, HS/12) = (3/16)\,(D^3/H)\, S^{1/3} \qquad \text{...A}$$

$G = 3H/4$ (from vertex) and $B = (3h/4) = (3/4)\, HS^{1/3}$ (from vertex)

$$\therefore \quad GB = (3/4)\, H\,(1 - S^{1/3}) \qquad \text{...B}$$

Equating A and B

$$(3D^2S^{1/3}/16H) = (3/4)\, H\,(1 - S^{1/3})$$

$$(D^2/H^2) = 4\,(1 - S^{1/3})/S^{1/3} \textbf{ or } \mathbf{H^2 = D^2S^{1/3}/4\,(1 - S^{1/3})}$$

In actual case H^2 should be less than this value for stability.

Problem 4.28 *A conical wooden block of 0.4 m dia and 0.6 m high has a relative density of 0.8 for a fluid in which it floats.* ***Determine whether it can float in a stable condition.*** For stability, the limiting value of H is given by

$H^2 = D^2\, S^{1/3}/4\,(1 - S^{1/3}) = 0.4^2 \times 0.8^{1/3}/4\,(1 - 0.8^{1/3}) = 0.518\ m^2$

$\therefore$ $H = 0.7197$ m. The actual value of is $0.6 < 0.7197$ and so the cone will float in a stable position.

Check: $H = 0.6$ m, $h = H\,S^{1/3} = 0.557$ m; $D = 0.4$ m, $d = DS^{1/3} = 0.371$ m

$I = \pi d^4/64 = 9.332 \times 10^{-4}\ m^4$; $V = (1/3)\,(\pi D^2/4)\,HS$

$V = (1/3)\,(\pi \times 0.4^2/4)\,0.6 \times 0.8 = 0.0201\ m^3$;

$\therefore$ $I/V = 0.04642$ m

$GB = (3/4)\,(0.6 - 0.557) = 0.03226$ m;

$MG = (I/V) - GB = 0.04642 - 0.03226 = 0.01416$ m.

This is positive and hence stable.

Problem 4.29 *A rectangular pontoon 10 m long, 8m wide and 3m deep weighs 6×10^5 N and carries a boiler of 4 m dia on its deck which weighs 4×10^5 N. The centre of gravity of each may be taken to be at the geometric centre.* ***Determine the value of the meta centric height of the combined unit,*** *when it floats in river water. Calculate also the restoring torque for a tilt of 5° from vertical.*

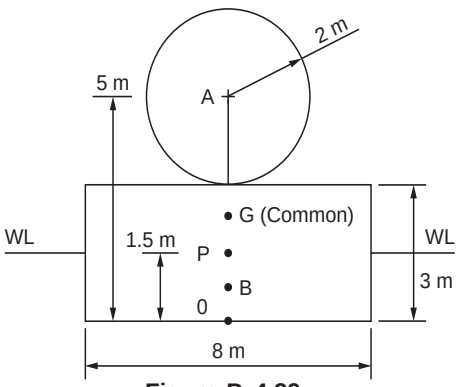


Figure P. 4.29

Assuming the centres to be on the vertical line for the combined unit, the position of the centre of gravity from base can be determined by taking moments about O.

$OP \times 6 \times 10^5 + OA \times 4 \times 10^5$

$= OG \times 10^6$; $OP = 1.5$ m; $OA = 5$ m;

Solving, $OG = 2.9$ m, Total weight $= 10^6$ N

Depth of immersion: $10^6 = 10 \times 8 \times h \times 9810$; $\therefore$ $h = 1.2742$ m

$OB = 1.2742/2 = 0.6371$ m; $GB = OG - OB = 2.2629$

$\mathbf{MG} = (I/V) - GB = [(1/12) \times 10 \times 8^3]/[10 \times 8 \times 1.2742] - 2.2629$

$\mathbf{= 1.9227\ m}$

This is positive and hence the unit is stable.

Restoring torque $= W\,MG\,\theta$ (θ in radian)

$= 10^6 \times 1.9227 \times (\pi \times 5/180) = \mathbf{1.68 \times 10^5\ Nm.}$

REVIEW QUESTIONS

1. Prove Archimedes principle from basics.
2. State the conditions for the stability of floating bodies.
3. Define centre of buoyancy.
4. Define metacentre and metacentric height.
5. Describe an experimental method to determine the metacentric height of a boat.
6. Derive an expression for the height to diameter ratio of a cylinder of specific gravity S to float with its axis vertical in a stable condition.
7. Describe how the density of liquid can be estimated using a cylindrical hydrometer.

OBJECTIVE QUESTIONS

O Q. 4.1 Fill in the blanks

1. When a body floats in water the buoyancy force equals _____.
2. The weight of volume of liquid displaced by a floating body equals _____.
3. The centre of buoyancy is defined as _____.
4. The statement of Archimedes Principle is _____.
5. The three states of equilibrium of a floating body are_____.
6. When a small tilt is given to a body floating in stable equilibrium it will ____.
7. When a small tilt is given to a body floating in neutral equilibrium it will ____.
8. When a small tilt is given to a body floating in unstable equilibrium it will ___.
9. If the centre of gravity coincides with the centre of buoyancy, the floating body will be in _____ equilibrium.
10. If the centre of gravity is below the centre of buoyancy the body will always be in _____ equilibrium.
11. If the centre of gravity is above the centre of buoyancy the metacentric height should be _____ stable equilibrium.
12. Metacentric height is equal to _____.
13. The righting moment due to a tilt of a floating body equals_____.
14. The condition for a cylinder of given diameter to length ratio to float vertically in stable equilibrium is _____.
15. The condition for a square prism of given side to length ratio to float vertically in stable equilibrium is _____.
16. The height to diameter ratio for stable floating condition of a cone is _____.
17. Metacenter is the point at which _____ cuts the body centre line.
18. The body displaces $1m^3$ of water when it floats. It's weight is _____.
19. As fluid density increases the hydrometer will sink by a _____ distance.

Answers

1. The weight of volume of water displaced 2. the weight of the body 3. the centre of gravity of the displaced volume 4. the buoyant force on a floating body equals the weight of the displaced volume and a floating body displaces it's own weight of liquid in which it floats 5. satble, neutral and unstable 6. return to the original position 7. remain in the new position 8. overturn 9. neutral 10. stable 11. positive 12. distance between the metacentre and centre of gravity 13. $W\,MG \sin\theta$, $W\,MG\,\theta$ 14. $(d/h) = 2[2s(1 - s)]^{0.5}$ 15. $(a/h) = [6s(1 - s)]^{-0.5}$ 16. $(H/D)^2 = s^{1/3}/[4(1 - s^{1/3})]$ 17. The line of action of the buoyant force in the displaced position 18. 9810 N 19. shorter

O Q. 4.2 Fill in the blanks with increases, decreases or remains constant

1. Stability of a floating body improves as the metacentric height ____
2. The position of a floating body will _____ when a small tilt is given if the metacentric height is positive.
3. As the density of the floating body increases the distance between the centre of gravity and centre of buoyancy _____
4. The stability of a floating body deteriotes as the metacentric height _____.
5. The volume of liquid displaced by a floating body of wieght W will _____ irrespective of the shape of the body.
6. When a given body floats in different liquids the volume displaced will _____ with increase in the specific gravity of the fluid.

7. For a given floating body in stable equilibrium the righting couple will _____ with increasing metacentric height.
8. For a given shape of a floating body the stability will improve when the density of the body_____
9. The metacentric height of a given floating body will _____ if the density of the liquid decreases.
10. For a body immersed in a fluid the buoyant force _____ with increase in density of the body.

Answers

Increases 1, 7, 8 Decreases 3, 4, 6, 9, 10 Remains constant 2, 5

O Q. 4.3 Indicate whether the following statements are correct or incorrect

1. A floating body will displace the same volume of liquid irrespective of the liquid in which it floats.
2. The buoyant force on a given body immersed in a liquid will be the same irrespective of the liquid.
3. A floating body will displace a volume of liquid whose weight will equal the weight of the body.
4. As the metacentric height increases the stability of a floating body will improve.
5. When the metacentric height is zero the floating body will be in stable equilibrium.
6. When the centre of buyoancy is below the metacenter the floating body will be in stable equilibrium.
7. When the centre of gravity is below the centre of buoyancy the floating body will be unstable.
8. When the metacentre is between the centre of gravity and centre of buoyancy the body will be unstable.
9. When the length of a square log is larger than the side of section the log will float horizontally.
10. A given cubic piece will float more stably in mercury than in water.

Answers

Correct : 3, 4, 6, 8, 9, 10 Incorrect 1, 2, 5, 7

O Q. 4.4 Choose the correct answer

1. If a body is in stable equilibrium the metacentric height should be
   (*a*) zero (*b*) positive
   (*c*) negative (*d*) depends on the fluid.
2. When a heavy object is immersed in a liquid completely the centre of byoyancy will be at
   (*a*) The centre of gravity of the object.
   (*b*) The centre of gravity of the volume of the liquid displaced.
   (*c*) Above the centre of gravity of the object.
   (*d*) Below the centre of gravity of the displaced volume.
3. An object with specific gravity 4 weighs 100N in air. When it is fully immersed in water its weight will be
   (*a*) 25 N (*b*) 75 N
   (*c*) 50 N (*d*) None of the above.
4. A solid with a specific weight 9020 N/m^3 floats in a fluid with a specific weight 10250 N/m^3. The percentage of volume submerged will be
   (*a*) 90% (*b*) 92%
   (*c*) 88% (*d*) 78%.
5. An object weighs 50 N in water. Its volume is 15.3 *l*. Its weight when fully immersed in oil of specific gravity 0.8 will be

(*a*) 40 N (*b*) 62.5 N
(*c*) 80 N (*d*) 65 N.

6. When a ship leaves a river and enters the sea
(*a*) It will rise a little (*b*) It will sink a little
(*c*) There will be no change in the draft. (*d*) It will depend on the type of the ship.
7. When a block of ice floating in water in a container begins to melt the water level in the container
(*a*) will rise (*b*) will fall
(*c*) will remains constant (*d*) will depend on the shape of the ice block.
8. Two cubes of equal volume but of specific weights of 0.8 and 1.2 are connected by a weightless string and placed in water.
(*a*) one cube will completely submerged and the other will be completely outside the surface.
(*b*) heavier cube will go down completely and the lighter one to 0.25 times its volume.
(*c*) will float in neutral equilibrium.
(*d*) heavier cube will submerge completely and the lighter one will submerge to 0.8 times its volume.
9. For *a* floating body to be in stable equilibrium (with usual notations)
(*a*) $I/V = GB$ (*b*) $I/V < GB$
(*c*) $I/V > GB$ (*d*) $I/V = MG$.
10. A cube of side, *a* floats in a mercury/water layers with half its height in mercury. Considering the relative density of mercury as 13.6, the relative density of the cube will be
(*a*) 6.3 (*b*) 7.3
(*c*) 6.8 (*d*) $a \times 13.6/2$

Answers

(1) *b*, (2) *b*, (3) *b*, (4) *c*, (5) *c*, (6) *a*, (7) *b*, (8) *c*, (9) *c*, (10) *b*

O Q. 4.5 Match the sets A and B

A	B
1. Metacentric height	(*a*) weight of displaced volume
2. *G* below *B*	(*b*) *CG* of displaced volume
3. Centre of buoyancy	(*c*) Stability
4. Buoyant force	(*d*) Always stabe.

Answers

(1) *c*, (2) *d*, (3) *b*, (4) *a*

EXERCISE PROBLEMS

E 4.1 Determine the buoyant force on a cube of 2 m side which stays afloat in water with its top face horizontal and 0.2 m above the free surface. **[70632 N]**

E 4.2 A hot air filled balloon of 8m diameter is used to support a platform. The surrounding air is at 20° C and 1 bar while the hot air inside the balloon is at a temperature of 70° C. Determine the buoyant force and the weight that may be supported by the balloon.

[3074.93 N, 403.37 N]

E 4.3 A closed cylindrical drum of 3 m dia and 2 m height is filled fully with oil of specific gravity 0.9 is placed inside an empty tank vertically. If water is filled in the tank, at what height of water level, the drum will start floating. Neglect the self weight of the drum.

[1.80 m from bottom]

E 4.4 A balloon is filled with hydrogen of density 0.08 kg/m^3. To support 50 N of weight in an atmospheric condition where the sir/density is 0.9 kg/m^3. Calculate the diameter of the balloon? **[2.28 m]**

E 4.5 A box of size 1m × 2m × 3m and weight 1000 N to lie just submerged in water is held down with a cubic block placed on it. If the density of the cubic block material is 2000 kg/m^3, find the dimension of the cubic block. The cubic block is also submerged in water. **[1.81 m]**

E 4.6 Determine the depth of immersion of a cubic block of 2 m side weighing 20 kN which floats in a liquid whose specific weight varies as 9810 (1 + depth in m). **[1 m]**

E 4.7 An object weighs 20 N when fully submerged in water. The same object weighs 35 N when fully submerged in an oil of specific gravity 0.8. Determine its volume and density.

[7.65 × 10^{-3} m^{-3}, 1265.88 kg/m^3]

E 4.8 Determine the specific gravity of a liquid when a hydrometer which is in the form of a sphere of 20 mm dia attached with a cylindrical stem of 5 mm dia and 200 mm length showed a depth of immersion of the stem of 100 mm. The total mass of hydrometer is 15 grams. **[0.8]**

E 4.9 Determine the metacentric height of a ship for rolling ($Y - Y$ aixs) and pitching ($X - X$ axis) whose plan view is in the form of an ellipse with major axis of 40 m and minor axis of 15 m. The weight of the ship is 9000 kN and the centre of buoyancy is 2 m below the water level and the centre of gravity is 0.5 m below the water level. Assume density of sea water as 1025 kg/m^3.

[5.9 m, 51.15 m]

E 4.10 Determine the metacentric height of a ship which displaces 5000 kN of water when it tilts by 6° due to the movement of 300 kN weight through 3 m from one side of center line to the other.

[1.72 m]

E 4.11 A cylinder with diameter 0.25 m and length 0.5 m floats in water. Determine its stability if its specific weight is 8000 N/m^3. **[unstable]**

E 4.12 A hollow cylinder with *ID* 0.8 m, *OD* 1.6 m and height 2 m floats in water. Check the stability of the cylinder if its specific gravity is 0.4. For stability of the cylinder, what is the required outer diameter? **[unstable, 2.479 m]**

E 4.13 A torus of D = 2 m and d = 0.5 with specific gravity 0.5 floats in water. Determine its metacentric height. **[2.6 m]**

E 4.14 Determine the D/h ratio for a stable floating log of circular cross section with density 800 kg/m^3. **[1.13]**

E 4.15 Determine the maximum density of a conical wooden block of 0.5 m dia and 0.8 m height to float stably in water. **[756 kg/m^3]**

E 4.16 Determine the metacentric height of the combined unit of a rectangular pontoon, 9 m long, 7 wide and 2 m deep weighing 500 kN carrying on its deck a boiler of 3 m dia weighing 300 kN. The centre of gravity of each unit may be taken to be at the geometric centre and along the same line. Also calculate the restoring torque for a tilt of 4° from vertical. Assume the centre to be on the vertical line. **[1.865 m, 1.0416 × 10^5 Nm]**

E 4.17 The stem of a hydrometer is of cylindrical shape of 2.8 mm dia and it weighs 0.0216 N. It floats 22.5 mm deeper in an oil than in alcohol of specific gravity 0.821. Determine the specific gravity of the oil. **[0.78]**

E 4.18 A metal piece floats in mercury of specific gravity 13.56. If the fraction of volume above the surface was 0.45, determine the specific gravity of the metal. **[7.458]**

E 4.19 A piece of material weighs 100 N in air and when immersed in water completely it weighs 60 N. Calculate the volume and specific gravity of the material. **[0.00408 m^3, 1.67]**

E 4.20 A wooden block when floating in glycerin projects 76 mm above the surface of the liquid. If the specific gravity of the wood was 0.667, how much of the block wil project above the surface in water. Specific gravity of glycerin is 1.6. **[50 mm]**

E 4.21 A long log of 2.5 m dia and 4.5 m length and of specific gravity of 0.45 floats in water. Calculate the depth of floatation.

E 4.22 A tank of 1.5 m dia and 2 m length open at one end is immersed in water with the open end in water. Water rises by 0.6 m inside. The water level is 1 m from the top. Determine the weight of the tank.

E 4.23 A ship with vertical sides near the water line weighs 4000 tons and the depth of immersion is 6.7056 m in sea water of specific gravity 1.026. When 200 tons or water ballast is discharged, the depth of immersion is 6.4 m. Calculate the depth of immersion in fresh water.

E 4.24 Determine whether a cylinder of 0.67 m dia and 1.3 m length will float vertically in stable condition in oil of specific gravity 0.83.

E 4.25 A sphere of 1.25 m dia floats half submerged in water. If a chain is used to tie it at the bottom so that it is submerged completely, determine the tension in the chain.

E 4.26 The distance between the markings of specific gravity of 1 and 1.1 is 10 mm, for a hydrometer of 10 mm dia. Determine the weight of the unit. **[0.08475 N]**

E 4.27 The difference in specific gravities of 1.1 and 1.4 is to be shown by 40 mm by a hyrometer of mass 25 gram. What should be the diameter of the stem. **[4.95 mm]**

E 4.28 A sphere of specific gravity 1.2 is immersed in a fluid whose sepcific gravity increases with depth y as $1 + 20 \times 10^{-6}\, y$, ($y$ in mm). Determine the location of the centre of the sphere when it will float in nuteral equilibrium.

E 4.29 A cube side 60 cm is made of two equal horizontal layers of specific gravity 1.4 and 0.6 and floats in a bath made of two layers of sepecific gravity 1.2 and 0.9, the top layer being 60 cm thick. Determine the location of the base from the liquid surface. **[0.2 m]**

E 4.30 A barge is of rectangular section of 26.7 m × 10 m and is 3 m in height. The mass is 453.62 tons with load. The centre of gravity is at 4 m from bottom. Determine the metacentric height for rotation along the 26.7 m centreline. Investigate the stability. Also determine the restoring torque if it is rotated by 5° about the axis. **[1.753 m, stable, 1904 kN/m]**

E 4.31 A wedge of wood of specific gravity 0.65 and base width 0.5 m and height 0.5 m is forced into water by 666 N. If the wedge is 50 cm wide, determine the depth of immersion.

E 4.32 A cube of side 40 cm weighing 1050 N is lowered into a tank containing water over a layer of mercury. Determine the position of the block under equilibrium.

E 4.33 An iceberg of specific gravity 0.92 floats in ocean water of specific gravity 1.02. If 3000 m^3 protrudes above the water level calculate the total volume of the iceberg.

E 4.34 Show that in the case of cylindrical hydrometer, the difference in the height of immersion in liquid of specific gravity S, over the height h of immersion in water is given by $h = V(S - 1) \times A \times S$ where V is the submerged volume in water, and A is the sectional area of the stem.

5 Fluid Flow—Basic Concepts—Hydrodynamics

5.0 INTRODUCTION

In the previous three chapters the pressure distribution in static fluids and its effect on surfaces exposed to the fluid was discussed. In this chapter the flow of ideal fluids will be discussed. The main attempt in this chapter is to visualise flow fields. A flow field is a region in which the flow is defined at all points at any instant of time. The means to that is to define the velocities at all the points at different times. It should be noted that the velocity at a point is the velocity of the fluid particle that occupies that point. In order to obtain a complete picture of the flow the fluid motion should be described mathematically. Just like the topography of a region is visualised using the contour map, the flow can be visualised using the velocity at all points at a given time or the velocity of a given particle at different times. It is then possible to also define the potential causing the flow.

Application of a shear force on an element or particle of a fluid will cause continuous deformation of the element. Such continuing deformation will lead to the displacement of the fluid element from its location and this results in fluid flow. The fluid element acted on by the force may move along a steady regular path or randomly changing path depending on the factors controlling the flow. The velocity may also remain constant with time or may vary randomly. In some cases the velocity may vary randomly with time but the variation will be about a mean value . It may also vary completely randomly as in the atmosphere. The study of the velocity of various particles in the flow and the instantaneous flow pattern of the flow field is called flow kinematics or hydrodynamics. Such a study is generally limited to ideal fluids, fluids which are incompressible and inviscid. In real fluid shows, beyond a certain distance from the surfaces, the flow behaves very much like ideal fluid. Hence these studies are applicable in real fluid flow also with some limitations.

5.1 LAGRANGIAN AND EULARIAN METHODS OF STUDY OF FLUID FLOW

In the Lagrangian method a single particle is followed over the flow field, the co-ordinate system following the particle. The flow description is particle based and not space based. A moving coordinate system has to be used. This is equivalent to the observer moving with the particle to study the flow of the particle. This method is more involved mathematically and is used mainly in special cases.

In the Eularian method, the description of flow is on fixed coordinate system based and the description of the velocity etc. are with reference to location and time *i.e.*, $V = V\ (x, y, z, t)$ and not with reference to a particular particle. Such an analysis provides a picture of various parameters at all locations in the flow field at different instants of time. This method provides an easier visualisation of the flow field and is popularly used in fluid flow studies. However the final description of a given flow will be the same by both the methods.

5.2 BASIC SCIENTIFIC LAWS USED IN THE ANALYSIS OF FLUID FLOW

(i) Law of conservation of mass: This law when applied to a control volume states that the net mass flow through the volume will equal the mass stored or removed from the volume. Under conditions of steady flow this will mean that the mass leaving the control volume should be equal to the mass entering the volume. The determination of flow velocity for a specified mass flow rate and flow area is based on the continuity equation derived on the basis of this law.

(ii) Newton's laws of motion: These are basic to any force analysis under various conditions of flow. The resultant force is calculated using the condition that it equals the rate of change of momentum. The reaction on surfaces are calculated on the basis of these laws. Momentum equation for flow is derived based on these laws.

(iii) Law of conservation of energy: Considering a control volume the law can be stated as "the energy flow into the volume will equal the energy flow out of the volume under steady conditions". This also leads to the situation that the total energy of a fluid element in a steady flow field is conserved. This is the basis for the derivation of Euler and Bernoulli equations for fluid flow.

(iv) Thermodynamic laws: are applied in the study of flow of compressible fluids.

5.3 FLOW OF IDEAL / INVISCID AND REAL FLUIDS

Ideal fluid is nonviscous and incompressible. Shear force between the boundary surface and fluid or between the fluid layers is absent and only pressure forces and body forces are controlling.

Real fluids have viscosity and surface shear forces are involved during flow. However the flow after a short distance from the surface is not affected by the viscous effects and approximates to ideal fluid flow. The results of ideal fluid flow analysis are found applicable in the study of flow of real fluids when viscosity values are small.

5.4 STEADY AND UNSTEADY FLOW

In order to study the flow pattern it is necessary to classify the various types of flow. The classification will depend upon the constancy or variability of the velocity with time. In the next three sections, these are described. In steady flow the property values at a location in the flow are constant and the values do not vary with time. The velocity or pressure at a point remains constant with time. These can be expressed as $V = V(x, y, z)$, $P = P(x, y, z)$ etc. In steady flow a picture of the flow field recorded at different times will be identical. In the case of unsteady flow, the properties vary with time or $V = V(x, y, z, t)$, $P = P(x, y, z, t)$ where t is time.

In unsteady flow the appearance of the flow field will vary with time and will be constantly changing. In turbulent flow the velocity at any point fluctuates around a mean value, but the mean value at a point over a period of time is constant. For practical purposes turbulent flow is considered as steady flow as long as the mean value of properties do not vary with time.

5.5 COMPRESSIBLE AND INCOMPRESSIBLE FLOW

If the density of the flowing fluid is the same all over the flow field at all times, then such flow is called incompressible flow. Flow of liquids can be considered as incompressible even if the density varies a little due to temperature difference between locations. Low velocity flow of gases with small changes in pressure and temperature can also be considered as incompressible flow. Flow through fans and blowers is considered incompressible as long as the density variation is below 5%. If the density varies with location, the flow is called compressible flow. In this chapter the study is mainly on incompressible flow.

5.6 LAMINAR AND TURBULENT FLOW

If the flow is smooth and if the layers in the flow do not mix macroscopically then the flow is called laminar flow. For example a dye injected at a point in laminar flow will travel along a continuous smooth line without generally mixing with the main body of the fluid. Momentum, heat and mass transfer between layers will be at molecular level of pure diffusion. In laminar flow layers will glide over each other without mixing.

In turbulent flow fluid layers mix macroscopically and the velocity/temperature/mass concentration at any point is found to vary with reference to a mean value over a time period. For example $u = \overline{u} + u'$ where u is the velocity at an instant at a location and $\overline{u}$ is the average velocity over a period of time at that location and u' is the fluctuating component. This causes

higher rate of momentum/heat/mass transfer. A dye injected into such a flow will not flow along a smooth line but will mix with the main stream within a short distance.

The difference between the flows can be distinguished by observing the smoke coming out of an incense stick. The smoke in still air will be found to rise along a vertical line without mixing. This is the laminar region. At a distance which will depend on flow conditions the smoke will be found to mix with the air as the flow becomes turbulent. Laminar flow will prevail when viscous forces are larger than inertia forces. Turbulence will begin where inertia forces begin to increase and become higher than viscous forces.

5.7 CONCEPTS OF UNIFORM FLOW, REVERSIBLE FLOW AND THREE DIMENSIONAL FLOW

If the velocity value at all points in a flow field is the same, then the flow is defined as uniform flow. The velocity in the flow is independent of location. Certain flows may be approximated as uniform flow for the purpose of analysis, though ideally the flow may not be uniform.

If there are no pressure or head losses in the fluid due to frictional forces to be overcome by loss of kinetic energy (being converted to heat), the flow becomes reversible. The fluid can be restored to its original condition without additional work input. For a flow to be reversible, no surface or fluid friction should exist. The flow in a venturi (at low velocities) can be considered as reversible and the pressures upstream and downstream of the venturi will be the same in such a case. The flow becomes irreversible if there are pressure or head losses.

If the components of the velocity in a flow field exist only in one direction it is called one dimensional flow and $V = V(x)$. Denoting the velocity components in x, y and z directions as u, v and w, in one dimensional flow two of the components of velocity will be zero. In two dimensional flow one of the components will be zero or $V = V(x, y)$. In three dimensional flow all the three components will exist and $V = V(x, y, z)$. This describes the general steady flow situation. Depending on the relative values of u, v and w approximations can be made in the analysis. In unsteady flow $V = V(x, y, z, t)$.

5.8 VELOCITY AND ACCELERATION COMPONENTS

The components of velocity can be designated as

$$u = \frac{dx}{dt},\ v = \frac{dy}{dt} \quad \text{and} \quad w = \frac{dz}{dt}$$

where t is the time and dx, dy, dz are the displacements in the directions x, y, z.

In general as $u = u(x, y, z, t),\ v = v(x, y, z, t)$ and $w = w(x, y, z, t)$

Defining acceleration components as

$$a_x = \frac{du}{dt},\ a_y = \frac{dv}{dt} \text{ and } a_z = \frac{dw}{dt}, \quad \text{as } u = u\,(x, y, z, t)$$

$$a_x = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial u}{\partial t}$$

$$= u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

Similarly, $$a_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}$$

and $$a_z = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}$$

The first three terms in each case is known as convective acceleration terms, because these represent the convective act of moving from one position to another. The last term is known as local accleration term, because the flow at a point is changing with time. Under steady flow conditions, only the convective acceleration terms will exist.

5.9 CONTINUITY EQUATION FOR FLOW—CARTESIAN CO-ORDINATES

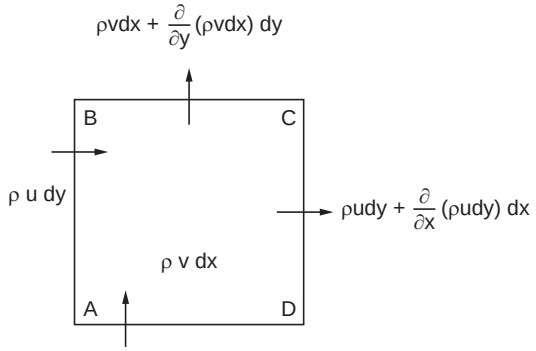


Figure. 5.9.1 Derivation of continuity equation

Consider an element of size dx, dy, dz in the flow as shown in Fig 5.9.1.

Applying the law of conservation of mass, for a given time interval,

The net mass flow into the element through all the surfaces

= The change in mass in the element.

First considering the $y - z$ face, perpendicular to the x direction and located at x, the flow through face during time dt is given by

$$\rho u\, dy\, dz\, dt \tag{5.9.1}$$

The flow through the $y - z$ face at $x + dx$ is given by

$$\rho u\, dy\, dz\, dt + \frac{\partial}{\partial x}(\rho\, u\, dy\, dz\, dt)\, dx \tag{5.9.2}$$

The net mass flow in the x direction is the difference between the quantities given by (5.9.1) and (5.9.2) and is equal to

$$\frac{\partial}{\partial x}(\rho u)\, dx\, dy\, dz\, dt \tag{5.9.3}$$

Similarly the net mass through the faces $z - x$ and $x - y$ in y and z directions respectively are given by

$$\frac{\partial}{\partial x}(\rho v)\, dx\, dy\, dz\, dt \tag{5.9.4}$$

$$\frac{\partial}{\partial x}(\rho w)\, dx\, dy\, dz\, dt \tag{5.9.5}$$

The change in the mass in the control volume equals the rate of change of density × volume × time or

$$\frac{\partial \rho}{\partial t}\, dx\, dy\, dz\, dt \tag{5.9.6}$$

The sum of these quantities should equal zero, cancelling common terms $dx\, dy\, dz\, dt$

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = \frac{\partial \rho}{\partial t} \tag{5.9.7}$$

This is the general equation. For steady flow this reduces to

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \tag{5.9.8}$$

For incompressible flow this becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{5.9.9}$$

Whether a flow is steady can be checked using this equation when the velocity components are specified. For two dimensional steady incompressible flow, the equation reduces to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{5.9.10}$$

For one dimensional flow with varying area, the first term of the general equation alone need be considered. For steady flow

$\frac{\partial(\rho u\, dy\, dz)}{\partial x} = 0$ as $dy\, dz = dA$. Integrating ρuA = constant. or

$$\rho_1 u_1 A_1 = \rho_2 u_2 A_2 \tag{5.9.11}$$

This equation is used to calculate the area, or velocity in one dimensional varying area flow, like flow in a nozzle or venturi.

5.10 IRROTATIONAL FLOW AND CONDITION FOR SUCH FLOWS

Irrotational flow may be described as flow in which each element of the moving fluid suffers no net rotation from one instant to the next with respect to a given frame of reference. In flow along a curved path fluid elements will deform. **If the axes of the element rotate equally towards or away from** each **other, then the flow will be irrotational**. This means that as long as the algebraic average rotation is zero, the flow is irrotational. The idea is illustrated in Fig. 5.10.1

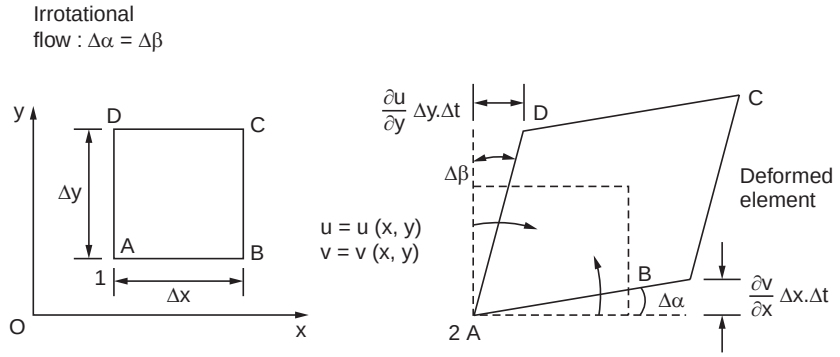


Figure 5.10.1 Rotation in Flow

An element is shown moving from point 1 to point 2 along a curved path in the flow field. At 1 the undeformed element is shown. As it moves to location 2 the element is deformed. The angle of rotation of x axis is given by $(\partial v/\partial y).\ \Delta y.\Delta t$. The angle of rotation of y axis is given by $(\partial u/\partial y).\ \Delta y.\Delta t$. (It is assumed that $\Delta x = \Delta y$. For irrotational flow, the angle of rotation of the axes towards each other or away from each other should be equal *i.e.,* the condition to be satisfied for irrotational flow is,

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \quad \text{or} \quad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \tag{5.10.1}$$

Another significance of irrotational flow is that it is defined by a potential function ϕ for the flow described in para 5.15.

In case there is rotation, then the rotation is given by (with respect to the Z axis in the case of two dimensional flow along x and y)

$$\omega_z = (1/2)\ (\partial v/\partial x - \partial u/\partial y) \tag{5.10.2}$$

and $\omega_z = 0$ for irrotational flow.

5.11 CONCEPTS OF CIRCULATION AND VORTICITY

Considering a closed path in a flow field as shown in Fig. 5.11.1, circulation is defined as the line integral of velocity about this closed path. The symbol used is Γ.

$$\Gamma = \oint_L u \; ds = \oint_L u \; \cos \beta \; dL$$

where dL is the length on the closed curve, u is the velocity at the location and β is the angle between the velocity vector and the length dL.

The closed path may cut across several stream lines and at each point the direction of the velocity is obtained from the stream line, as its tangent at that point.

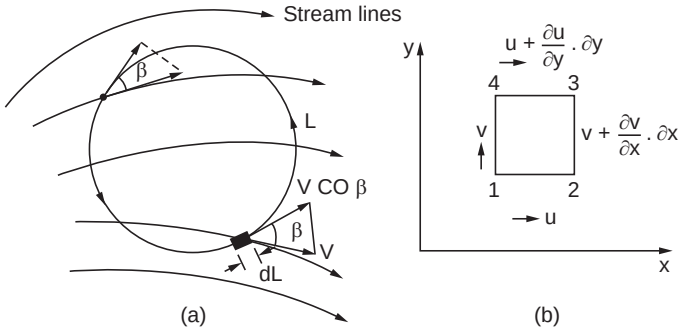


Figure. 5.11.1 Circulation in flow

The integration can be performed over an element as shown in Fig. 5.11.1 (*b*).

In the cartesian co-ordinate if an element $dx.\,dy$ is considered, then the circulation can be calculated as detailed below:

Consider the element 1234 in Fig. 5.3*b*. Starting at 1 and proceeding counter clockwise,

$$d\Gamma = u\,dx + [v+(\partial v/\partial x)dx]dy - [u+(\partial u/\partial y).dy]\,dx - v dy$$

$$= [\partial v/\partial x - \partial u/\partial y]dxdy \qquad (5.11.1)$$

Vorticity is defined as circulation per unit area. i.e.,

Vorticity = circulation per unit area, here area is $dx\,dy$, so

$$\text{Vorticity} = \frac{d\Gamma}{dxdy} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \qquad (5.11.2)$$

For irrotational flow, vorticity and circulation are both zero. In polar coordinates

$$\text{Vorticity} = \frac{\partial v_\theta}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\theta}{r}$$

5.12 STREAM LINES, STREAM TUBE, PATH LINES, STREAK LINES AND TIME LINES

The analytical description of flow velocity is geometrically depicted through the concept of stream lines. The velocity vector is a function of both position and time. If at a fixed instant of time a curve is drawn so that it is tangent everywhere to the velocity vectors at these locations

then the curve is called a stream line. Thus stream line shows the mean direction of a number of particles in the flow at the same instant of time. **Stream lines are a series of curves drawn tangent to the mean velocity vectors of a number of particles in the flow. Since stream lines are tangent to the velocity vector at every point in the flow field, there can be no flow across a stream line.**

A bundle of neighbouring stream lines may be imagined to form a passage through which the fluid flows. Such a passage is called a stream tube. Since the stream tube is bounded on all sides by stream lines, there can be no flow across the surface. Flow can be only through the ends. A stream tube is shown diagrammatically in Figure 5.12.1.

Under steady flow condition, the flow through a stream tube will be constant along the length.

Path line is the trace of the path of a single particle over a period of time. Path line shows the direction of the velocity of a particle at successive instants of time. In steady flow path lines and stream lines will be identical.

Streak lines provide an instantaneous picture of the particles, which have passed through a given point like the injection point of a dye in a flow. In steady flow these lines will also coincide with stream lines.

Path lines and streak lines are shown in Figure 5.12.1.

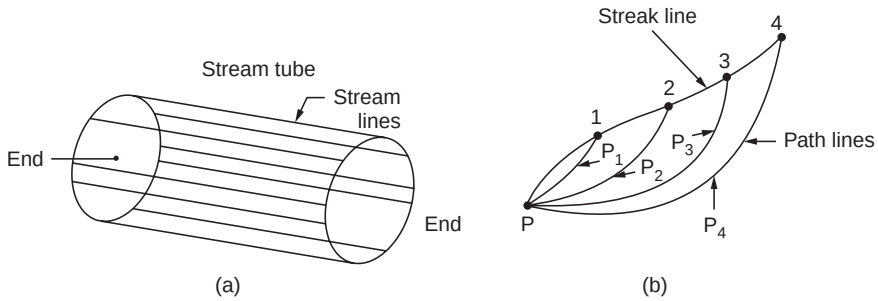


Figure 5.12.1 Stream tube, Path lines and Streak lines

Particles P_1, P_2, P_3, P_4, starting from point P at successive times pass along path lines shown. At the instant of time considered the positions of the particles are at 1, 2, 3 and 4. A line joining these points is the streak line.

If a number of adjacent fluid particles in a flow field are marked at a given instant, they form a line at that instant. This line is called time line. Subsequent observations of the line may provide information about the flow field. For example the deformation of a fluid under shear force can be studied using time lines.

5.13 CONCEPT OF STREAM LINE

In a flow field if a continuous line can be drawn such that the tangent at every point on the line gives the direction of the velocity of flow at that point, such a line is defined as a stream line. In steady flow any particle entering the flow on the line will travel only along this line. This leads to visualisation of a stream line in laminar flow as the path of a dye injected into the flow.

There can be no flow across the stream line, as the velocity perpendicular to the stream line is zero at all points. The flow along the stream line can be considered as one dimensional flow, though the stream line may be curved as there is no component of velocity in the other directions. Stream lines define the flow paths of streams in the flow. The flow entering between two stream lines will always flow between the lines. The lines serve as boundaries for the stream.

In the cartesian co-ordinate system, along the stream line in two dimensional flow it can be shown that

$$\frac{dx}{u} = \frac{dy}{v} \quad \text{or} \quad v\,dx - u\,dy = 0 \tag{5.13.1}$$

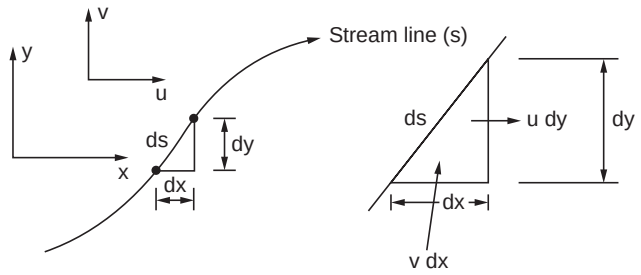


Figure 5.13.1 Velocity components along a stream line

Referring to Fig. 5.5 considering the velocity at a point and taking the distance *ds* and considering its *x* and *y* components as *dx* and *dy*, and noting that the net flow across *ds* is zero,

the flow along *y* direction = $dx\ v$

the flow along *x* direction = $dy\ u$

These two quantities should be equal for the condition that the flow across *ds* is zero, thus proving the equation (5.13.1).

In the next para, it is shown that stream lines in a flow can be described by a stream function having distinct values along each stream line.

5.14 CONCEPT OF STREAM FUNCTION

Refer to Fig. 5.14.1 showing the flow field, co-ordinate system and two stream lines.

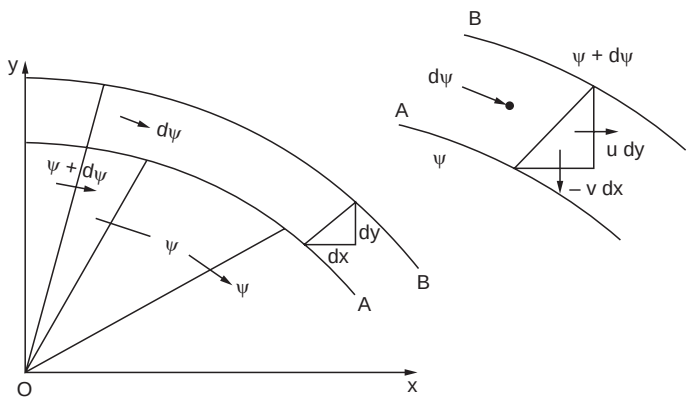


Figure. 5.14.1 Stream function—Definition

Stream function is a mathematical expression that describes a flow field. The definition is based on the continuity principle. It provides a means of plotting and interpreting flow fields. Considering the stream line A in figure, the flow rate across any line joining 0 and any point on A should be the same as no flow can cross the stream line A. Let the slow rate be denoted as ψ. Then ψ is a constant of the streamline A. If ψ can be described by an equation in x and y then stream line A can be plotted on the flow field. Consider another stream line B close to A. Let the flow between stream lines A and B be $d\psi$. The flow across any line between A and B will be $d\psi$. Now taking components in the x and y directions,

$$d\psi = u\, dy - v\, dx \tag{5.14.1}$$

If the stream function ψ can be expressed as $\psi = \psi(x, y)$ (as it has a value at every point) then

$$d\psi = \frac{\partial \psi}{\partial x}\, dx + \frac{\partial \psi}{\partial y}\, dy \tag{5.14.2}$$

and comparing the above two equations, it is seen that

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x} \tag{5.14.3}$$

In the practical point of view equation 5.14.3 can be considered as the definition of stream function.

As a result of the definition, if the stream function for a stream line is known, then the velocity at each point can be determined and vice versa.

If the velocity is expressed for a flow field in terms of x and y then the stream function value can be obtained by integrating equation 5.14.1.

$$\psi = \int \frac{\partial \psi}{\partial x}\, dx + \int \frac{\partial \psi}{\partial y}\, dy + c \tag{5.14.4}$$

The constant provides the difference in flow between various stream lines. By substituting for the values of u and v in the continuity equation 5.9.10 in terms of ψ,

$$\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0 \tag{5.14.5}$$

As the value of the derivative is the same irrespective of the order in which it is taken the continuity equation is automatically satisfied by the stream function.

If the value of stream function is expressed in terms of x and y, stream lines can be plotted and the flow values can also be obtained between the stream lines. There are only a limited number of flows which are simple enough that stream function can be easily obtained. Many real flows can be obtained by the combination of the simple flows. It is also possible to combine two flows and then obtain the stream lines for the combined flow. This technique of superposition is found very useful in the analysis of more complex flows, with complex boundary conditions.

5.15 POTENTIAL FUNCTION

Flow is caused by a driving potential. It will be useful to have an idea of the potential at various locations.

If a fluid flow is irrotational, then equation 5.10.1 is satisfied

i.e., $$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$$

Fluid flows which approximate to this condition are found to be large in number. Converging flows, and flows outside the boundary layer are essentially irrotational.

If this condition is satisfied everywhere in a flow except at a few singular points, it is mathematically possible to define a velocity potential function ϕ as

$$u = -\frac{\partial \phi}{\partial x}, \quad v = -\frac{\partial \phi}{\partial y} \tag{5.15.1}$$

The negative sign indicates that ϕ decreases in the direction of velocity increase.

These partial derivatives are known as potential gradients and give the flow velocity in the direction of the gradient. **Potential functions exist only in irrotational flow whereas stream functions can be written for all flows**. Substituting these in the continuity equation, an equation known as Laplace's equation results. Considering the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial x} = -\frac{\partial^2 \phi}{\partial x^2}, \frac{\partial v}{\partial y} = -\frac{\partial^2 \phi}{\partial y^2}$$

Substituting, $$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{5.15.2}$$

This is similar to heat conduction equation with temperature T replacing ϕ as potential.

Substituting this in equation 5.10.1 (irrotational flow)

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

$$\frac{\partial v}{\partial x} = -\frac{\partial^2 \phi}{\partial x \partial y} \cdot \frac{\partial u}{\partial y} = -\frac{\partial^2 \phi}{\partial y \partial x} \tag{5.15.3}$$

as these two quantities are equal the irrotationality condition is satisfied.

Potential function and stream functions are orthogonal to each other. The proof is given in solved problem 5.1. The method of determination of potential function given the velocities or the stream function is described under solved problems 5.11, 5.12 and 5.13.

5.16 STREAM FUNCTION FOR RECTILINEAR FLOW FIELD (POSITIVE X DIRECTION)

It is often found necessary to analyse flow fields around immersed bodies. The extent of the approaching flow is often large and possesses straight and parallel stream lines, and the velocity distribution is uniform at a distance from the object. Such a flow is termed as **rectilinear flow** and is of practical importance. The flow can be described by the condition, **u = constant and v = 0.**

$$\therefore \qquad \psi = \int \text{const.}\, dy + \int (0)\, dx = c_1 y + c_2$$

where c_1 and c_2 are constants. By applying the boundary at $y = 0$, *i.e.,* coincident with x axis, $c_2 = 0$.

So $y = Cy = uy$ (5.16.1. *a*)

and $\phi = cx = ux$ (5.16.1. *b*)

In polar coordinates

$\psi = ur \sin\theta, \quad \phi = ur \cos\theta$ (5.16.1. *c*)

Considering uniform flow at an angle α with x-axis

$\psi = u(y \cos\alpha - x \sin\alpha), \quad \phi = u(x \cos\alpha + y \sin\alpha)$ (5.16.1. *d*)

The stream lines are shown in Fig. 5.16.1.

The circulation Γ around any closed curve will be zero in this flow (check) Potential function is orthogonal to stream function.

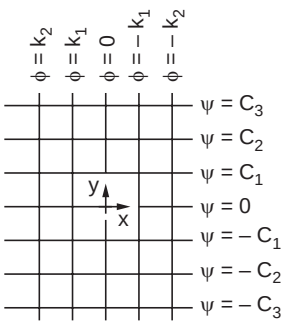


Figure. 5.16.1 Rectilinear flow stream and potential lines

It y distances are equally spaced, with distance 'a' then $C_0 = 0$, $C_1 = ua$, $C_3 = 2ua$ etc.

5.17 TWO DIMENSIONAL FLOWS—TYPES OF FLOW

There are only a few types of flow for which stream and potential functions can be determined directly. For other flows can be generally approximated as combinations of these flows. In this section, the simple flows are described.

5.17.1 Source Flow

A source flow consists of a symmetrical flow field with radial stream lines directed outwards from a common point, the origin from where fluid is supplied at a constant rate q. As the area increases along the outward direction, the velocity will decrease and the stream lines will spread out as the fluid moves outwards. The velocity at all points at a given radial distance will be the same.

The equations describing the flow are:

Velocity at radius r for flow rate of strength q is given by

$$u_r = q/2\pi r \tag{5.17.1}$$

The velocity in the tangential direction is zero

$$u_\theta = 0 \tag{5.17.2}$$

Stream function is represented by

$$\psi = (q/2\pi)\theta \tag{5.17.3}$$

The potential function is represented by

$$\phi = -(q/2\pi) \ln r \tag{5.17.4}$$

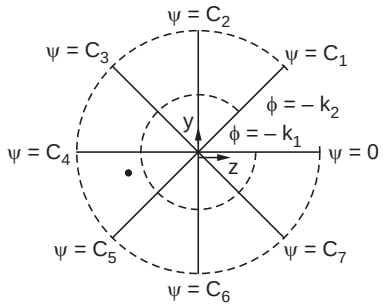


Figure 5.17.1 Potential and stream lines for source flow

The origin is a singular point. The circulation Γ around any closed curve is zero.

The stream lines are shown in Fig. 5.17.1.

Here C_1, C_2 etc are simply $\left(\frac{q}{2\pi}\right)\theta$, where θ is the angle of the stream line.

5.17.2 Sink Flow

Sink is the opposite of source and the radial streamlines are directed inwards to a common point, origin, where the fluid is absorbed at a constant rate. The velocity increases as the fluid moves inwards or as the radius decreases, the velocity will increase. In this case also the velocity at all points at a given radial distance from the origin will be the same. The origin is a singular point. The circulation around any closed curve is zero. The equations describing the flow are

$$u_r = -(q/2\pi r)\ ,\ u_\theta = 0, \tag{5.17.5}$$

$$\psi = -(q/2\pi)\theta\ ,\ \phi = (q/2\pi) \ln r \tag{5.17.6}$$

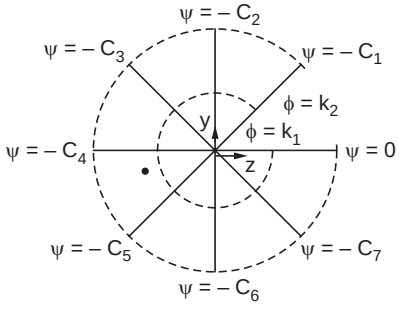


Figure 5.17.2 Stream and potential lines for sink flow

The stream lines are shown in Fig. 5.17.2.

5.17.3 Irrotational Vortex of Strength *K*

(Free vortex, counter clockwise is taken as +ve. The origin is at the centre and is a singular point).

The equations describing the flow are

$$u_r = 0,\ u_\theta = (K/2\pi r) \tag{5.17.7}$$

$$\psi = -(K/2\pi) \ln r,\ \phi = -(K/2\pi)\theta \tag{5.17.8}$$

Circulation $\Gamma = K$ for closed curve enclosing origin and $\Gamma = 0$ for any other closed curves.

In this case the velocity varies inversely with radius. At r = o, velocity will tend to be ∞ and that is why the centre is a singular point.

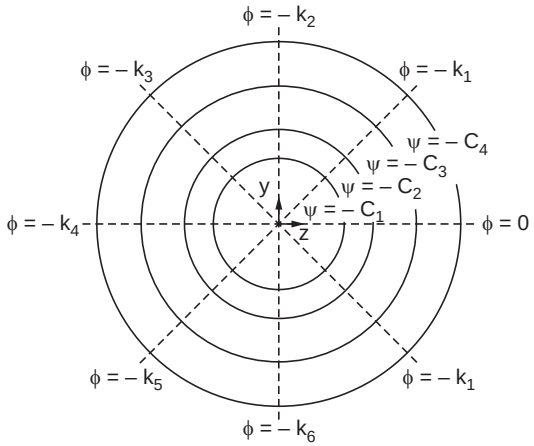


Figure 5.17.3 Irrotational vortex

Forced vortex is discussed in solved problem 5.3.

5.17.4 Doublet of Strength Λ

The centre is at the origin and is a singular point. Such a flow is obtained by allowing a source and sink of equal strengths merge and

$\Lambda = q\ ds/2\pi$, where ds is the distance between them.

The equations describing the flow are

$$u_r = -(\Lambda/r^2)\cos\theta\ ,\ u_\theta = -(\Lambda/r^2)\sin\theta \tag{5.17.9}$$

$$\psi\ = -(\Lambda\sin\theta/r),\ \phi = -(\Lambda\cos\theta/r) \tag{5.17.10}$$

The equation and the plot are for the limiting condition, $ds \to 0$. In this case Λ takes a definite value.

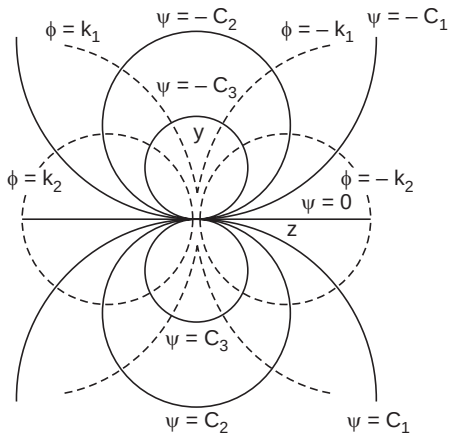


Figure 5.17.4 Potential and stream line for doublet

5.18 PRINCIPLE OF SUPERPOSING OF FLOWS (OR COMBINING OF FLOWS)

Some of the practical flow problems can be more easily described by combination of the simple flows discussed in previous article. For example, if in uniform flow a cylinder like body is interposed, the flow area reduces. The stream lines nearer the body move closer to each other and the flow far removed from the body is still uniform. This flow can be visualised by the combination of uniform flow and a source. The wake flow (behind the body) can be visualised by means of a sink and uniform flow. As equations for stream lines are available for flows like uniform flow, source, sink etc, it is found useful to study such combination of flows.

The simple rule for such a combination of two flows A and B is

$$\psi = \psi_A + \psi_B$$

where ψ describes the combined flow and ψ_A and ψ_B describe the component flows. Similarly $\phi = \phi_A + \phi_B$

Some of the examples follow.

5.18.1 Source and Uniform Flow (Flow Past a Half Body)

The combined stream lines are shown in Fig. 5.18.1.

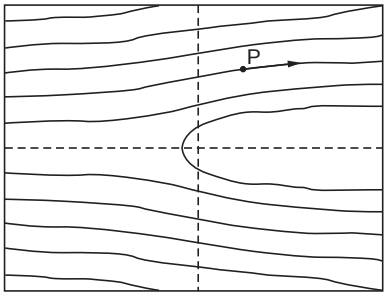


Figure 5.18.1 Source and uniform flow

The velocity in uniform flow along the x direction is u and along y direction is zero. The flow rate of the source is q.

The equations describing the flows are,

For source flow $\psi_1 = (q/2\pi)\,\theta$, For uniform flow $\psi_2 = cy = uy$

$\therefore\ \psi = \psi_1 + \psi_2 = (q/2\pi)\,\theta + uy$, In polar coordinates

$\psi = (q/2\pi)\theta + ur \sin\theta$

For uniform flow $\phi_2 = -ux$, For source flow $\phi_1 = -(q/2\pi) \ln r$, Combining

$$\phi = \phi_1 + \phi_2 = -(q/2\pi) \ln r - ux, \text{ in polar coordinates.}$$

$\phi = -(q/2\pi) \ln r - ur \cos\theta$

5.18.2 Source and Sink of Equal Strength with Separation of 2a Along x-axis

For source flow $\psi_1 = (q/2\pi)\,\theta_1$, for sink flow $\psi_2 = -(q/2\pi)\,\theta_2$,

Combining $\psi = \psi_1 + \psi_2 = (q/2\pi)\theta_1 - (q/2\pi)\theta_2 = (q/2\pi)(\theta_1 - \theta_2)$

Similarly using

$\phi_1 = -(q/2\pi) \ln r_1$ and $\phi_2 = (q/2\pi) \ln r_2$

$\phi = \phi_1 + \phi_2 = (q/2\pi) \ln (r_2/r_1)$

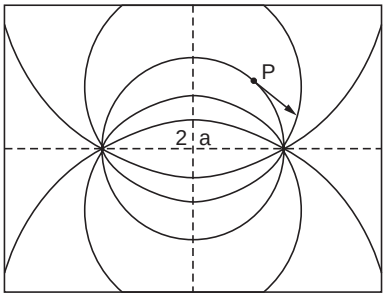


Figure 5.18.2 Source and sink of equal strength

5.18.3 Source and Sink Displaced at 2a and Uniform Flow (Flow Past a Rankine Body)

In this case refer para 5.18.1 and 2,

$$\psi_1 = (q/2\pi)\,(\theta_1 - \theta_2),\ \psi_2 = uy$$

$$\therefore \quad \psi = (q/2\pi)(\theta_1 - \theta_2) + uy$$

$$= (q/2\pi)(\theta_1 - \theta_2) + ur \sin\theta$$

$$\phi_1 = (q/2\pi) \ln (r_2/r_1) \text{ and } \phi_2 = -ux$$

$$\therefore \quad \phi = (q/2\pi) \ln (r_2/r_1) - ur \cos\theta$$

here r is the distance from the origin to the point and θ is the angle made by this line with x axis.

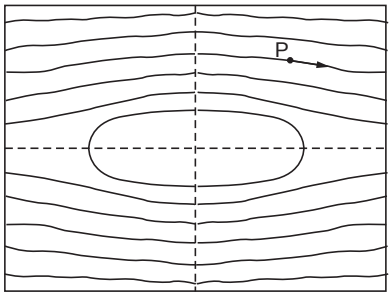


Figure 5.18.3 Source, sink and uniform flow

5.18.4 Vortex (Clockwise) and Uniform Flow

Refer results of section 5.17.3 for the vortex $\psi = (K/2\pi) \ln r$ (clockwise)

For uniform flow

$$\psi = uy$$

$$\therefore \quad \psi = (K/2\pi) \ln r + uy,$$

In polar coordinates,

$$\psi = (K/2\pi) \ln r + ur \sin\theta$$

For vortex $\phi_1 = (K/2\pi)\theta$, For uniform flow

$$\phi_2 = -ux$$

$$\therefore \quad \phi = (K/2\pi)\theta - ux$$

In polar coordinates,

$$\phi = (K/2\pi)\theta - ur \cos\theta$$

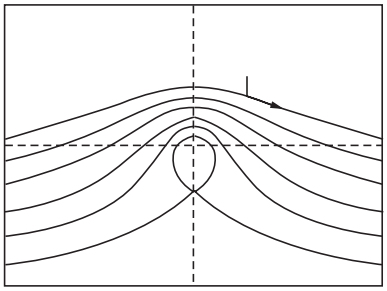


Figure 5.18.4 Vortex and uniform flow

5.18.5 Doublet and Uniform Flow (Flow Past a Cylinder)

Refer results of para 5.17.4. For doublet $\psi_1 = \Lambda \sin\theta/r$,

For uniform flow $\psi_2 = uy = ur \sin\theta$

$$\therefore \quad \psi = (\Lambda \sin\theta/r) + ur \sin\theta$$

defining $a^2 = \Lambda/u$, $\psi = ur\,[1 - (a^2/r^2)] \sin\theta$

$$\phi_1 = -(\Lambda \cos\theta/r),$$

$$\phi_2 = -ux = -ur \cos\theta$$

$$\therefore \quad \phi = -ur\,[1 + (a^2/r^2)] \cos\theta$$

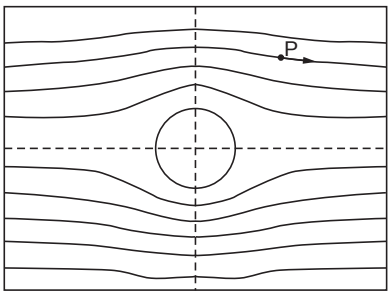


Figure 5.18.5 Flow past a cylinder

5.18.6 Doublet, Vortex (Clockwise) and Uniform Flow

Refer results of para 15.18.5 and 15.17.3

$$\psi = ur\,[1 - (a^2/r^2)] \sin\theta + (K/2\pi) \ln r$$

$$\phi = -ur\,[1 + (a^2/r^2)] \cos\theta + (K/2\pi)\theta$$

where $a^2 = \Lambda/u$, and for $K < 4\pi au$

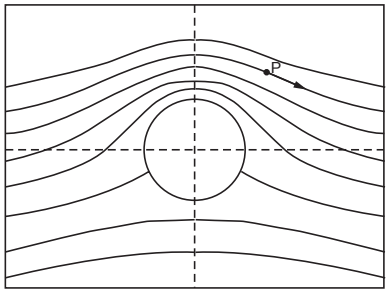


Figure 5.18.6 Doublet, vortex and uniform flow

5.18.7 Source and Vortex (Spiral Vortex Counterclockwise)

Refer results of para 5.17.1 and 3

$$\psi = (q/2\pi)\,\theta - (K/2\pi)\ln r$$

$$\phi = -(q/2\pi)\ln r - (K/2\pi)\,\theta$$

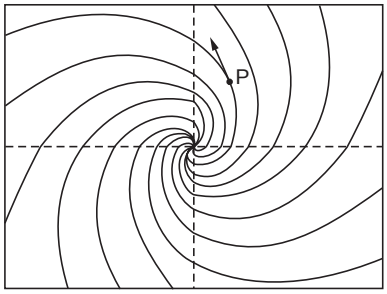


Figure 5.18.7 Source and vortex

5.18.8 Sink and Vortex (Spiral Vortex Counterclockwise)

Refer results of para 15.17.2 and 3

$$\psi = -(q/2\pi)\theta - (K/2\pi)\ln r$$

$$\phi = (q/2\pi)\ln r - (K/2\pi)\theta$$

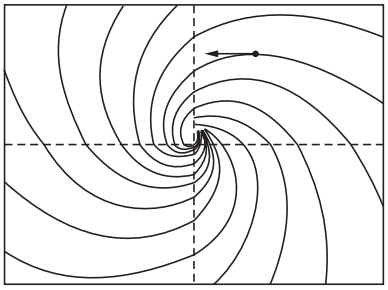


Figure 5.18.9 Sink and vortex

5.18.9 Vortex Pair (Equal Strength, Opposite Rotation, Separation by 2a)

Refer results of para 15.17.3

$$\psi = (K/2\pi)\ln(r_2/r_1),$$

$$\phi = (K/2\pi)(\theta_2 - \theta_1)$$

Many more actual problems can be modelled by the use of this basic principle.

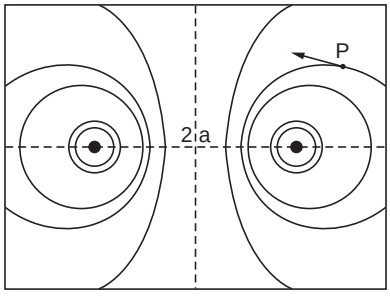


Figure 5.18.8 Vortex pair

5.19 CONCEPT OF FLOW NET

The plot of stream lines and potential flow lines for a flow in such a way that these form curvilinear squares is known as flow net. The idea that stream lines and potential lines are orthogonal is used in arriving at the plot.

Such a plot is useful for flow visualisation as well as calculation of flow rates at various locations and the pressure along the flow. The lines can be drawn by trial or electrical or magnetic analogue can also be used.

An example is shown in Fig. 5.19.1 for flow through a well rounded orifice in a large tank. The flow rate along each channel formed by the stream lines will be equal. The pressure drop between adjacent potential lines will also be equal.

With the advent of computer softwares for flow analysis, the mechanical labour in the plotting of such flow net has been removed. However the basic idea of flow net is useful.

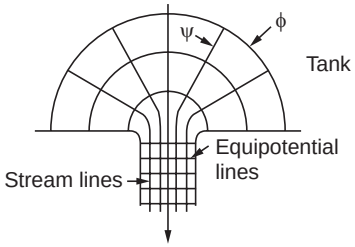


Figure 5.19.1 Flow Net

SOLVED PROBLEMS

Problem 5.1. *Prove that the stream function and potential function lead to orthogonality of stream lines and equipotential flow lines.*

$$\psi = \psi(x, y)$$

$$\therefore \quad d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy$$

Substituting from definition of $\frac{\partial \psi}{\partial x}$ and $\frac{\partial \psi}{\partial y}$ as $-v$ and u

$$\therefore \quad \partial\psi = -vdx + udy$$

as ψ is constant along a stream line $d\psi = 0$,

$$\therefore \quad vdx = udy$$

The slope of the stream line at this point is thus given by

$$\frac{dy}{dx} = \frac{v}{u} \tag{1}$$

Similarly, $\phi = \phi(x, y)$

$$\therefore \quad d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy$$

Substituting for $\frac{\partial \phi}{\partial x}$ and $\frac{\partial \phi}{\partial y}$ as $-u$ and $-v$

$$d\phi = -udx - vdy$$

as $d\phi = 0$ along an equipotential line. $udx = -vdy$

$$\therefore \quad \frac{\partial y}{\partial x} = -u/v \tag{2}$$

These values of slopes show that the two sets of lines are perpendicular to each other. Hence stream lines and equipotential lines are orthogonal.

Problem 5.2. *Determine the stream function in the case of free vortex.*

In free vortex flow the stream lines are concentric circles around the singular point. The flow is irrotational except at the singularity. This condition can be used to show that $v_t\, r$ = constant where v_t is the tangential velocity. This is done by considering the circulation around the element going along 1234.

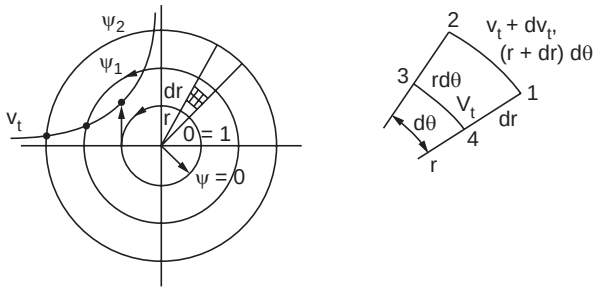


Figure P. 5.2

$$d\Gamma = (v_t + dv_t)\,(r + dr)\,d\,\theta + 0 - v_t\, r d\,\theta + 0 = 0$$

neglecting second order terms and simplifying

$$v_t.\, dr + r\, dv_t = d\,(rv_t) = 0 \quad \text{or} \quad v_t\, r = \text{constant for the flow}$$

This relationship holds at all locations except the centre (singular point)

The circulation along any streamline can be calculated by the usual procedure of

$$\oint_L v\, dL \quad \text{and} \quad \Gamma = 2\pi r v_t$$

As $v_t\, r$ = constant for the flow, circulation is constant for the vortex and Γ is known as vortex strength. The stream function can be determined by integration

$$v_t = \frac{\Gamma}{2\pi r} = \frac{-\partial\psi}{\partial r},\ v_r = 0 \qquad \text{(radial velocity)}$$

$$\therefore \qquad \psi = \int -\frac{\Gamma}{2\pi r}\, dr + \int (0) r d\theta + C$$

taking that $\psi = 0$ for the stream line at r =1

$$\psi = \frac{\Gamma}{2\pi} \ln r \qquad (1)$$

For a clockwise vortex

$$\psi = \frac{\Gamma}{2\pi} \ln r$$

Problem 5.3. *Determine the stream function for a forced vortex.*

A forced vortex is obtained by rotating the fluid as a whole. The flow is characterised by the equation $v_t = -\omega r$ and $v_r = 0$ (– sign for clockwise vortex)

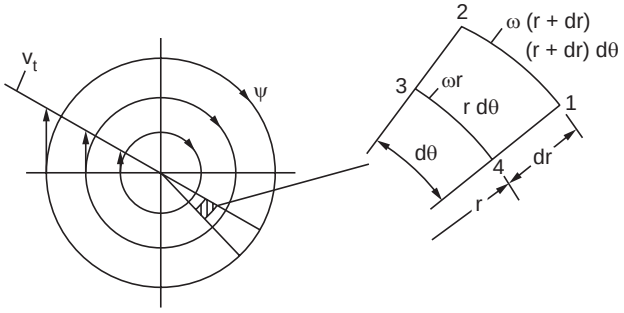


Figure P. 5.3

$$\therefore \qquad \psi = \int (\omega r)\, dr + \int (0)\, rd\, \varphi + C = \omega r^2/2 + C$$

taking $\psi = 0$ for the stream line at $r = 0$, C will vanish and so $\psi = \omega r^2/2$

An important aspect of the flow is that the flow is rotational

This can be shown by considering an element in the flow as shown in Fig. P.5.3 and calculating the circulation and then vorticity

$$d\,\Gamma = -\,\omega\,(r + dr)\,(r + dr)\,d\,\theta + 0 + \omega\, r\, rd\,\theta + 0 = -\,2\,\omega\, rd\, rd\,\theta$$

as vorticity = $d\,\Gamma$/area and as area = $r\,d\theta\,dr$

Vorticity = $-\,2\,\omega$. Hence flow is rotational. The vorticity is directly related to the angular velocity of the mass.

Problem 5.4. *In a two dimensional flow the x and y directional velocities u and v are given by*

$$u = -\frac{x}{x^2 + y^2}, \qquad v = -\frac{y}{x^2 + y^2}$$

1. *Show that the flow is steady and* 2. *Check whether the flow is irrotational*

To check for steady flow, the continuity equation should be used.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial x} = -\left[\frac{(x^2 + y^2) - 2x^2}{(x^2 + y^2)^2}\right] = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$\frac{\partial v}{\partial y} = -\left[\frac{(x^2 + y^2) - 2y^2}{(x^2 + y^2)^2}\right] = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

The sum is zero and this satisfies the continuity equation and so the flow is steady. To check for irrotationality, the condition to be satisfied is,

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \quad \frac{\partial v}{\partial x} = \frac{2xy}{(x^2 + y^2)^2}\ , \ \frac{\partial u}{\partial y} = \frac{2xy}{(x^2 + y^2)^2},$$

The difference is zero and hence the flow is irrotational.

Problem 5.5. *Check whether the following velocity relations satisfy the requirements for steady irrotational flow.*

(*i*) $u = x + y$, $v = x - y$ (*ii*) $u = xt^2 + 2y$, $v = x^2 - y\,t^2$ (*iii*) $u = xt^2$, $v = xyt + y^2$

To check for steady flow use continuity equation:

(*i*) $\frac{\partial u}{\partial x} = 1, \frac{\partial v}{\partial y} = -1$ $\therefore$ $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ So the flow is steady

(*ii*) $\frac{\partial u}{\partial x} = t^2, \frac{\partial v}{\partial y} = -t^2$ $\therefore$ satisfies the continuity equation and flow is steady

(*iii*) $\frac{\partial u}{\partial x} = t^2, \frac{\partial v}{\partial y} = xt + 2y$

This does not satisfy the requirements for steady flow

To Check for irrotational flow: $\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0$

(*i*) $\frac{\partial u}{\partial y} = 1, \frac{\partial v}{\partial x} = 1$ $\therefore$ flow is irrotational

(*ii*) $\frac{\partial u}{\partial y} = 2, \frac{\partial v}{\partial x} = 2x$ $\therefore$ flow is not irrotational

(*iii*) $\frac{\partial u}{\partial y} = 0, \frac{\partial v}{\partial x} = yt$ $\therefore$ flow is not irrotational

Problem 5.6. *Check whether the following flows are (i) steady and (ii) irrotational*

(i) $u = 2y$, $v = -3x$, *(ii)* $u = 3xy$, $v = 0$, *(iii)* $u = -2x$, $v = 2y$

(*a*) *Check for steady flow:* $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

(*i*) $\frac{\partial u}{\partial x} = 0, \frac{\partial v}{\partial y} = 0$ So steady flow prevails

(*ii*) $\frac{\partial u}{\partial x} = 3y, \frac{\partial v}{\partial y} = 0$ This is not a steady flow

(*iii*) $\frac{\partial u}{\partial x} = -2, \frac{\partial v}{\partial y} = 2$ So steady flow prevails

(*b*) *Check for irrotational flow:* $\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0$

(*i*) $\frac{\partial u}{\partial y} = 2, \frac{\partial v}{\partial x} = -3$ Hence not irrotational

(*ii*) $\frac{\partial u}{\partial y} = 3x, \frac{\partial v}{\partial x} = 0$ Hence not irrotational

(*iii*) $\frac{\partial u}{\partial y} = 0, \frac{\partial v}{\partial x} = 0$ Hence irrotational

Problem 5.7. *The stream function for a flow is given by* $\psi = xy$. *Is the flow irrotational? Determine (i) u, v (ii) the vorticity and (iii) circulation.*

(*i*) From the definition stream function, $u = \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial x}$, as $\psi = xy$,

$$v = -\frac{\partial \psi}{\partial x} = -y, \; u = \frac{\partial \psi}{\partial y} = x$$

$\therefore$ $u = x, \quad v = -y$

For irrotational flow

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}, \quad \frac{\partial u}{\partial y} = 0, \; \frac{\partial v}{\partial x} = 0, \; \therefore \text{Flow is irrotational}$$

(*ii*) vorticity and (*iii*) circulation will be zero for irrotational flow.

Problem 5.8. *Describe the method of determination of the stream function given the velocity relationship and also determine the stream function given*

$$u = 4xy \quad and \quad v = c - 2y^2$$

The method used for the determination of stream function is described below

(1) First check for continuity

$$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0,$$

(2) Let $u = f_1(x, y)$

$\therefore$ $u = \frac{\partial \psi}{\partial y} = f_1(x, y)$

$\therefore$ $\psi = \int f_1(x, y)\, dy + f(x)$

where the second terms is a function of x only

(3) Let $-v = \frac{\partial \psi}{\partial x} = f_2(x, y)$, using equation 1 determine the derivative.

$$\frac{\partial \psi}{\partial x} = \frac{\partial}{\partial x} \left[\int f_1(x, y)\, dy \right] + f'(x) = f_2(x, y) = -v$$

comparing the terms with $f_2(x, y)$, $f'(x)$ can be obtained

(4) $$\psi = \int f(x)dx + \int f_1(x, y)dy + \text{constant}$$

(1) $u = 4xy$, $v = c - 2y^2$

$$\frac{\partial u}{\partial x} = 4y, \frac{\partial v}{\partial y} = -4y, \quad \therefore \text{ continuity is satisfied}$$

(2) $$u = \frac{\partial \psi}{\partial y} = 4xy$$

$\therefore$ $$\psi = \int 4xy\, dy = 2xy^2 + f(x) \qquad \text{(A)}$$

where $f(x)$ is a function of x only

(3) $$\frac{\partial \psi}{\partial x} = -v = 2y^2 - c$$

$$= 2y^2 + f'(x) \qquad \text{(Using equation } A\text{)}$$

Differentiating equation (A) *w.r.t.* x and comparing $f'(x) = -c, \quad f(x) = -cx$

(4) Now substitute for $f(x)$ in A

$$\psi = 2xy^2 - cx + \text{constant} \qquad \text{(B)}$$

Check (use equation *B)*

$$u = \frac{\partial \psi}{\partial y} = 4xy, v = -\frac{\partial \psi}{\partial x} = -2y^2 + c$$

Problem 5.9. *Determine the stream function given,* $u = 2x + y$, $v = x - 2y$

(1) Check for continuity

$$\frac{\partial u}{\partial x} = 2, \frac{\partial v}{\partial y} = -2 \quad \therefore \quad \text{Satisfies continuity}$$

(2) $$\frac{\partial \psi}{\partial y} = u = 2x + y$$

$$\psi = \int (2x + y)\, dy + f(x)$$

$$= 2xy + (y^2/2) + f(x) \qquad \text{(A)}$$

(3) Using A, $$\frac{\partial \psi}{\partial x} = 2y + f'(x) = -v = 2y - x$$

$$f'(x) = -x, f(x) = -x^2/2$$

$\therefore$ $$\psi = 2xy + (y^2/2) - (x^2/2) + \text{Constant}$$

Check $$u = \frac{\partial \psi}{\partial y} = 2x + y, v = -\frac{\partial \psi}{\partial x} = -2y + x$$

Problem 5.10. *Explain how the validity of a given potential function ϕ is established. Validate the potential function given as (i) $\phi = y^2 - x^2$ (ii) $\phi = xy$*

A potential function should satisfy the laplace equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

It should also satisfy the condition for irrotational flow

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$$

Case (i) $$\frac{\partial \phi}{\partial x} = -2x, \frac{\partial^2 \phi}{\partial x^2} = -2, \frac{\partial \phi}{\partial y} = 2y, \frac{\partial^2 \phi}{\partial x^2} = 2$$

Hence Laplace equation is satisfied. To check for irrotational flow

$$u = -\frac{\partial \phi}{\partial x} = -\frac{\partial (y^2 - x^2)}{\partial x} = 2x, \qquad \therefore \quad \frac{\partial u}{\partial y} = 0$$

$$v = -\frac{\partial \phi}{\partial y} = -\frac{\partial (y^2 + x^2)}{\partial y} = -2y, \qquad \therefore \quad \frac{\partial v}{\partial x} = 0$$

So the flow is irrotational and hence the function is valid.

Case (ii) $$\phi = xy, \frac{\partial \phi}{\partial x} = y \quad \therefore \quad \frac{\partial^2 \phi}{\partial x^2} = 0, \frac{\partial \phi}{\partial y} = x, \quad \therefore \quad \frac{\partial^2 \phi}{\partial y^2} = 0$$

Hence valid. $$u = -\frac{\partial \phi}{\partial x} = -y \quad \therefore \quad \frac{\partial u}{\partial y} = -1, v = -\frac{\partial \psi}{\partial y} = -x \quad \therefore \quad \frac{\partial v}{\partial x} = -1$$

Hence irrotationality is satisfied. The function is a valid potential function.

Problem 5.11. *Explain how the potential function can be obtained if the stream function for the flow is specified.*

(1) Irrotational nature of the flow should be checked first. Stream function may exist, but if the flow is rotational potential function will not be valid.

(2) The values of u and v are obtained from the stream function as

$$\frac{\partial \psi}{\partial y} = u \text{ and } \frac{\partial \psi}{\partial x} = -v$$

(3) From the knowledge of u and v, ϕ can be determined using the same procedure as per the determination of stream function

$$u = -\frac{\partial \phi}{\partial x}$$

$$\therefore \qquad \phi = -\int u dx - f(y) \qquad \text{(A)}$$

where $f(y)$ is a function of y only

$\frac{\partial \phi}{\partial y}$ is determined and equated to $-v$

Comparing $f'(y)$ is found and then $f(y)$ is determined and substituted in equation A

$$\phi = -\int u dx - f(y) + C$$

Problem 5.12. *For the following stream functions, determine the potential function*

(i) $\psi = (3/2)(x^2 - y^2)$ *(ii)* $\psi = -8xy$ *(iii)* $\psi = x - y$

(*i*) $$u = \frac{\partial \psi}{\partial y} = -3y\,, \quad u = -3y, \quad -v = \frac{\partial \psi}{\partial x} = 3x, \quad v = -3x$$

To check for irrotationality $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$, here both are – 3, so checks

(2) $$u = -3y, \quad \text{also} \quad u = -\frac{\partial \phi}{\partial x} \quad \therefore \quad \phi = \int 3y dx + f(y)$$

$$\phi = 3xy + f(y) \qquad (A)$$

Differentiating equation A with respect to y and equating to v,

$$\frac{\partial \phi}{\partial y} = 3x + f'y = -v = 3x$$

$\therefore$ $f'(y) = 0$ and so $f(y) =$ constant

Substituting in A, $\phi = 3xy +$ constant

check $$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \ \frac{\partial \phi}{\partial y} = 3y, \ \frac{\partial^2 \phi}{\partial x^2} = 0\,, \text{ So also } \frac{\partial^2 \phi}{\partial y^2} = 0$$

So checks, $$u = -\frac{\partial \phi}{\partial x} = -3y, \ v = -\frac{\partial \phi}{\partial y} = -3x \text{ also checks,}$$

(*ii*) $$\psi = -8xy$$

$$u = \frac{\partial \psi}{\partial y} = -8x \quad \therefore \quad u = -8x, \quad -v = \frac{\partial \psi}{\partial x} = -8y$$

$$\therefore \quad v = 8y$$

Check for irrotationality : $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$, in this case both are zero. Hence flow is irrotational.

$$\frac{\partial \phi}{\partial x} = -u = 8x \quad \phi = \int 8x dx + f(y) = 4x^2 + f(y)$$

differentiating this expression with respect to y, $\frac{\partial \phi}{\partial y} = f'(y) = -v = -8y$

$\therefore$ $f(y) = -4y^2$ $\therefore$ $\phi = 4x^2 - 4y^2$

Check for countinuity, Laplace equation etc. as an exercise. Also calculate u and v from this expression and check

(*iii*) $\psi = x - y,$

$$u = \frac{\partial \psi}{\partial y} = -1, \quad v = -\frac{\partial \psi}{\partial x} = -1 \quad \therefore \quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = 0,$$

Hence flow is irrotational.

$$\frac{\partial \phi}{\partial x} = -u = 1, \quad \phi = \int 1\, dx + f(y) = x + f(y)$$

Differentiating *w.r.t.* y,

$$\frac{\partial \phi}{\partial y} = f'(y) = -v = 1 \qquad \therefore \quad f(y) = y$$

$\therefore$ $\phi = x + y$ (Check for other conditions)

Note: In all cases, a constant can be added to the function ψ as well as ϕ.

A similar procedure is to be adopted to obtain stream function when potential function is specified. In this case stream function will always exist. Only in the case of potential function, the condition for irrotationality is to be checked.

Problem 5.13. *Given that $u = x^2 - y^2$ and $v = -2xy$, determine the stream function and potential function for the flow.*

Check for continuity;

$$\frac{\partial u}{\partial x} = 2x, \ \frac{\partial v}{\partial y} = -2x.$$

Hence satisfies the condition $\dfrac{\partial v}{\partial y} + \dfrac{\partial u}{\partial x} = 0$

Check for rotation:

$$\frac{\partial u}{\partial y} = -2y, \ \frac{\partial v}{\partial x} = -2y \qquad \therefore \quad \text{flow is irrotational}$$

(*i*) To determine the stream function

$$u = \frac{\partial \psi}{\partial y} = x^2 - y^2 \quad \psi = x^2 y - (y^3/3) + f(x)$$

Differentiating this expression w.r.t. x,

$$\frac{\partial \psi}{\partial x} = 2xy + f'(x) = -v = 2xy$$

$\therefore$ $f'(x) = 0$ and $f(x) =$ constant.

$\therefore$ $\psi = x^2 y - y^3/3 +$ constant

Check: $\dfrac{\partial \psi}{\partial y} = x^2 - y^2 = u$, checks. $\dfrac{\partial \psi}{\partial x} = 2xy = -v$, checks

Also $\dfrac{\partial^2\psi}{\partial x^2} + \dfrac{\partial^2\psi}{\partial y^2} = 0$ *i.e.*, $2y - 2y = 0$

(*ii*) To determine the potential function

$$\frac{\partial\phi}{\partial x} = -u = -x^2 + y^2\,,\ \phi = -x^3/3 + y^2x + f(y)$$

Differentiating w.r.t. y,

$$\frac{\partial\psi}{\partial y} = 2xy + f'(y) = -v = 2xy$$

$\therefore$ $f'(y) = 0$ and $f(y)$ = constant

$\therefore$ $\phi = -x^3/3 + y^2x + c$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = -2x + 2x = 0.$$ Hence checks.

(also check calculating u, v) and for orthogonality.

Problem 5.14. *Derive an expression for the stream function for (i) uniform flow of 10 m/s along the x direction (ii) uniform flow of 5 m/s parallel to the negative y direction (iii) the combination of the two.*

(*i*) When the flow is uniform along the x direction with velocity u

$\psi_1 = uy$ and in polar coordinates $\psi_1 = ur\cos\theta$ $\quad\therefore\quad \psi_1 = 10y$

(*ii*) For uniform flow along negative y direction with velocity v

$$\psi_2 = -(-vx) = vx = 5x$$

(*iii*) Combining ψ_1 and ψ_2, $\psi = 10y + 5x$

The combined streamlines are shown in Fig. P. 5.14.

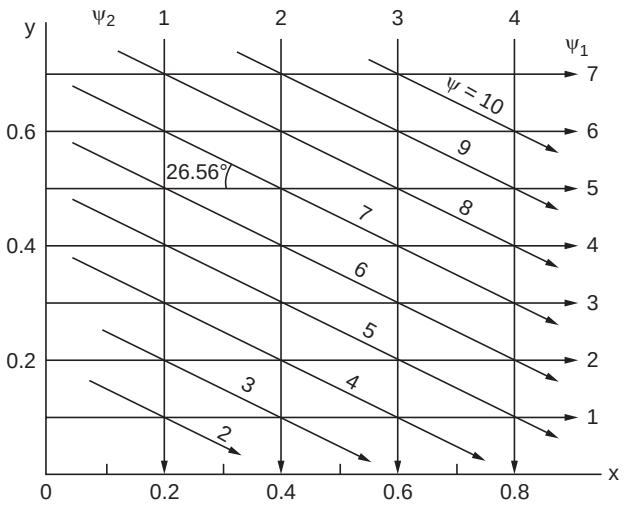


Figure P. 5.14

At any point the resultant velocity is

$$(u^2 + v^2)^{0.5} = (10^2 + 5^2)^{0.5} = 11.18 \text{ m/s}$$

The direction is given by $\theta = \tan^{-1}(5/10) = 26.6^0$ to the x axis

Poblem 5.15. *Determine the stream function for a uniform flow in the negative x direction towards the origin at 5m/s combined with a sink flow of strength 12.*

Show the resulting stream lines.

For uniform flow $\psi_1 = -uy = -5y$,

For the sink $\psi_2 = -12\ \theta/2\pi$

For the combined stream lines

$$\psi = \psi_1 + \psi_2 = -5y - (12\theta/2\pi)$$

The combined flow is shown in Fig. P. 5.15

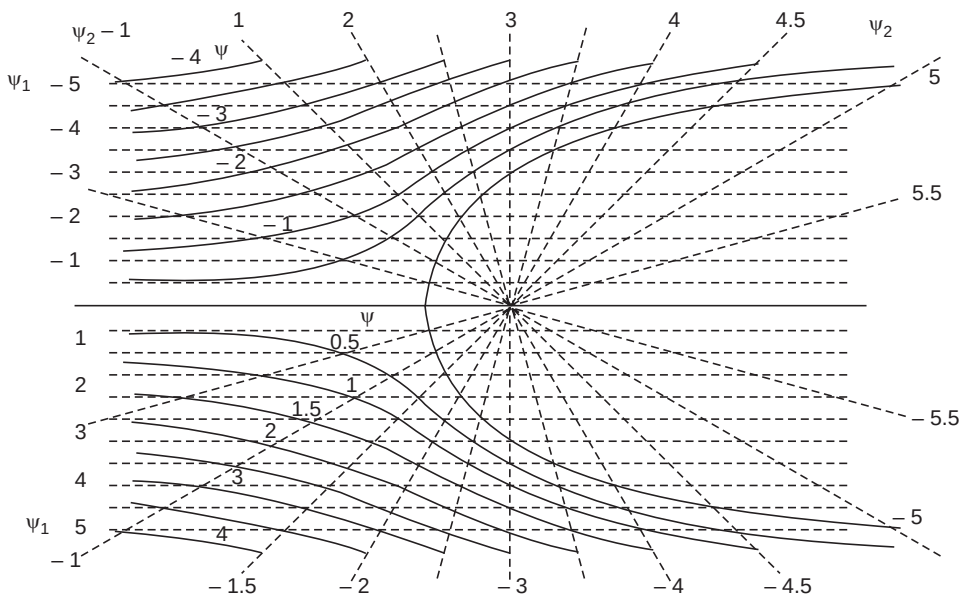


Figure P. 5.15

Problem 5.16. *Determine the stream function for a uniform x directional flow towards the origin from the positive x direction at 5m/s and a source of strength 12 m*

For uniform flow $\psi_1 = -uy = -5y$,

For the sink $\psi_2 = 12\ \theta/2\pi$

For the combined stream lines

$$\psi = \psi_1 + \psi_2 = -5y + (12\theta/2\pi)$$

The combined flow is shown in Fig. P. 5.16

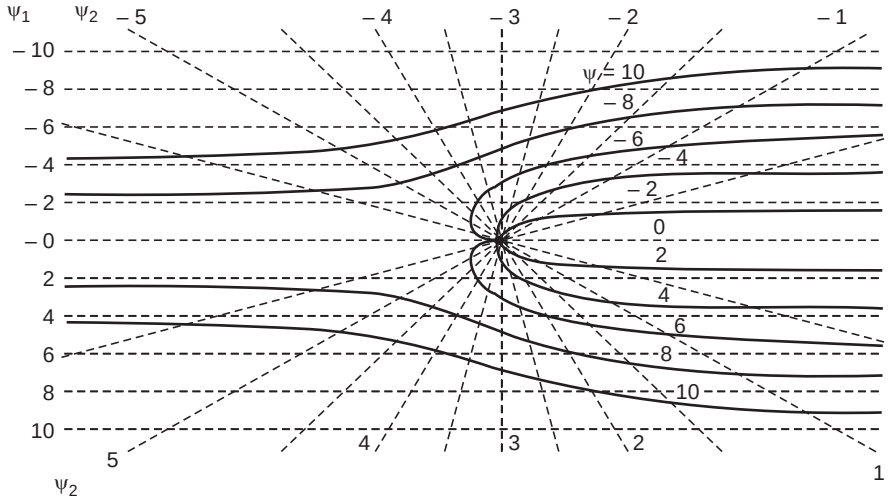


Figure P. 5.16

Problem 5.17. The velocity components at point (2, 2) is specified by the equation $u = x^2 + 3y$ and $v = -2xy$. Determine the accelerations and vorticity at this point.

$$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial u}{\partial t} = (x^2 + 3y)(2x) + (-2xy)\,3 + 0$$

$$= 2x^3 + 6xy - 6xy = 2x^3 = 16 \text{ units}$$

$$a_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial v}{\partial t} = (x^2 + 3y)(-2y) + (-2xy)(-2x) + 0$$

$$= -2x^2y - 6y^2 + 4x^2y = 2x^2y - 6y^2$$

$$= 2 \times 8 - 6 \times 4 = -8 \text{ units}$$

$$w_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}(-2y - 3) = -(y + 1.5)$$

$$= -3.5 \text{ units}$$

Problem 5.18. *In a two dimensional flow, determine a possible x component given* $v = 2y^2 + 2x - 2y$. *Assume steady incompressible flow.*

The continuity equation is $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0$

$$\therefore \qquad \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} = -[4y - 2] = 2 - 4y.$$

$$\therefore \qquad u = \int (2 - 4y)\,dx = 2x - 4xy + f(y)$$

There are numerous possibilities for $f(y)$.

One possibility is $f(y) = 0$.

$\therefore \quad u = 2x - 4xy.$

Problem 5.19. *The velocity components in a flow are given by $u = 4x$, $v = -4y$. Determine the stream and potential functions. Plot these functions for $\psi = 60, 120, 180,$ and 240 and $\phi = 0, -60, -120, -180, +60, +120, +180$. Check for continuity*

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 4 - 4 = 0 \text{ checks}$$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

flow is irrotational and so ϕ exists

$$\psi = \int 4x\, dy + f(x) = 4xy + f(x)\text{, differentiating w.r.t. } y$$

$$\frac{\partial \psi}{\partial y} = -v = -(-4y) = 4y + f'(x) \quad \therefore \quad f'(x) = \text{constant}$$

$$\psi = 4xy + \text{constant or } \psi = 4xy$$

$$\phi = \int -4x\, dx + f(y) = -2y^2 + f(y),$$

$$\frac{\partial \psi}{\partial y} = -v = 4y = f'(y) \quad \therefore \quad f(y) = 2y^2$$

$$\phi = 2y^2 - 2x^2 + c \quad \text{or} \quad \phi = 2y^2 - 2x^2$$

To plot the stream function, the values of y are calculated for various values of x, using $\psi = 4xy$ or $y = \psi/4x$. The calculated values of y for $x = 1$ to 15 and $\psi = 60$ to 240 are tabulated below.

x / ψ	1	2	3	4	5	10	15
60.000	15.00	7.50	5.00	3.75	3.00	1.50	1.00
120.00	30.00	15.00	10.00	7.50	6.00	3.00	2.00
180.00	45.00	22.50	15.00	11.25	9.00	4.50	3.00
240.00	60.00	30.00	20.00	15.00	12.00	6.00	4.00

These values are shown plotted in Fig P. 5.19

To plot the potential function the values of x or y are calculated for given values y or x using

$$x = \pm\sqrt{y^2 - (\phi/2)} \quad \text{or} \quad y = \pm\sqrt{x^2 + (\phi/2)}$$

The values of y so calculated are tabulated below.

The range used is $x \rightarrow 1$ to 15, $\phi \rightarrow 0, -60, -120, -180$.

x / ϕ	1	2	3	4	5	10	15
– 60.0	5.57	5.83	6.25	6.75	7.42	11.4	15.97
– 120.0	7.81	8.00	8.31	8.72	9.22	12.65	16.88
– 180.0	9.54	9.70	9.95	10.30	10.72	13.78	17.75

y / ϕ	0	1	2	3	4	5	10	15
60.00	5.48	5.57	5.83	6.25	6.75	7.42	11.4	15.97
120.00	7.75	7.81	8.00	8.31	8.72	9.22	12.65	16.88
180.00	9.49	9.54	9.70	9.95	10.30	10.72	13.78	17.75

(when values of x or y exceeded 15, then the corresponding values of y or x for exact values of 15 is calculated and used)

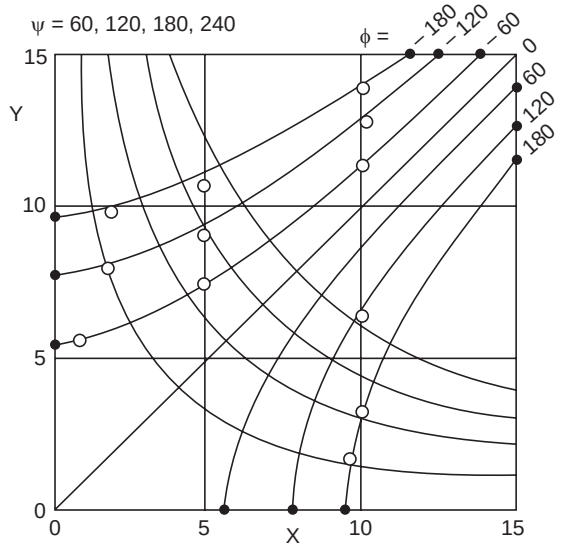


Figure P. 5.19 Flux plot for plot of potential and stream lines

OBJECTIVE QUESTIONS

O Q. 5.1. Fill in the blanks:

1. Ideal fluid is defined as _____.
2. Real fluids exhibit _____.
3. Steady flow is defined as flow where the flow parameters _____.
4. Incompressible flow is defined as flow when _____ does not vary _____.
5. Under unsteady flow conditions the flow parameters vary with _____.
6. Hydrodynamics deals with _____.
7. Irreversibility in flow is due to _____.
8. The laws of thermodynamics apply to _____ flow.
9. The various laws applicable for steady incompressible flow are _____.
10. In turbulent flow the velocity at a point _____.
11. In laminar flow momentum transfer takes place at _____ level.

Answers

1. fluid with zero viscosity 2. viscosity 3. does not vary with time 4. density, with location 5. time 6. ideal fluid flow providing mathematical model for such flow 7. Frictional effects 8. compressible 9. Law of conservation of mass, Newtons laws of motion, Law of conservation of energy 10. varies with time about a mean velocity 11. molecular/microscopic level.

O Q. 5.2. Fill in the blanks:

1. Stream line is defined as the line along which _____ at any point is _____ to the line.
2. Path line is defined as _____.
3. Streak line is defined as _____.
4. Irrotational flow is defined as _____.
5. Circulation is defined as _____.
6. Vorticity is defined as _____.
7. Stream function is defined by _____
8. Potential flow function is defined by _____.
9. Potential flow function exists only if the flow is _____.
10. Continuity equation is derived using the law of _____.
11. A doublet is defined as a combination of _____.
12. The equation for a free vortex is _____.
13. The slope for stream line is _____.
14. The slope for velocity potential line is _____.

Answers

1. the velocity vector is, tangent. 2. line described over time by a particle which has passed through a given point. 3. the line showing the location of various particles that passed through a specified point. 4. there is no net rotation of the fluid particles along the flow – equal deformation along the axes as the flow proceeds. 5. the line integral over a closed path, the product of differential length on the path and the velocity component along the length. 6. circulation per unit area. 7. a function describing the flow field in terms of velocities at various locations – a

function describing the stream lines for the flow field ($\partial\psi/dy = u$, $\partial\psi/\partial x = -v$) 8. a function which describes the flow field potential – a function describing the equipotential lines ($\partial\phi/\partial x = -u$, $\partial\phi/\partial y = -v$). 9. irrotational. 10. law of conservation of mass. 11. a combination of a source and sink of equal strength. 12. $u_\theta r$ = constant 13. v/u 14. $-(u/v)$

O Q. 5.3. Fill in the blanks

1. Rectilinear flow is defined as _____.
2. A source is defined as _____.
3. A sink is defined as _____.
4. The stream function for rectilinear flow is _____.
5. The stream function for source/sink is _____.
6. The stream lines and equipotential lines for a flow field are _____.
7. The x and y directional velocities in a flow is specified by the stream function by _____.
8. The x and y directional velocities in a flow field is given by the potential function as _____.
9. The condition to satisfied by irrotational flow is _____.
10. The stream function for a combination of flows with ψ_A and ψ_B is _____.

Answers

1. a flow having stream lines parallel to one of the axes axis 2. flow with radial stream lines, directed outwards 3. flow with radial stream lines directed towards the centre 4. $\psi = cy$ where c is a constant equal to the velocity 5. $\psi = q\,\theta/2\pi$, q = total flow, θ = angle (in polar co-ordinate), $\psi = -q\,\theta/2\pi$ 6. perpendicular to each other. 7. $u = \partial\psi/\partial y$ $v = -\partial\psi/\partial y$ (8) $u = -\partial\phi/\partial x$, $= -\partial\phi/\partial y$ 9. $\partial v/\partial x = \partial u/\partial y$ 10. $\psi = \psi_A + \psi_B$.

O Q. 5.4. Indicate whether the statements are correct or incorrect.

1. An ideal fluid flow is a good approximation for real fluid flow if viscosity is small.
2. Compressible flow is flow of gases.
3. Turbulent flow is unsteady flow.
4. A stream line shows the path of a particle in any flow.
5. For every stream function a potential function should exist.
6. For every potential function a stream function should exist.
7. Stream function can exist only for irrotational flow.
8. Potential function can exist only for irrotational flow.
9. Circulation will be zero for irrotational flow.
10. Free vertex flow is irrotational.
11. Forced vertex flow is irrotational.

Answers

Correct: 1, 2, 6, 8, 9, 10 Incorrect : 3, 4, 5, 7, 11

O Q. 5.5. Choose the correct answer:

1. A flow is defined by $u = 2(1 + t)$, $v = 3(1 + t)$ where t is the time. The velocyity at $t = 2$ is
   (*a*) 6 (*b*) 9 (*c*) 10.82 (*d*) 6.7.
2. The value of local acceleration in the x direction for flow with $u = 2(1 + t)$ is given by
   (*a*) 0 (*b*) $2t$ (*c*) 2 (*d*) t.
3. The value of x directional convective acceleration in the case of flow with $u = 2(1 + t)$ and $v = 3(1 + t)$ equals
   (*a*) 0 (*b*) 5 (*c*) 1 (*d*) $5t$.

4. When $u = 3 + 2xy + 4t^2$, $v = xy^2 + 3t$. The x directional acceleration is given by
   (*a*) $2x^2y^2 + 4xy^2 + 6y + 8t^2y + 6tx$ (*b*) $4xy$
   (*c*) x/y (*d*) $2x^2y^2 + 6tx$.
5. The continuity equation is satisfied by
   (*a*) $u = A \sin xy$, $v = -A \sin xy$ (*b*) $u = x + y$. $v = x - y$
   (*c*) $u = 2x^2 + cy$, $v = 3y^2$ (*d*) $u = x + 2y$, $v = 2x + y$.
6. The following represent steady incompressible flow.
   (*a*) $u = 4xy + 2y^2$, $v = 6xy + 3x^2$ (*b*) $u = x^2 + y^2$, $v = -2xy + 7$
   (*c*) $u = x/y$, $v = y/x$ (*d*) $u = 2x + y$, $v = 4y + x$.
7. If $\psi = 3x^2y - y^3$. The values u and v are
   (*a*) $6xy$, $3x^2 - 3y^2$ (*b*) $3x^2 - 3y^2$, $6xy$
   (*c*) $(3x^2 - 3y^2)$, $-6xy$ (*d*) $3y^2 - 3x^2$, $6xy$.
8. This is a valid potential function
   (*a*) $\phi = c \ln x$ (*b*) $\phi = c \cos x$
   (*c*) $\phi = 3xy$ (*d*) $\phi = c\,(x^2 + y^2)$.
9. The continuity equation for incompressible two dimensional steady flow is
   (*a*) $\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} = 0$, (*b*) $\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial t} = 0$,
   (*c*) $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0$, (*d*) $\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial t} = 0$.
10. The velocity components in the x and y directions in terms of stream function ψ is given by
    (*a*) $u = \dfrac{\partial \psi}{\partial x}$, $v = \dfrac{\partial \psi}{\partial y}$ (*b*) $u = \dfrac{\partial \psi}{\partial y}$, $v = -\dfrac{\partial \psi}{\partial x}$
    (*c*) $u = -\dfrac{\partial \psi}{\partial y}$, $v = \dfrac{\partial \psi}{\partial x}$ (*d*) $u = -\dfrac{\partial \psi}{\partial x}$, $v = \dfrac{\partial \psi}{\partial y}$.

Answers

1. *c*, 2. *c*, 3. *a*, 4. *a*, 5. *b*, 6. *b*, 7. *c*, 8. *c*, 9. *c*, 10. *b*.

O Q. 5.6. Choose the correct answer

1. The velocity components in the x and y directions in terms of potential function ϕ is given by
   (*a*) $u = -\dfrac{\partial \phi}{\partial y}$, $v = \dfrac{\partial \phi}{\partial x}$ (*b*) $u = \dfrac{\partial \phi}{\partial y}$, $v = -\dfrac{\partial \phi}{\partial x}$
   (*c*) $u = \dfrac{\partial \phi}{\partial x}$, $v = \dfrac{\partial \phi}{\partial y}$ (*d*) $u = -\dfrac{\partial \phi}{\partial x}$, $v = -\dfrac{\partial \phi}{\partial y}$.
2. The condition for irrotational flow is
   (*a*) $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ (*b*) $\dfrac{\partial u}{\partial x} = -\dfrac{\partial v}{\partial y}$
   (*c*) $\dfrac{\partial v}{\partial x} = \dfrac{\partial u}{\partial y}$ (*d*) $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$.

3. The equation of a stream line in two dimensional steady flow can be expressed as

   (*a*) $\frac{u}{dx} = \frac{dy}{v}$ (*b*) $\frac{dy}{u} = \frac{dx}{v}$

   (*c*) $-\frac{dy}{u} = \frac{dx}{v}$ (*d*) $-\frac{dx}{u} = \frac{dy}{v}$.

4. The flow rate between stream lines with values ψ_1 and ψ_2 is given by

   (*a*) $\psi_1 + \psi_2$ (*b*) $\psi_1 + C\psi_2$

   (*c*) $\psi_2 - \psi_1$ (*d*) $C\,\psi_1 + \psi_2$.

5. The continuity equation is the result of application of the following law to the flow field

   (*a*) First law of thermodynamics (*b*) Conservation of energy

   (*c*) Newtons second law of motion (*d*) Conservation of mass.

6. A path line describes

   (*a*) The velocity direction at all points on the line

   (*b*) The path followed by particles in a flow

   (*c*) The path over a period of times of a single particle that has passed out at a point

   (*d*) The instantaneous position of all particles that have passed a point.

7. The relationship between stream and potential functions ψ and ϕ is

   (*a*) $\frac{\partial\psi}{\partial y} = \frac{\partial\phi}{\partial x}$ (*b*) $\frac{\partial\phi}{\partial x} = -\frac{\partial\psi}{\partial y}$

   (*c*) $\frac{\partial\phi}{\partial y} = \frac{\partial\psi}{\partial x}$ (*d*) $\frac{\partial^2\phi}{\partial x^2} = \frac{\partial^2\psi}{\partial y^2}$.

8. The value that will satisfy potential function ϕ is

   (*a*) $\phi = x^2 + y^2$ (*b*) $\phi = \sin x$

   (*c*) $\phi = x + y$ (*d*) $\phi = \ln (x + y)$.

9. The stream function is

   (*a*) constant along an equipotential line (*b*) along a stream line

   (*c*) defined only in irrotational flow (*d*) defined only for incompressible flow.

10. A potential function

    (*a*) is constant along a stream line

    (*b*) is definable if a stream function is available for the flow

    (*c*) describes the flow if it is rotational

    (*d*) describes the flow if it is irrotational.

Answers

1. *d*, 2. *c*, 3. *d*, 4. *c*, 5. *d*, 6. *c*, 7. *c*, 8. *c*, 9. *b*, 10. *d*.

O Q. 5.7. Match the pairs

1.

Set *A*	Set *B*
A. Ideal fluid	1. Irreversible flow
B. Steady flow	2. Incompresible flow
C. Low velocity gas flow	3. Zero viscosity
D. Friction	4. Velocity at a point is constant.

Answers

$A - 3$, $B - 4$, $C - 2$, $D - 1$.

2.

Set A	Set B
A. Stream line	1. Path of particles that passed a point
B. Streak line	2. The path of a single particle that passed a point
C. Equipotential line	3. Shows velocity
D. Path line	4. Described by potential function.

Answers

$A - 3$, $B - 1$, $C - 4$, $D - 2$.

EXERCISE PROBLEMS

E 1. Given $u = kx$ in a two dimensional flow determine v. **[– *ky* + *f*(*x*)]**

E 2. Given velocity potential, determine the velocity components u and v.

(a) $\phi = \ln xy$ **[– (1/x), (1/y)]**

(b) $\phi = 3\,(x^2 + y^2)$ **[– 6x , – 6y)]**

(c) $\phi = a \cos xy$ **[aysinxy, axsinxy]**

E 3. Given stream function $\psi = 3x - 4y$, calculate the slope of the line and also the value of resultant velocity. Does it satisfy continuity equation? Is the flow irrotational?

[θ = 37°, v = 5 units]

E 4. Given $\phi = x(2y - 1)$, determine ψ. **[– y^2 + y + f(y)]**

E 5. Given $\psi = 4x^2 - 4y^2$, find ϕ. **[ϕ = 8xy + c]**

E 6. Find the relationship between a and b if in steady flow $u = bx$ and $v = ay$, **[b = – a]**

E 7. Show that for two dimensional steady flows with velocity components u and v,

$$a_x = u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} \quad \text{and} \quad a_y = u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y}.$$

E 8. Given $u = 2y$, $v = x$, sketch the flow. Also find a_x, a_y and a.

E 9. If $u = 3 + 2xy$ and $v = xy^2$ determine a_x, a_y and a.

E 10. If $u = 3y$ and $v = 2$, determine a_x and a_y.

E 11. If $u = 0$, $v = 3xy$, determine a_x and a_y.

E 12. If $u = -2y$, $v = 3xy$, determine a_x and a_y.

E 13. Determine the normal and tangential components of acceleration for a circular stream line.

E 14. Find ϕ and ψ given $u = 2x$ and $v = -2y$. $[\psi = 2xy + c_1,\ \phi = -(x^2 - y^2) + c^2]$

E 15. If $u = 2$, $v = 8x$ determine ψ.

E 16. Tabulate the values of x and y for $\psi = 0, 1, 2, 3$ given (i) $\psi = 10y$.

(ii) $\psi = -20x$, (iii) $\psi = 10y - 20x$.

E 17. Determine the stream function if it exists. Also check for irrotationality.

(i) $u = 5,\ v = 6$ (ii) $u = 3 + x$, $v = 4$

(iii) $u = 3xy$, $v = 1.5x^2$ (iv) $u = 3x$, $v = 3y$

(v) $u = 4 + 2x,\ v = -y$.

E 18. Calculate the values of x and y for stream lines. $\psi = 0, 1, 2, 3$ and 4, given $\psi = 1.2xy$ (for one quadrant).

E 19. Tabulate and plot $\psi = 1.5x^2 + y^2$ for positive values of x, y.

E 20. If $\psi = x^2 - y$, find u and v and also the vorticity.

E 21. A source discharging 1 m^3/sm is at (– 1, 0) and a sink taking in $1m^3/sm$ is at (+1, 0). If this is combined with uniform flow of u = 1.5 m/s, left to right, calculate the length of the resolution of closed body contour.

E 22. Two sources one of strength 8π and the other of 16π are located at (2, 0) and (– 3, 0) respectively. For the combined flow field, calculate the location of the stagnation point. Also plot $\psi = 4\pi$, $\psi = 8\pi$ and $\psi = 0$.

E 23. A source of $20m^3/sm$ at (0, 0) is combined with a uniform flow with $u = 3m/s$ from left to right. Determine ψ for the flow.

E 24. Given $u = x^2 + 2x - y$ and $v = -2xy - 2y$, determine ψ. Also compute vorticity.

6 Bernoulli Equation and Applications

6.0 INTRODUCTION

In chapter five flow of ideal fluids was discussed. The main idea was the study of flow pattern. The determination of equal flow paths and equal potential lines was discussed. No attempt was made to determine the numerical value of these quantities.

In this chapter the method of determination of the various energy levels at different locations in the flow is discussed. In this process first the various forms of energy in the fluid are identified. Applying the law of conservation of energy the velocity, pressure and potential at various locations in the flow are calculated. Initially the study is limited to ideal flow. However the modifications required to apply the analysis to real fluid flows are identified.

The material discussed in this chapter are applicable to many real life fluid flow problems. The laws presented are the basis for the design of fluid flow systems.

Energy consideration in fluid flow:

Consider a small element of fluid in flow field. The energy in the element as it moves in the flow field is conserved. This principle of conservation of energy is used in the determination of flow parameters like pressure, velocity and potential energy at various locations in a flow. The concept is used in the analysis of flow of ideal as well as real fluids.

Energy can neither be created nor destroyed. It is possible that one form of energy is converted to another form. The total energy of a fluid element is thus conserved under usual flow conditions.

If a stream line is considered, it can be stated that the total energy of a fluid element at any location on the stream line has the same magnitude.

6.1 FORMS OF ENERGY ENCOUNTERED IN FLUID FLOW

Energy associated with a fluid element may exist in several forms. These are listed here and the method of calculation of their numerical values is also indicated.

6.1.1 Kinetic Energy

This is the energy due to the motion of the element as a whole. If the velocity is V, then the kinetic energy for m kg is given by

$$KE = \frac{mV^2}{2g_o} \text{ Nm} \tag{6.1.1}$$

The unit in the SI system will be Nm also called Joule (J)

{(kg m^2/s^2)/(kg m/N s^2)}

The same referred to one kg (specific kinetic energy) can be obtained by dividing 6.1.1 by the mass m and then the unit will be Nm/kg.

$$KE = \frac{V^2}{2g_o}, \text{ Nm/kg} \tag{6.1.1}b$$

In fluid flow studies, it is found desirable to express the energy as the head of fluid in m. This unit can be obtained by multiplying equation (6.1.1) by g_o/g.

$$\text{Kinetic head} = \frac{V^2}{2g_o}\frac{g_o}{g} = \frac{V^2}{2g} \tag{6.1.2}$$

The unit for this expression will be $\frac{m^2s^2}{s^2m}$ = m

Apparantly the unit appears as metre, but in reality it is Nm/N, where the denominator is weight of the fluid in N.

The equation in this form is used at several places particularly in flow of liquids. But the energy associated physically is given directly only be equation 6.1.1.

The learner should be familiar with both forms of the equation and should be able to choose and use the proper equation as the situation demands. **When different forms of the energy of a fluid element is summed up to obtain the total energy, all forms should be in the same unit.**

6.1.2 Potential Energy

This energy is due to the position of the element in the gravitational field. While a zero value for *KE* is possible, the value of potential energy is relative to a chosen datum. The value of potential energy is given by

$$PE = mZ\, g/g_o \text{ Nm} \tag{6.1.3}$$

Where m is the mass of the element in kg, Z is the distance from the datum along the gravitational direction, in m. The unit will be (kg m m/s^2) × (Ns^2/kgm) *i.e.,* Nm. The specific potential energy (per kg) is obtained by dividing equation 6.1.3 by the mass of the element.

$$PE = Z\, g/g_0 \text{ Nm/kg} \tag{6.1.3.}\ b)$$

This gives the physical quantity of energy associated with 1 kg due to the position of the fluid element in the gravitational field above the datum. As in the case of the kinetic energy, the value of *PE* also is expressed as head of fluid, *Z*.

$$PE = Z\,(g/g_o)\,(g_o/g) = Z\ m. \tag{6.1.4}$$

This form will be used in equations, but as in the case of KE, one should be familiar with both the forms and choose the suitable form as the situation demands.

6.1.3 Pressure Energy (Also Equals Flow Energy)

The element when entering the control volume has to flow against the pressure at that location. The work done can be calculated referring Fig. 6.1.1.

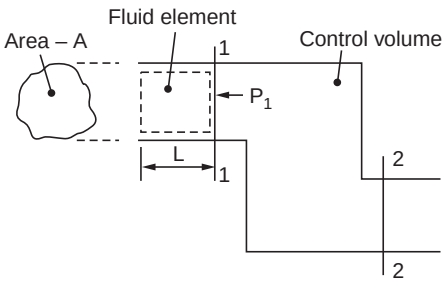


Figure 6.1.1 Flow work calculation

The boundary of the element of fluid considered is shown by the dotted line, Force = P_1 A, distance to be moved = L, work done = $P_1AL = P_1\ mv$ as AL = volume = mass × specific volume, v. ∴ flow work = $P\ mv$.

The pressure energy per kg can be calculated using $m = 1$. The flow energy is given by

$$FE = P.v = P/\rho, \text{ Nm/kg} \tag{6.1.5}$$

Note: $$\frac{N}{m^2}\frac{m^3}{kg} \rightarrow \frac{Nm}{kg}$$

As in the other cases, the flow energy can also expressed as head of fluid.

$$FE = \frac{P}{\rho}\frac{g_o}{g}, \text{ m} \tag{6.1.5a}$$

As specific weight $\gamma = \rho\, g/g_o$, the equation is written as,

$$FE = P/\gamma, \text{ m} \tag{6.1.5b}$$

It is important that in any equation, when energy quantities are summed up consistent forms of these set of equations should be used, that is, all the terms should be expressed either as head of fluid or as energy (J) per kg. These are the three forms of energy encountered more often in flow of incompressible fluids.

6.1.4 Internal Energy

This is due to the thermal condition of the fluid. This form is encountered in compressible fluid flow. For gases (above a datum temperature) $IE = c_v\ T$ where T is the temperature above the datum temperature and c_v is the specific heat of the gas at constant volume. The unit for internal energy is J/kg (Nm/kg). When friction is significant other forms of energy is converted to internal energy both in the case of compressible and incompressible flow.

6.1.5 Electrical and Magnetic Energy

These are not generally met with in the study of flow of fluids. However in magnetic pumps and in magneto hydrodynamic generators where plasma flow in encountered, electrical and magnetic energy should also be taken into account.

6.2 VARIATION IN THE RELATIVE VALUES OF VARIOUS FORMS OF ENERGY DURING FLOW

Under ideal conditions of flow, if one observes the movement of a fluid element along a stream line, the sum of these forms of energy will be found to remain constant. However, there may be an increase or decrease of one form of energy while the energy in the other forms will decrease or increase by the same amount. For example when the level of the fluid decreases, it is possible that the kinetic energy increases. When a liquid from a tank flows through a tap this is what happens. In a diffuser, the velocity of fluid will decrease but the pressure will increase. In a venturimeter, the pressure at the minimum area of cross section (throat) will be the lowest while the velocity at this section will be the highest.

The total energy of the element will however remain constant. In case friction is present, a part of the energy will be converted to internal energy which should cause an increase in temperature. But the fraction is usually small and the resulting temperature change will be so small that it will be difficult for measurement. From the measurement of the other forms, it will be possible to estimate the frictional loss by difference.

6.3 EULER'S EQUATION OF MOTION FOR FLOW ALONG A STREAM LINE

Consider a small element along the stream line, the direction being designated as *s*.

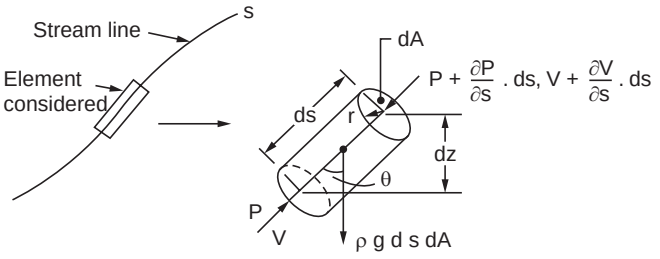


Figure 6.3.1 Euler's equation of Motion – Derivation

The net force on the element are the body forces and surface forces (pressure). These are indicated in the figure. Summing this up, and equating to the change in momentum.

$$PdA - \{P + (\partial P/\partial s)\, dA - \rho g\, dA\, ds \cos\theta = \rho\, dA\, ds\, a_s \qquad (6.3.1)$$

where a_s is the acceleration along the *s* direction. This reduces to,

$$\frac{1}{\rho}\frac{\partial P}{\partial s} + g\cos\theta + a_s = 0 \qquad (6.3.2)$$

(**Note**: It will be desirable to add g_o to the first term for dimensional homogeneity. As it is, the first term will have a unit of N/kg while the other two terms will have a unit of m/s^2. Multiplying by g_o, it will also have a unit of m/s^2).

$a_s = dV/dt$, as velocity, $V = f(s, t)$, (t = time).

$$dV = \frac{\partial V}{\partial s} ds + \frac{\partial V}{\partial t} dt \quad \text{dividing by } dt,$$

$$\frac{dV}{dt} = \frac{\partial V}{\partial s}\frac{ds}{dt} + \frac{\partial V}{\partial t} \qquad \text{As } \frac{ds}{dt} = V,$$

and as $\cos\theta = dz/ds$, equation 6.3.2 reduces to,

$$\frac{1}{\rho}\frac{\partial P}{\partial s} + g\frac{\partial z}{\partial s} + V\frac{\partial V}{\partial s} + \frac{\partial V}{\partial t} = 0 \qquad (6.3.2.\ a)$$

For steady flow $\partial V/\partial t = 0$. Cancelling ∂s and using total derivatives in place of partials as these are independent quantities.

$$\frac{dp}{\rho} + gdz + VdV = 0 \qquad (6.3.3)$$

(**Note**: in equation 6.3.3 also it is better to write the first term as $g_o.dp/\rho$ for dimensional homogeneity).

This equation after dividing by g, is also written as,

$$\frac{dp}{\gamma} + d\left(\frac{V^2}{2g}\right) + dz = 0 \quad \text{or} \quad d\left[\frac{P}{\gamma} + \frac{V^2}{2g} + z\right] = 0 \qquad (6.3.4)$$

which means that the quantity within the bracket remains constant along the flow.

This equation is known as Euler's equation of motion. The assumptions involved are:

1. Steady flow
2. Motion along a stream line and
3. Ideal fluid (frictionless)

In the case on incompressible flow, this equation can be integrated to obtain Bernoulli equation.

6.4 BERNOULLI EQUATION FOR FLUID FLOW

Euler's equation as given in 6.3.3 can be integrated directly if the flow is assumed to be incompressible.

$$\frac{dP}{\rho} + gdz + VdV = 0, \quad \text{as } \rho = \text{constant}$$

$$\frac{P}{\rho} + gz + \frac{V^2}{2} = \text{const. or } \frac{P}{\rho} + z\left(\frac{g}{g_0}\right) + \frac{V^2}{2g_0} = \text{Constant} \qquad (6.4.1)$$

The constant is to be evaluated by using specified boundary conditions. The unit of the terms will be energy unit (Nm/kg).

In SI units the numerical value of g_o = 1, kg m/N s^2. Equation 6.4.1 can also be written as to express energy as head of fluid column.

$$\frac{P}{\gamma} + z + \frac{V^2}{2g} = \text{constant} \tag{6.4.2}$$

(γ is the specific weight N/m^3). In this equation all the terms are in the unit of head of the fluid.

The constant has the same value along a stream line or a stream tube. The first term represents (flow work) pressure energy, the second term the potential energy and the third term the kinetic energy.

This equation is extensively used in practical design to estimate pressure/velocity in flow through ducts, venturimeter, nozzle meter, orifice meter etc. In case energy is added or taken out at any point in the flow, or loss of head due to friction occurs, the equations will read as,

$$\frac{P_1}{\rho} + \frac{V_1^2}{2g_o} + \frac{z_1 g}{g_o} + W - \frac{h_f g}{g_o} = \frac{P_2}{\rho} + \frac{V_2^2}{2g_o} + \frac{z_2 g}{g_o} \tag{6.4.3}$$

where W is the energy added and h_f is the loss of head due to friction.

In calculations using SI system of units g_o may be omitting as its value is unity.

Example 6.1 *A liquid of specific gravity 1.3 flows in a pipe at a rate of 800 l/s, from point 1 to point 2 which is 1 m above point 1. The diameters at section 1 and 2 are 0.6 m and 0.3 m respectively. If the pressure at section 1 is 10 bar,* ***determine the pressure at section 2.***

Using Bernoulli equation in the following form (6.4.2)

$$\frac{P}{\gamma} + z + \frac{V^2}{2g} = \text{constant},$$

Taking the datum as section 1, the pressure P_2 can be calculated.

$V_1 = 0.8 \times 4/\pi \times 0.6^2 = 2.83$ *m/s*, $V_2 = 0.8 \times 4/\pi \times 0.3^3 = 11.32$ *m/s*

$P_1 = 10 \times 10^5$ N/m^2, γ = sp. gravity × 9810. Substituting.

$$\frac{10 \times 10^5}{9810 \times 1.3} + 0 + \frac{2.83^2}{2 \times 9.81} = \frac{P_2}{9810 \times 1.3} + 1 + \frac{11.32^2}{2 \times 9.81}$$

Solving, **P_2 = 9.092 bar** (9.092×10^5 N/m^2).

As P/γ is involved directly on both sides, gauge pressure or absolute pressure can be used without error. However, it is desirable to use absolute pressure to avoid negative pressure values (or use of the term vacuum pressure).

Example 6.2 *Water flows through a horizontal venturimeter with diameters of 0.6 m and 0.2 m. The guage pressure at the entry is 1 bar.* ***Determine the flow rate*** *when the throat pressure is 0.5 bar (vacuum). Barometric pressure is 1 bar.*

Using Bernoulli's equation in the form,

$$\frac{P_1}{\gamma} + Z_1 + \frac{V_1^2}{2g} = \frac{P_2}{\gamma} + Z_2 + \frac{V_2^2}{2g}$$

and noting $Z_1 = Z_2$, $P_1 = 2 \times 10^5$ N/m^2 (absolute)

$P_2 = 0.5 \times 10^5$ N/m^2 (absolute), $\gamma = 9810$ N/m^3

$V_1 = Q \times 4/(\pi \times 0.60^2) = 3.54\,Q$, $V_2 = Q \times 4/(\pi \times 0.20^2) = 31.83Q$

$$\frac{2 \times 10^5}{9810} + 0 + \frac{3.54^2}{2 \times 9.81} Q^2 = \frac{0.5 \times 10^5}{9810} + 0 + \frac{31.83^2\, Q^2}{2 \times 9.81}$$

Solving, $\mathbf{Q = 0.548\ m^3/s}$, $V_1 = 1.94$ m/s, $V_2 = 17.43$ m/s.

Example 6.3 *A tap discharges water evenly in a jet at a velocity of 2.6 m/s at the tap outlet, the diameter of the jet at this point being 15 mm. The jet flows down vertically in a smooth stream.* ***Determine the velocity and the diameter*** *of the jet at 0.6 m below the tap outlet.*

The pressure around the jet is atmospheric throughout. Taking the tap outlet as point 1 and also taking it as the datum using Bernoulli equation.

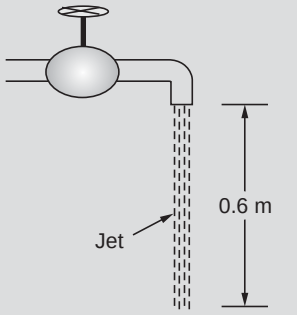


Figure Ex. 6.3 Problem model

$$\frac{P_1}{\gamma} + Z_1 + \frac{V_1^2}{2g} = \frac{P_2}{\gamma} + Z_2 + \frac{V_2^2}{2g},$$

$P_1 = P_2$, $Z_2 = 0$,

$Z_2 = -0.6$ m, $V_1 = 2.6$ m/s

$$\therefore \quad \frac{2.6^2}{2 \times 9.81} = -0.6 + \frac{V_2^2}{2 \times 9.81}$$

$$\therefore \quad \mathbf{V_2 = 4.3\ m/s.}$$

using continuity equation (one dimensional flow) and noting that density is constant.

$$A_1 V_1 = A_2 V_2$$

$$\frac{\pi \times 015^2}{4} \times 2.6 = \frac{\pi \times D^2}{4} \times 4.3, \quad \therefore \quad \mathbf{D = 0.01166\ m \text{ or } 11.66\ mm}$$

As the potential energy decreases, kinetic energy increases. As the velocity is higher the flow area is smaller.

Entrainment of air may increase the diameter somewhat.

Example 6.4 *Water flows in a tapering pipe vertically as shown in Fig. Ex.6.4.* ***Determine the manometer reading "h"****. The manometer fluid has a specific gravity of 13.6. The flow rate is 100 l/s*

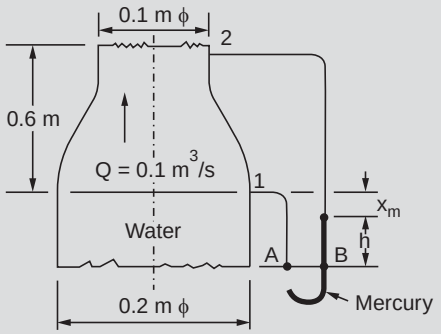


Figure Ex. 6.4 Problem model

The velocities at sections 1 and 2 are first calculated.

$$V_1 = 4 \times 0.1/ (\pi \times 0.2^2) = 3.183 \text{ m/s},$$

$$V_2 = 4 \times 0.1/ (\pi \times 0.1^2) = 12.732 \text{ m/s}$$

It is desired to determine $P_1 - P_2$. Rearranging Bernoulli equation for this flow,

$$\frac{P_1 - P_2}{\gamma} = 0.6 + (12.732^2 - 3.183^2)/(2 \times 9.81) = 8.346 \text{ m of water}$$

For water $\gamma = 9810$ N/m^3. For the manometer configuration, considering the level AB and equating the pressures at A and B

$$\frac{P_1}{\gamma} + x + h = \frac{P_2}{\gamma} + 0.6 + x + sh$$

(where x, h are shown on the diagram and s is specific gravity)

$$\therefore \quad \frac{P_1 - P_2}{\gamma} = 0.6 + h(s - 1), \text{ substituting the values,}$$

$$8.346 = 0.6 + h(13.6 - 1)$$

$$\therefore \quad \mathbf{h = 0.6148 \text{ m or } 61.48 \text{ cm}}$$

6.5 ENERGY LINE AND HYDRAULIC GRADIENT LINE

The total energy plotted along the flow to some specified scale gives the energy line. When losses (frictional) are negligible, the energy line will be horizontal or parallel to the flow direction. For calculating the total energy kinetic, potential and flow (pressure) energy are considered.

Energy line is the plot of $\frac{P}{\gamma} + Z + \frac{V^2}{2g}$ along the flow. It is constant along the flow when losses are negligible.

The plot of $\frac{P}{\gamma} + Z$ along the flow is called the hydraulic gradient line. When velocity increases this will dip and when velocity decreases this will rise. An example of plot of these lines for flow from a tank through a venturimeter is shown in Fig. 6.5.1.

The hydraulic gradient line provides useful information about pressure variations (static head) in a flow. The difference between the energy line and hydraulic gradient line gives the value of dynamic head (velocity head).

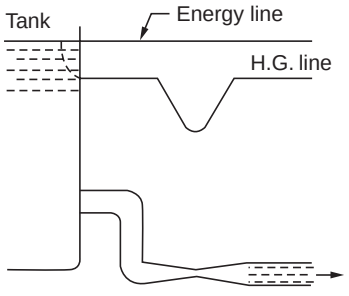


Figure 6.5.1 Energy and hydraulic gradient lines

6.6 VOLUME FLOW THROUGH A VENTURIMETER

Example 6.5 *Under ideal conditions show that the volume flow through a venturimeter is given by*

$$Q = \frac{A_2}{\left\{1-(A_2/A_1)^2\right\}^{0.5}}\left[2g\left(\frac{P_1-P_2}{\gamma}+(Z_1-Z_2)\right)\right]^{0.5}$$

where suffix 1 and 2 refer to the inlet and the throat.

Refer to Fig. Ex. 6.5

Volume flow $= A_1 V_1 = A_2 V_2$

$$\therefore \quad V_1 = \frac{A_2}{A_1}V_2,\ V_1^2 = \left(\frac{A_2}{A_1}\right)^2 . V_2^2,$$

$$\therefore \quad (V_2^2 - V_1^2) = V_2^2\left[1-\left(\frac{A_2}{A_1}\right)^2\right]$$

Applying Bernoulli equation to the flow and considering section 1 and 2,

$$\frac{P_1}{\gamma}+Z_1+\frac{V_1^2}{2g}=\frac{P_2}{\gamma}+Z_2+\frac{V_2^2}{2g}$$

Rearranging,

$$\left[2g\left\{\frac{P_1-P_2}{\gamma}+(Z_1-Z_2)\right\}\right]^{0.5} = V_2\left[1-\left(\frac{A_2}{A_1}\right)^2\right]^{0.5}$$

$$V_2 = \frac{1}{[1-(A_2/A_1)^2]^{0.5}}\left[2g\left\{\frac{P_1-P_2}{\gamma}+(Z_1-Z_2)\right\}\right]^{0.5}$$

$\therefore$ Volume flow is

$$A_2V_2 = \frac{A_2}{[1-(A_2/A_1)^2]^{0.5}}\left[2g\left\{\frac{P_1-P_2}{\gamma}+(Z_1-Z_2)\right\}\right]^{0.5} \qquad (6.6.1)$$

This is a general expression and can be used irrespective of the flow direction, inclination from horizontal or vertical position. This equation is applicable for orifice meters and nozzle flow meters also.

In numerical work consistent units should be used.

Pressure should be in N/m^2, Z in m, A in m^2 and then volume flow will be m^3/s.

A coeficient is involved in actual meters due to friction.

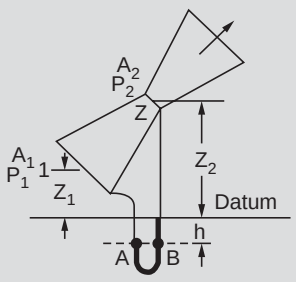


Figure Ex. 6.5 Venturimeter-flow

Example 6.6 *Show that when a manometric fluid of specific gravity S_2 is used to measure the head in a venturimeter with flow of fluid of specific gravity S_1, if the manometer shows a reading of hm, the volume flow is given by*

$$Q = \frac{A_2}{\left[1-(A_2/A_1)^2\right]^{0.5}}\left[2gh\left(\frac{S_2}{S_1}-1\right)\right]^{0.5}$$

Comparing the equation (6.6.1) with the problem at hand, it is seen that it is sufficient to prove,

$$h\left(\frac{S_2}{S_1}-1\right) = \frac{P_1-P_2}{\gamma_1} + (Z_2 - Z_1)$$

Considering the plane *A–B* in the manometer and equating the pressures at *A* and *B* Fig. Ex. 6.5 : The manometer connection at the wall measures the static pressure only)

$$P_1 + Z_1\,\gamma_1 + h\gamma_1 = P_2 + Z_2\gamma_1 + h\gamma_2$$

$$(P_1 - P_2) + (Z_1 - Z_1)\,\gamma_1 = h(\gamma_2 - \gamma_1), \text{ dividing by } \gamma_1,$$

$$\frac{P_1-P_2}{\gamma_1} + (Z_1 - Z_2) = h\left(\frac{\gamma_2}{\gamma_1}-1\right) = h\left(\frac{S_2}{S_1}-1\right)$$

Hence volume flow,

$$Q = \frac{A_2}{[1-(A_2/A_1)^2]^{0.5}}\left[2gh\left(\frac{S_2}{S_1}-1\right)\right]^{0.5} \qquad (6.6.2)$$

This equation leads to another conclusion. The fluid head, H, causing the flow is equal to the manometer reading $h[(S_2/S_1) - 1]$ and flow is independent of the inclination if the reading of the manometer and the fluids are specified.

i.e., As the manometer reading converted to head of flowing fluid, $H = h[(S_2/S_1) - 1]$

$$Q = \frac{A_2}{[1-(A_2/A_1)^2]^{0.5}}[2gH]^{0.5} \qquad (6.6.3)$$

If the pressure at various locations are specified, these equations are applicable for orifice and nozzle meters also.

Example. 6.7 ***Determine the flow rate*** *through the siphon Fig. Ex. 6.7 when flow is established. Also determine the pressure at A.*

The pressure at C and B are atmospheric. Considering locations C and B and taking the datum at B, applying Bernoulli equation, noting that the velocity at water surface at C = 0.

$$0 + 0 + V_B^2/2g = 3 + 0 + 0$$

$$\therefore \quad V_B = 7.672 \text{ m/s.}$$

$$\therefore \quad \textbf{Flow rate} = (\pi D^2/4) \times V$$

$$= (\pi \times 0.1^2/4) \times 7.672$$

$$= \mathbf{0.06\ m^3/s}$$

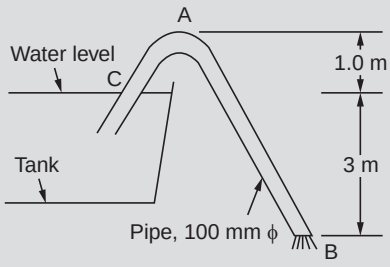


Figure Ex. 6.7 Problem model

The velocity at A is the same as velocity at B. Now considering locations C and A,

$$3 + 0 + 0 = 4 + (P_A/\gamma) + 7.672^2/(2 \times 9.81)$$

$\therefore$ $P_A/\gamma = -4$m or -4m of water head or 4m water-head below atmospheric pressure.

Check: Consider points A and B

$$4 + \frac{P_A}{\gamma} + \frac{V_A^{\,2}}{2g} = \frac{V_B^{\,2}}{2g} + 0 + 0 \quad \text{as } V_A = V_B, \quad \frac{\mathbf{P_A}}{\gamma} = \mathbf{-4\ m} \text{ checks.}$$

Example. 6.8 *Water flows in at a rate of 80 l/s from the pipe as shown in Fig. Ex. 6.8 and flows outwards through the space between the top and bottom plates. The top plate is fixed.* ***Determine the net force acting on the bottom plate****. Assume the pressure at radius r = 0.05 m is atmospheric.*

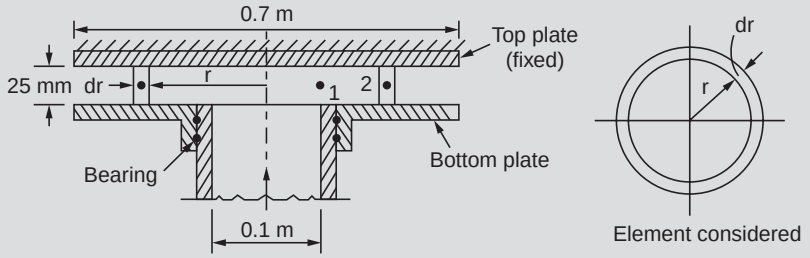


Figure Ex. 6.8 Problem model

Consider an element area of width dr (annular) in the flow region at a distance r as shown in figure. The pressure at this location as compared to point 1 can be determined using Bernoulli equation.

$$\frac{P_1}{\gamma} + Z_1 + \frac{V_1^{\,2}}{2g} = \frac{P_2}{\gamma} + Z_2 + \frac{V_2^{\,2}}{2g},\ P_1 \text{ is atmospheric}$$

As $Z_1 = Z_2$, $$P_2 - P_1 = \frac{\gamma}{2g}(V_1^{\,2} - V_2^{\,2})$$

$$V_1^{\,2} = (0.08/2\pi \times 0.05 \times 0.025)^2 = 103.75$$

$$V_2^{\,2} = (0.08/2\pi \times 0.025 \times r)^2 = 0.2594/r^2$$

$(P_2 - P_1)$ is the pressure difference which causes a force at the area $2\pi rdr$ at r.

The force on the element area of the bottom plate = $2\pi rdr\,(P_2 - P_1)$

Substituting and nothing $\gamma = \rho g/g_0$, the elemental force dF is given by,

$$dF = \rho\,\pi rdr\left[103.75 - \frac{0.2594}{r^2}\right],$$

Integrating between the limits $r = 0.05$ to 0.35,

$$\textbf{Net force} = 1000 \times \pi\left[(103.75\,(0.35^{\,2} - 0.05^2)/2) - \left(0.2594 \ln \frac{0.35}{0.05}\right)\right] = \mathbf{17970\ \ N}$$

6.7 EULER AND BERNOULLI EQUATION FOR FLOW WITH FRICTION

Compared to ideal flow the additional force that will be involved will be the shear force acting on the surface of the element. Let the shear stress be τ, the force will equal $\tau\, 2\pi r\, ds$ (where r is the radius of the element, and $A = \pi r^2$)

Refer **Para 6.3** and **Fig. 6.3.1.** The Euler equation 6.3.3 will now read as

$$\frac{dP}{\rho} + VdV + gdZ - \frac{2\tau ds}{\rho r} = 0$$

$$\frac{dP}{\gamma} + d\left(\frac{V^2}{2g}\right) + dZ - \frac{2\tau ds}{\gamma\, r} = 0$$

ds can also be substituted in terms of Z and θ

Bernoulli equation will now read as (taking s as the length)

$$\frac{P_1}{\gamma} + \frac{V_1^2}{2g} + Z_1 = \frac{P_2}{\gamma} + \frac{V_2^2}{2g} + Z_2 + \frac{2\tau s}{\gamma r}$$

The last term is the loss of head due to friction and is denoted often as h_L,h_f in head of fluid in metre height (check for the unit of the last term).

Example 6.9 *The delivery line of a pump is 100 mm ID and it delivers water at a height of 12 m above entry. The pipe ends in a nozzle of diameter 60 mm. The total head at the entry to the pipe is 20 m.* ***Determine the flow rate if losses in the pipe is given by 10 $V_2^2/2g$****. where V_2 is the velocity at nozzle outlet. There is no loss in the nozzle.*

Equating the total energy at inlet and outlet,

$$20 = 12 + \frac{V_2^2}{2g} + 10\,\frac{V_2^2}{2g},$$

$$\therefore \qquad V_2^2 = \frac{8 \times 2 \times 9.81}{11}\,,\ V_2 = 3.777 \text{ m/s}$$

$$\textbf{Flow} = A_2\, V_2 = \frac{\pi \times 0.06^2}{4} \times 3.777 = 0.01068 \text{ m}^3\text{/s} = \textbf{0.64 m}^3\textbf{/min.}$$

(If losses do not occur then, V_2 = 12.53 m/s and flow will be 2.13 m^3/min)

Example 6.10 *A tank with water level of 12 m has a pipe of 200 mm dia connected from its bottom which extends over a length to a level of 2 m below the tank bottom.* ***Calculate the pressure at this point*** *if the flow rate is 0.178 m^3/s. The losses due to friction in the pipe length is accounted for by 4.5 $V_2^2/2g$.*

Taking location of the outlet of the pipe as the datum, using Bernoulli equation and accounting for frictional drop in head (leaving out the atmospheric pressure which is the same at the water level and at outlet).

$$14 = \frac{P_2}{\gamma} + \frac{V_2^2}{2g} + 4.5\,\frac{V_2^2}{2g}$$

$$\therefore \quad V_2 = 0.178/\pi \times 0.1 \times 0.1 = 5.67 \text{ m/s}$$

$$14 - 5.5 \times \frac{5.67^2}{2 \times 9.81} = \frac{P_2}{\gamma} = 5 \text{ m of water head.}$$

$$\therefore \quad P_2 = 9810 \times 5 \text{ N/m}^2 = \textbf{0.49 bar} \text{ (above atmospheric pressure)}$$

Example 6.11 *A vertical pipe of diameter of 30 cm carrying water is reduced to a diameter of 15 cm. The transition piece length is 6 m. The pressure at the bottom is 200 kPa and at the top it is 80 kPa. If frictional drop is 2 m of water head,* ***determine the rate*** *of flow.*

Considering the bottom as the datum,

$$\frac{200 \times 10^3}{9810} + 0 + \frac{V_1^2}{2g} = \frac{80 \times 10^3}{9810} + 6 + \frac{V_2^2}{2g} + 2$$

$$V_2^2 = V_1^2\,(0.3/0.15)^4 = 16V_1^2$$

$$\therefore \quad \frac{120 \times 10^3}{9810} - 8 = 15\frac{V_1^2}{2g}, \text{ Solving, } V_1 = 2.353 \text{ and } V_2 = 9.411 \text{ m/s}$$

$$\therefore \quad \textbf{Flow rate} = A_1 V_1 = A_2 V_2 = \textbf{0.166 m}^3\textbf{/s}$$

6.8 CONCEPT AND MEASUREMENT OF DYNAMIC, STATIC AND TOTAL HEAD

In the Bernoulli equation, the pressure term is known as static head. It is to be measured by a probe which will be perpendicular to the velocity direction. Such a probe is called static probe. The head measured is also called Piezometric head. (Figure 6.8.1 (*a*))

The velocity term in the Bernoulli equation is known as dynamic head. It is measured by a probe, one end of which should face the velocity direction and connected to one limb of a manometer with other end perpendicular to the velocity and connected to the other limb of the manometer. (Figure 6.8.1 (*b*))

The total head is the sum of the static and dynamic head and is measured by a single probe facing the flow direction. (Figure 6.8.1 (*c*))

The location of probes and values of pressures for the above measurements are shown in Fig. 6.8.1.

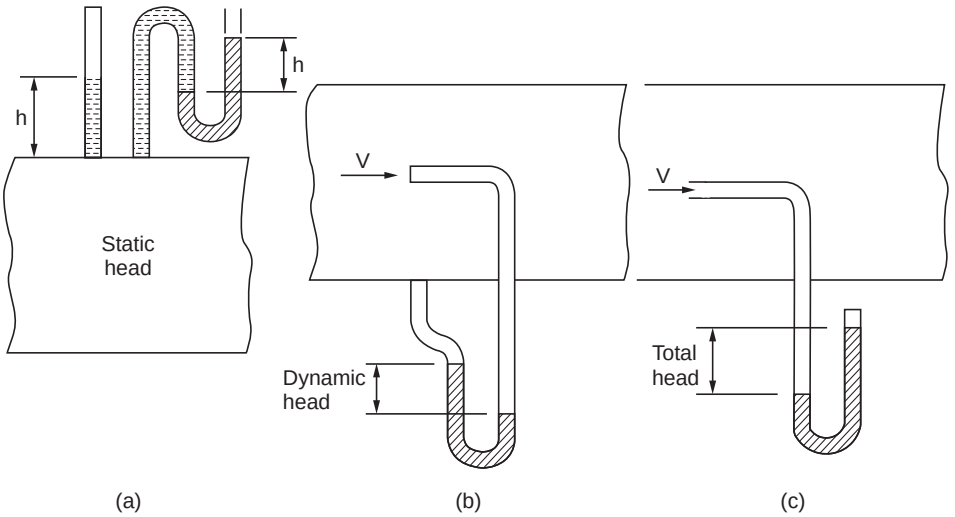


Figure 6.8.1 Pressure measurement

6.8.1 Pitot Tube

The flow velocity can be determined by measuring the dynamic head using a device known as pitot static tube as shown in Fig. 6.8.2. The holes on the outer wall of the probe provides the static pressure (perpendicular to flow) and hole in the tube tip facing the stream direction of flow measures the total pressure. The difference gives the dynamic pressure as indicated by the manometer. The head will be $h\ (s - 1)$ of water when a differential manometer is used ($s > 1$).

The velocity variation along the radius in a duct can be conveniently measured by this arrangement by traversing the probe across the section. This instrument is also called pitot–static tube.

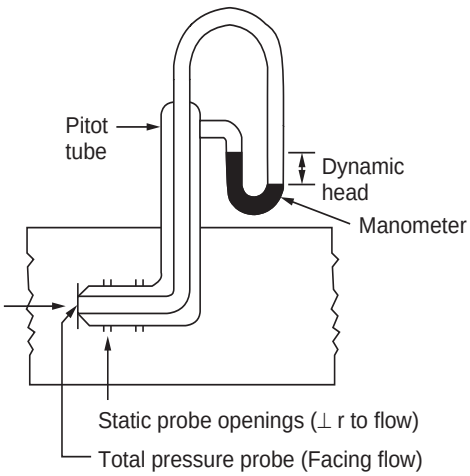


Figure 6.8.2 Pitot-Static tube

Example 6.12 *The dynamic head of a water jet stream is measured as 0.9 m of mercury column.* ***Determine the height to which the jet will rise*** *when it is directed vertically upwards.*

Considering the location at which the dynamic head is measured as the datum and converting the column of mercury into head of water, and noting that at the maximum point the velocity is zero,

$$0.9 \times 13.6 + 0 + 0 = 0 + 0 + Z \quad \therefore \mathbf{Z = 12.24\ m}$$

Note. If the head measured is given as the reading of a differential manometer, then the head should be calculated as $0.9\ (13.6 - 1)$ m.

Example 6.13 *A diverging tube connected to the outlet of a reaction turbine (fully flowing) is called "Draft tube". The diverging section is immersed in the tail race water and this provides additional head for the turbine by providing a pressure lower than the atmospheric pressure at the turbine exit. If the turbine outlet is open the exit pressure will be atmospheric as in Pelton wheel. In a draft tube as shown in Fig. Ex. 6.13,* ***calculate the additional head provided by the draft tube.*** *The inlet diameter is 0.5 m and the flow velocity is 8 m/s. The outlet diameter is 1.2 m. The height of the inlet above the water level is 3 m. Also calculate the pressure at the inlet section.*

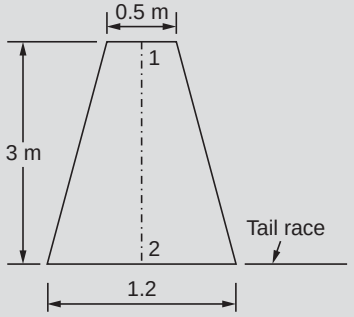


Figure Ex. 6.13 Draft tube

Considering sections 1 and 2

$$\frac{P_1}{\gamma} + \frac{V_1^2}{2g} Z_1 = \frac{P_1}{\gamma} + \frac{V_1^2}{2g} + Z_2$$

Considering tail race level, 2 as the datum, and calulating the velocities

$$V_1 = 8 \text{ m/s}, \; V_2 = 8 \times \frac{0.5^2}{1.2^2} = 1.39 \text{ m/s}.$$

$$P_2 = \text{atmospheric pressure}, \; Z_2 = 0, \; Z_1 = 3$$

$$\frac{P_1}{\gamma} + \frac{8^2}{2 \times 9.81} + 3 = \frac{1.39^2}{2 \times 9.81}$$

$$\therefore \qquad \frac{P_1}{\gamma} = \textbf{– 6.16 m of water.} \text{ (Below atmospheric pressure)}$$

Additional head provided due to the use of draft tube will equal 6.16 m of water

Note: This may cause cavitation if the pressure is below the vapour pressure at the temperature condition. Though theoretically the pressure at turbine exit can be reduced to a low level, cavitation problem limits the design pressure.

SOLVED PROBLEMS

Problem 6.1 *A venturimeter is used to measure the volume flow. The pressure head is recorded by a manometer. When connected to a horizontal pipe the manometer reading was h cm. If the reading of the manometer is the same when it is connected to a vertical pipe with flow upwards and (ii) vertical pipe with flow downwards, discuss in which case the flow is highest.*

Consider equation 6.6.2

$$Q = \frac{A_2}{\left[1 - (A_2/A_1)^2\right]^{0.5}} \left[2gh\left(\frac{S_2}{S_1} - 1\right)\right]^{0.5}$$

As long as '*h*' remains the same, the volume flow is the same for a given venturimeter as this expression is a general one derived without taking any particular inclination.

This is because of the fact that the manometer automatically takes the inclination into account in indicating the value of $(Z_1 - Z_2)$.

Problem 6.2 *Water flows at the rate of 600 l/s through a horizontal venturi with diameter 0.5 m and 0.245 m. The pressure gauge fitted at the entry to the venturi reads 2 bar.* ***Determine the throat pressure****. Barometric pressure is 1 bar.*

Using Bernoulli equation and neglecting losses

$$\frac{P_1}{\gamma} + \frac{V_1^2}{2g} + Z_1 = \frac{P_2}{\gamma} + \frac{V_2^2}{2g} + Z_2,\ P_1 = 2 \text{ bar (gauge)} = 3 \text{ bar (absolute)}\ 3 \times 10^5 \text{ N/m}^2$$

$$V_1 = \frac{Q}{(\pi \times d^2/4)} = \frac{0.6}{(\pi \times 0.5^2/4)} = 3.056 \text{ m/s} \quad \text{can also use}$$

$$V_2 = V_1 \left(\frac{D_2}{D_1}\right)^2$$

$$V_2 = \frac{0.6}{(\pi \times 0.245^2/4)} = 12.732 \text{ m/s, Substituting}$$

$$\frac{3 \times 10^5}{9810} + \frac{3.056^2}{2 \times 9.81} + 0 = \frac{P_2}{9810} + \frac{12.732^2}{2 \times 9.81} + 0$$

$\therefore$ $\mathbf{P_2} = 223617 \text{ N/m}^2 = \mathbf{2.236 \text{ bar (absolute)} = 1.136 \text{ bar (gauge)}}$

Problem 6.3 *A venturimeter as shown in Fig P. 6.3 is used measure flow of petrol with a specific gravity of 0.8. The manometer reads 10 cm of mercury of specific gravity 13.6.* ***Determine the flow rate.***

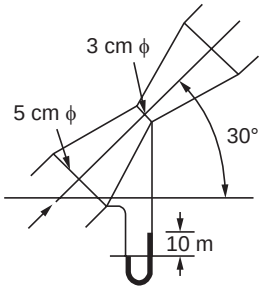


Figure P. 6.3 Problem model

Using equation 6.6.2

$$Q = \frac{A_2}{\left[1 - (A_2/A_1)^2\right]^{0.5}} \left[2gh\left(\frac{S_2}{S_1} - 1\right)\right]^{0.5}$$

$A_2 = (\pi/4)\ 0.03^2$ as $D_2 = 3$ cm

$\therefore$ $(A_2/A_1)^2 = (D_2/D_1)^4 = (0.03/0.05)^4$,

$h = 0.10$ m $S_2 = 13.6$, $S_1 = 0.8$, Substituting,

$$\boldsymbol{Q} = \frac{\left(\pi \times 0.03^2/4\right)}{\left[1 - (0.03/0.05)^4\right]^{0.5}} \left[2 \times 9.81 \times 0.1\left(\frac{13.6}{0.8} - 1\right)\right]^{0.5}$$

$= \mathbf{4.245 \times 10^{-3}\ m^3/s \text{ or } 15.282\ m^3/hr}$ or 4.245 l/s or 15282 l/hr or 3.396 kg/s

Problem 6.4 *A liquid with specific gravity 0.8 flows at the rate of 3 l/s through a venturimeter of diameters 6 cm and 4 cm. If the manometer fluid is mercury (sp. gr = 13.b)* ***determine the value of manometer reading, h.***

Using equation (6.6.2)

$$Q = \frac{A_2}{\left[1-(A_2/A_1)^2\right]^{0.5}}\left[2gh\left(\frac{S_2}{S_1}-1\right)\right]^{0.5}$$

$$A_1 = \frac{\pi \times 0.06^2}{4} = 2.83 \times 10^{-3}\ \text{m}^2\ ;$$

$$A_2 = \frac{\pi \times 0.04^2}{4} = 1.26 \times 10^{-3}\ \text{m}^2$$

$$3 \times 10^{-3} = \frac{1.26 \times 10^{-3}}{\left[1-\left(\frac{1.26 \times 10^{-3}}{2.83 \times 10^{-3}}\right)^2\right]^{0.5}}\left[2 \times 9.81 \times h\left(\frac{13.6}{0.8}-1\right)\right]^{0.5}$$

Solving, **h = 0.0146 m = 14.6 mm.** of mercury column.

Problem 6.5 *Water flows upwards in a vertical pipe line of gradually varying section from point 1 to point 2, which is 1.5m above point 1, at the rate of 0.9m³/s. At section 1 the pipe dia is 0.5m and pressure is 800 kPa. If pressure at section 2 is 600 kPa,* ***determine the pipe diameter at that location.*** *Neglect losses.*

Using Bernoulli equation,

$$\frac{P_1}{\gamma} + \frac{V_1^2}{2g} + Z_1 = \frac{P_2}{\gamma} + \frac{V_2^2}{2g} + Z_2$$

$$\frac{800 \times 10^3}{9810} + \frac{\left(0.9 \times 4/\pi \times 0.5^2\right)}{2 \times 9.81} + 0 = \frac{600 \times 10^3}{9810} + \frac{V_2^2}{2 \times 9.81} + 1.5$$

Solving, $V_2 = 19.37$ m/s.

$$\text{Flow = area} \times \text{velocity, } \frac{\pi \times d_2^2}{4} \times 19.37 = 0.9\ \text{m}^3/\text{s}$$

Solving for d_2, **Diameter of pipe at section 2 = 0.243 m**

As (p/γ) is involved directly on both sides, gauge pressure or absolute pressure can be used without error. However it is desirable to use absolute pressure to aviod nagative pressure values.

Problem 6.6 ***Calculate the exit diameter, if*** *at the inlet section of the draft tube the diameter is 1 m and the pressure is 0.405 bar absolute. The flow rate of water is 1600 l/s. The vertical distance between inlet and outlet is 6 m.*

Applying Bernoulli equation between points 1 and 2, neglecting losses

$$\frac{P_1}{\gamma} + \frac{V_1^2}{2g} + Z_1 = \frac{P_2}{\gamma} + \frac{V_2^2}{2g} + Z_2$$

$$V_1 = \frac{Q \times 4}{\pi \times D_1^{\,2}} = \frac{1600 \times 10^{-3} \times 4}{\pi \times 1^2} = 2.04 \text{ m/s}$$

P_2 = atmospheric pressure; $Z_2 = 0$ (datum); $Z_1 = 6$ m

$$\frac{0.405 \times 10^5}{9810} + \frac{2.04^2}{2 \times 9.81} + 6 = \frac{1.013 \times 10^5}{9810} + \frac{V_2^2}{2 \times 9.81} + 0 \quad \therefore V_2 = 0.531 \text{ m/s}$$

$$\frac{A_2}{A_1} = \frac{V_1}{V_2} = \frac{D_2^2}{1^2} = \frac{2.04}{0.531} \quad \therefore \quad \mathbf{D_2 = 1.96 \text{ m}}$$

0.405 bar absolute means vacuum at the inlet section of the draft tube. This may cause "cavitation" if this pressure is below the vapour pressure at that temperature. Though theoretically the pressure at turbine exit, where the draft tube is attached, can be reduced to a vary low level, cavitation problem limits the pressure level.

Problem 6.7 *Water flows at the rate of 200 l/s upwards through a tapered vertical pipe. The diameter at the bottom is 240 mm and at the top 200 mm and the length is 5m. The pressure at the bottom is 8 bar, and the pressure at the topside is 7.3 bar.* ***Determine the head loss through the pipe****. Express it as a function of exit velocity head.*

Applying Bernoulli equation between points 1 (bottom) and 2 (top) and considering the bottom level as datum.

$$\frac{P_1}{\gamma} + \frac{V_1^2}{2g} + Z_1 = \frac{P_2}{\gamma} + \frac{V_2^2}{2g} + Z_2 + \text{losses}$$

$$\frac{8 \times 10^2}{9810} + \frac{(200 \times 10^{-3} \times 4)/\pi \times 0.24^2)^2}{2 \times 9.81} + 0$$

$$= \frac{7.3 \times 10^5}{9810} + \frac{(200 \times 10^{-3} \times 4)/(\pi \times 0.2^2)^2}{2 \times 9.81} + 5 + \text{losses}$$

$\therefore$ **Losses = 1.07 m**

$$1.07 = X\frac{V_2^2}{2g} = X\left[\frac{200 \times 10^{-3} \times 4}{\pi \times 0.22}\right]^2 /2 \times 9.81 \quad \therefore X = 0.516,$$

$$\textbf{Loss of head} = 0.516\ \frac{\mathbf{V_2^2}}{\mathbf{2g}}$$

Problem 6.8 ***Calculate the flow rate of oil*** *(sp. gravity, 0.8) in the pipe line shown in Fig. P. 6.8. Also calculate* ***the reading "h"*** *shown by the differential manometer* ***fitted to the pipe*** *line which is filled with mercury of specific gravity 13.6.*

Applying Bernoulli equation (neglecting losses) between points 1 and 2

$$\frac{P_1}{\gamma} + \frac{V_1^2}{2g} + Z_1 = \frac{P_2}{\gamma} + \frac{V_2^2}{2g} + Z_2$$

$P_1 = 2 \times 10^5$ N/m^2; $P_2 = 0.8 \times 10^5$ N/m^2;

$Z_1 = 0$, $Z_2 = 2$ m

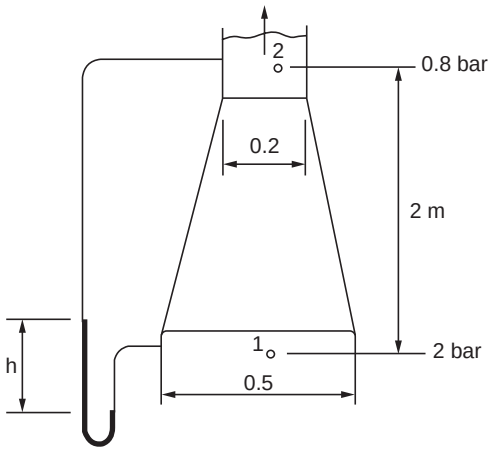


Figure P. 6.8

Applying continuity equation between points 1 and 2

$$A_1V_1 = A_2V_2,\ V_2 = V_1\frac{A_1}{A_2} = V_1\left(\frac{\pi \times 0.5^2/4}{\pi \times 0.2^2/4}\right) = 6.25\ V_1$$

$$\frac{2\times10^5}{9810\times0.8} + \frac{V_1^2}{2\times9.81} + 0 = \frac{0.8\times10^5}{9810\times0.8} + \frac{(6.25V_1)^2}{2\times9.81} + 2 \quad \therefore V_1 = 2.62 \text{ m/s}$$

Flow rate, $\mathbf{Q} = A_1\ V_1 = \dfrac{\pi \times 0.5^2}{4} \times 2.62 = 0.514$ m^3/s $=$ **514 l/s**

Using equation (6.6.2) (with $A_2 = 0.031$ m^2, $A_1 = 0.196$ m^2)

Flow rate, $$Q = \frac{A_2}{\left[1-(A_2/A_1)^2\right]^{0.5}}\left[2gh\left(\frac{S_2}{S_1}-1\right)\right]^{0.5}$$

$$0.514 = \frac{0.031}{\left[1-\left(\frac{0.031}{0.196}\right)^2\right]^{0.5}}\left[2\times9.81\times h\left(\frac{13.6}{0.8}-1\right)\right]^{0.5}$$

Solving, **h = 0.854 m**

Problem 6.9 *Water flows at the rate of 400 l/s through the pipe with inlet (1) diameter of 35 cm and (2) outlet diameter of 30 cm with 4m level difference with point 1 above point 2. If* $P_1 = P_2 = 2$ *bar absolute,* ***determine the direction of flow.***

Consider datum as plane 2

Total head 1, $\dfrac{2\times10^5}{9810} + \dfrac{\left(0.4\times4/\pi\times0.35^2\right)^2}{2\times9.81} + 4 = 25.27$ m water column

Total head at 2, $\dfrac{2\times10^5}{9810} + \dfrac{\left(0.4\times4/\pi\times0.3^2\right)^2}{2\times9.81} + 0 = 22.02$ m of water column

The total energy at all points should be equal if there are no losses. This result shows that there are losses between 1 and 2 as the total energy at 2 is lower. **Hence the flow will take place from points 1 to 2**.

Problem 6.10 *Petrol of relative density 0.82 flows in a pipe shown Fig. P.6.10. The pressure value at locations 1 and 2 are given as 138 kPa and 69 kPa respectively and point 2 is 1.2m vertically above point 1.* ***Determine the flow rate****. Also calculate* ***the reading of the differential manometer*** *connected as shown. Mercury with S = 13.6 is used as the manometer fluid.*

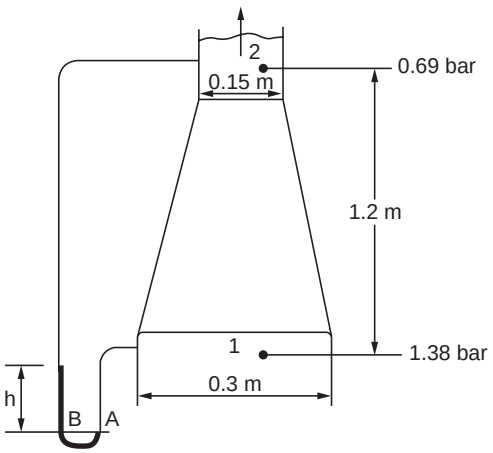


Figure P. 6.10 Problem Model

Considering point 1 as a datum and using Bernoulli equation.

$$\frac{P_1}{\gamma} + \frac{V_1^2}{2g} + Z_1 = \frac{P_2}{\gamma} + \frac{V_2^2}{2g} + Z_2,\ Z_1 = 0,\ Z_2 = 1.2\ \text{m},\ V_2 = V_1\frac{A_1}{A_2} = V_1\left(\frac{D_1^2}{D_2^2}\right)$$

$$\therefore \qquad V_2^{\,2} = V_1^{\,2}\left(\frac{D_1^4}{D_2^4}\right) = 16\,V_1^{\,2} \text{ as } D_1/D_2 = 2$$

$$\frac{138\times10^3}{0.82\times9810}+\frac{V_1^2}{2g}+0=\frac{69\times10^3}{0.82\times9810}+16\left(\frac{V_1^2}{2g}\right)+1.2$$

$$\frac{(138-69)10^3}{0.82\times9810}-1.2=15\frac{V_1^2}{2g}.\quad \text{Solving, } V_1=3.106 \text{ m/s}$$

$$\therefore \qquad \textbf{Volume flow} = \frac{\pi\times0.3^2}{4}\times3.106 = \textbf{0.22 m}^3\textbf{/s or 180 kg/s}$$

The flow rate is given by equation 6.6.2

$$Q=\frac{A_2}{\left[1-\left(\frac{A_2}{A_1}\right)^2\right]^{0.5}}\left[2gh\left(\frac{S_2}{S_1}-1\right)\right]^{0.5},\ \frac{S_2}{S_1}=\frac{13.6}{0.82}$$

$$0.22=\frac{\pi\times0.15^2/4}{\left[1-\left(\frac{0.15}{0.3}\right)^4\right]^{0.5}}\left[2\times9.81\times h\left(\frac{13.6}{0.82}-1\right)\right]^{0.5}$$

Solving, **h = 0.475 m** of mercury column

Problem 6.11 *Water flows downwards in a pipe as shown in Fig. P.6.11. If pressures at points 1 and 2 are to be equal,* ***determine the diameter of the pipe at point 2****. The velocity at point 1 is 6 m/s.*

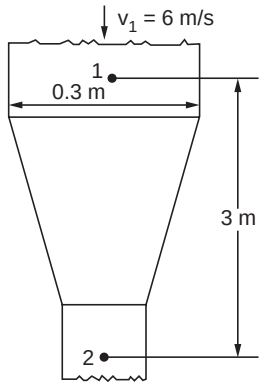


Figure P. 6.11 Problem model

Applying Bernoulli equation between points 1 and 2 (taking level 2 as datum)

$$\frac{P_1}{\gamma}+\frac{6^2}{2\times9.81}+3=\frac{P_2}{\gamma}+\frac{V_2^2}{2g}+0$$

as $P_1 = P_2$, $\mathbf{V_2 = 9.74\ m/s}$

Using the relation $A_1 V_1 = A_2 V_2$,

$$\frac{\pi\times0.3^2\times6}{4}=\frac{\pi\times d^2\times9.74}{4}$$

$$\therefore \qquad \textbf{d = 0.2355 m.}$$

Problem 6.12 *A siphon is shown in Fig P. 6.12. Point A is 1m above the water level, indicated by point 1. The bottom of the siphon is 8m below level A. Assuming friction to be negligible,* ***determine the speed of the jet at outlet and also the pressure at A.***

Using Bernoulli equation, between 1 and 2.

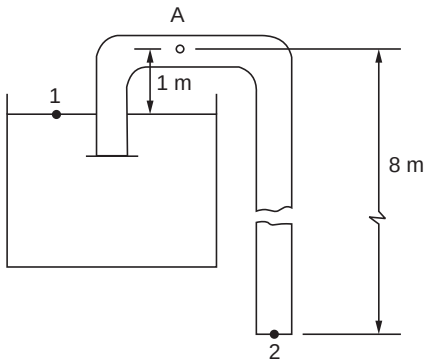


Figure P. 6.12 Problem model

$$\frac{P_1}{\gamma} + \frac{V_1^2}{2g} + Z_1 = \frac{P_2}{\gamma} + \frac{V_2^2}{2g} + Z_2 ,$$

$P_1 = P_2$ = atmospheric pressure.

Consider level 1 as datum. The velocity of water at the surface is zero.

$$\therefore \qquad 0 + 0 = \frac{V_2^2}{2g} - 7$$

$$\therefore \qquad \mathbf{V_2} = \sqrt{7 \times 2 \times 9.81} = \mathbf{11.72\ m/s = V_A}$$

Considering surface 1 and level *A*. As flow is the same,

$$\frac{P_1}{\gamma} + 0 + 0 = \frac{P_A}{\gamma} + 1 + \frac{V_A^2}{2g}$$

Considering $P_1/\gamma = 10.3$ m of water,

$$\frac{\mathbf{P_A}}{\boldsymbol{\gamma}} = \frac{P_1}{\gamma} - 1 - \frac{V_2^2}{2g} = 10.3 - 1 - 7$$

= **2.3 m of water column** (absolute)

Problem 6.13 *A pipe line is set up to draw water from a reservoir. The pipe line has to go over a barrier which is above the water level. The outlet is 8 m below water level.* ***Determine the maximum height of the barrier if the pressure*** *at this point should not fall below 1.0 m of water to avoid cavitation. Atmospheric pressure is 10.3 m.*

Considering outlet level 3 as datum and water level as 1 and appyling Bernoulli equation,

$$Z_3 = 0,\ Z_1 = 8,\ V_1 = 0,\ P_1 = P_3$$

$$\therefore \qquad 8 = \frac{V_3^2}{2g} \quad \therefore \quad V_3 = \sqrt{8 \times 9.81 \times 2} = 12.53 \text{ m/s}$$

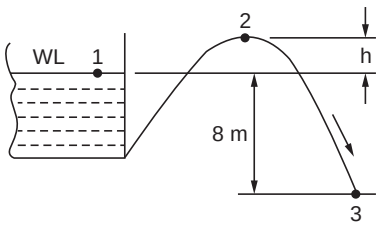


Figure P. 6.13 Problem model

Considering the barrier top as level 2

$$\frac{P_2}{\gamma} + \frac{V_2^2}{2g} + Z_2 = \frac{P_3}{\gamma} + \frac{V_3^2}{2g} + Z_3, \quad \text{As } V_2 = V_3,\ Z_3 = 0,\ P_2/\gamma = 1$$

$$1 + Z_2 = 10.3$$

$\therefore$ **Z_2 = 9.3 m.** Therefore the barrier can **be 1.3 m above water level.**

Problem 6.14 ***Determine the flow rate of water*** *across the shutter in an open canal if the water level upstream of shutter is 5m and downstream is 2m. The width of the canal is 1m and flow is steady.*

Applying Bernoulli equation between point 1 in the upstream and point 2 in the downstream on both sides of the shutter, both surface pressures being atmospheric.

$$\frac{V_1^2}{2g} + 5 = \frac{V_2^2}{2g} + 2 \tag{1}$$

Applying continuity equation, flow rate, $Q = A_1V_1 = A_2V_2$

$(1 \times 5)\ V_1 = (1 \times 2)\ V_2$, $\because$ $V_2 = 2.5\ V_1$, Substituting in equation (1),

$$\frac{V_1^2}{2 \times 9.81} + 5 = \frac{(2.5V_1)^2}{2 \times 9.81} + 2,$$

$\therefore$ $V_1 = 3.35$ m/s, $V_2 = 8.37$ m/s. **Q = 16.742 m³/s**,

Problem 6.15 *Uniform flow rate is maintained at a shutter in a wide channel. The water level in the channel upstream of shutter is 2m. Assuming uniform velocity at any section if the flow rate per m length is 3m³/s/m,* ***determine the level downstream***.

Assume velocities V_1 and V_2 upstream and downstream of shutter and the datum as the bed level. Using Bernoulli equation

$$2 + \frac{V_1^2}{2g} = h_2 + \frac{V_2^2}{2g} \tag{A}$$

Considering unit width from continuity $1 \times 2 \times V_1 = 1 \times h_2 \times V_2$ (B)

$\therefore$ $V_2 = (2/h_2)\ V_1$, from flow rate $V_1 = 3/2 = 1.5$ m/s $\therefore$ $V_2 = \dfrac{3}{h_2}$

Substituting

$$2 + \frac{1.5^2}{2 \times 9.81} = h_2 + \frac{3^2}{h^2 \times 2 \times 9.81}$$

Simplifying, this reduces to $h_2{}^3 - 2.1147\, h_2{}^2 + 0.4587 = 0$

Solving, h_2 can be **2 m, – 0.425 m, 0.54 m**

h_2 = 0.54 m is the acceptable answer. 2m being trivial.

Using B, $0.54 \times V_2 \times 1 = 2 \times 1.5 = 3$. $\therefore V_2 = 5.56$ m/s.

check using A, $2 + 0.1147 = 0.54 + 1.57$ checks.

The difference between the dynamic head values will equal the difference between the datum heads. This may be checked using the calculated velocity values.

Problem 6.16 *A pump with centre line 2m above the sump water level develops 50m head of water. The suction pipe is of 150 mm ID. The loss of head in the suction line is given by 5 $V_s^2/2g$. The delivery line is of 100 mm dia and the loss in the line is 12 $V_d^2/2g$. The water is delivered through a nozzle of 75 mm dia. The delivery is at 30m above the pump centre line.* ***Determine the velocity at the nozzle outlet and the pressure at the pump inlet***.

Let the velocity at the nozzle be V_n

Velocity in the delivery pipe $= V_d = V_n \times \frac{75^2}{100^2} = \frac{9}{16} V_n$

Velocity in suction pipe $V_s = V_n \left(\frac{75}{150}\right)^2 = \frac{V_n}{4}$

Kinetic head at outlet $= \frac{V_n^2}{2g}$

Loss in delivery pipe $= \frac{V_d^2}{2g} = 12 \times \left(\frac{9}{16}\right)^2 \frac{V_n^2}{2g} = 3.797 \frac{V_n^2}{2g}$

Loss in suction pipe $= \frac{V_s^2}{2g} = \frac{5}{16} \frac{V_n^2}{2g} = 0.3125 \frac{V_n^2}{2g}$

Equating the head developed to the static head, losses and kinetic head,

$$50 = 30 + 2 + \frac{V_n^2}{2g} [1 + 3.797 + 0.3125]$$

$$18 \times 2 \times 9.81 = V_n{}^2 [5.109]$$

$\therefore$ **Velocity at the nozzle V_n = 8.314 m/s**

Pressure at suction : Taking datum as the water surface and also the velocity of the water to be zero at the surface,

P_1 as atmospheric, 10.3 m of water column, Kinetic head $V^2/2g$, loss $5V^2/2g$

$$10.3 = \frac{P_2}{\gamma} + 2 + \left(\frac{(8.314/4)^2}{2 \times 9.81} \right) \times (5 + 1) \quad (\text{as } V_s = V_n/4)$$

$$\therefore \quad \frac{\mathbf{P_2}}{\boldsymbol{\gamma}} = 10.3 - 3.321 \text{ m} = \mathbf{6.979 \text{ m absolute}}$$

or 3.321 m below atmospheric pressure.

Problem 6.17 *A liquid jet at a velocity V_0 is projected at angle θ. Describe the path of the free jet. Also calculate the* ***maximum height and the horizontal distance travelled****.*

The horizontal component of the velocity of jet is $V_{xo} = V_o \cos \theta$. The vertical component $V_{zo} = V_o \sin \theta$.

In the vertical direction, distance travelled, Z, during time t, (using the second law of Newton)

$$Z = V_{zo}\, t - (1/2)\, gt^2 \tag{A}$$

The distance travelled along x direction

$$X = V_{xo}\, t \text{ or } t = X/V_{xo} \tag{B}$$

Solving for t from B and substituting in A,

$$Z = \frac{V_{zo}}{V_{xo}} X - \frac{1}{2} \frac{g}{V_{xo}^2} X^2 \tag{C}$$

Z value can be maximised by taking dz/dx and equating to zero

$$\frac{dz}{dx} = \frac{V_{zo}}{V_{xo}} - \frac{1}{2} \frac{g}{V_{xo}^2} 2X, \quad \frac{V_{zo}}{V_{xo}} = \frac{gX}{V_{xo}^2} \quad \therefore \quad X = V_{zo} V_{xo}/g$$

Substituting in C,

$$Z_{max} = \frac{V_{zo}}{V_{xo}} \cdot \frac{V_{zo} V_{xo}}{g} - \frac{1}{2} \frac{g}{V_{xo}^2} \cdot \frac{V_{zo}^2 V_{xo}^2}{g^2}$$

$$= \frac{1}{2} \frac{V_{zo}^2}{g} = \frac{V_0^{\,2} \sin^2 \theta}{2g}, \quad Z_{max} = V_o^{\,2} \sin^2 \theta / 2g \tag{D}$$

The maximum height is achieved when $\theta = 90°$.

$$\therefore \quad \mathbf{X_{mas} = 2 \text{ times x as } Z_{max.}}$$

$$X_{max} = 2V_o^{\,2} \sin \theta \cos \theta / g = V_o^{\,2} \sin 2\theta / g \tag{E}$$

Maximum horizontal reach is at $\theta = 45°$ or $2\theta = 90°$ and for this angle it will reach half the vertical height.

This describes an inverted parabola as shown in Fig. P.6.17

Bernoulli equation shows that $Z_t + V_t^2/2g$ = constant along the rejectory. V_t is the velocity at that location when air drag is neglected. Pressure is assumed to be uniform all over the trejectory as it is exposed to atmosphere all along its travel. Hence

$Z_t + V_t^2/2g$ = constant for the jet.

(**Note:** Velocity at time $t = V_{zo}$ $t = V_0 \sin\theta + a \times t$, where $a = -g$, so the velocity decreases, becomes zero and then turns – ve)

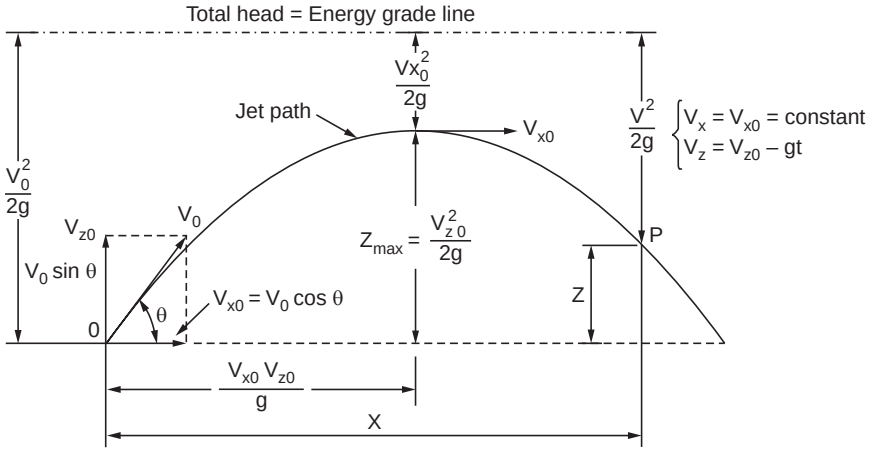


Figure P. 6.17 Jet trejectory

Problem 6.18 *A jet issuing at a velocity of 20 m/s is directed at 30° to the horizontal.* ***Calculate the height cleared*** *by the jet at 25m from the discharge location? Also* ***determine the maximum height the jet will clear*** *and the corresponding horizontal location.*

Ref Fig. P. 6.17

$$V_{xo} = V_o \cos 30 = 20 \cos 30 = 17.32 \text{ m/s};$$

$$V_{zo} = V_o \sin 30 = 20 \sin 30 = 10 \text{ m/s};$$

at time t, $X = V_{xo}t$; $Z = V_{zo}\, t - (1/2)\, gt^2$, Substituting for t as X/V_{xo} with $X = 25$ m

$$Z = \frac{V_{zo}}{V_{xo}} X - \frac{1}{2}\frac{g}{V_{xo}^2} X^2 \qquad \text{(A)}$$

Height cleared, $\quad \boldsymbol{Z_{25}} = \dfrac{10}{17.32} \times 25 - \dfrac{1}{2}\dfrac{9.81}{17.32^2} \times 25^2 = \mathbf{4.215\ m}$

Maximum height of the jet trajectory $= \dfrac{V_{zo}^2}{2g} = \dfrac{10^2}{2 \times 9.81} = \mathbf{5.097\ m}$

Corresponding horizontal distance $= \dfrac{V_{xo} V_{zo}}{g} = \dfrac{17.32 \times 10}{9.81} = \mathbf{17.66\ m}$

Total horizontal distance is twice the distance travelled in reaching

$$\mathbf{Z_{max} = 35.32\ m}$$

It would have crossed this height also at 10.43 m from the starting point (check using equations derived in Problem 6.17).

Problem 6.19 ***Determine the velocity of a jet*** *directed at 40° to the horizontal to clear 6 m height at a distance of 20m. Also determine the maximum height this jet will clear and the total horizontal travel. What will be the horizontal distance at which the jet will be again at 6m height.*

From basics, referring to Fig. P. 6.17,

$$V_{xo} = V_o \cos 40, \quad V_{zo} = V_o \sin 40,$$

$$X = V_{xo}\, t,\ t = \frac{X}{V_{xo}}, \quad Z = V_{zo}\, t - (1/2)\, gt^2$$

Substituting for t as X/V_{xo}

$$Z = \frac{V_{zo}}{V_{xo}} X - \frac{1}{2} \frac{g}{V_{xo}^{\ 2}} X^2 \qquad (A)$$

Substituting the values,

$$6 = \frac{V_o \sin 40}{V_o \cos 40} \times 20 - \frac{1}{2} \times \frac{9.81 \times 20^2}{V_o^{\ 2} \cos^2 40}$$

$$6 = 20 \tan 40 - \frac{1}{2} \frac{9.81 \times 20^2}{V_o^{\ 2} \cos^2 40} \qquad (B)$$

$$\therefore \qquad V_o^{\ 2} = \frac{9.81 \times 20^2}{2 \cos^2 40 (20 \tan 40 - 6)} = 310 \quad \therefore \quad \mathbf{V_o = 17.61\ m/s.}$$

Maximum height reached

$$= V_{zo}^{\ 2}/2g = (V_o \sin 40)^2/2g$$

$$= (17.61 \times \sin 40)^2 / 2 \times 9.81 = \mathbf{6.53\ m}$$

The X value corresponding to this is, (**half total horizontal travel**)

$$X = V_{xo} V_{zo}/g = 17.61^2 \sin 40 \cos 40/9.81 = \mathbf{15.56\ m.}$$

This shows that the jet clears 6m height at a distance of 20 m as it comes down. The jet would have cleared this height at a distance less than 15.56 m also. By symmetry, this can be calculated as – (20 – 15.56) + 15.56 = 11.12 m

check by substituting in equation B.

$$11.12 \tan 40 - \frac{1}{2} \times \frac{9.81 \times 11.12^2}{17.61^2 \cos^2 40} = 6$$

When both Z and X are specified unique solution is obtained. Given V_o and Z, two values of X is obtained from equation A.

Problem 6.20 ***Determine the angle at which a*** *jet with a given velocity is to be projected for obtaining maximum horizontal reach.*

Refer Problem 6.17. $\qquad X = V_{xo}\, t,\ Z = V_{zo}\, t - (1/2)\, gt^2$

The vertical velocity at any location/time is given by,

$$V_{zt} = \frac{dz}{dt} = V_{zo} - gt$$

The horizontal distance travelled will be half the total distance travelled when

$$V_{zt} = 0 \text{ or } t = V_{zo}/g$$

Total X distance travelled during time $2t$.

$$X = 2\, V_{xo}\, V_{zo}/g = 2\, V_o^2 \cos\theta \sin\theta/g = V_o^2 \sin 2\theta/g$$

For X to be maximum sin 2θ should be maximum or 2θ = 90° or

θ = 45°. For maximum horizontal reach, the projected angle should be 45°.

The maximum reach, $X = V_o^2/g$ as sin 2θ = 1.

Problem 6.21 ***Determined the angle at which*** *a jet with an initial velocity of 20 m/s is to be projected to clear 4m height at a distance of 10* m.

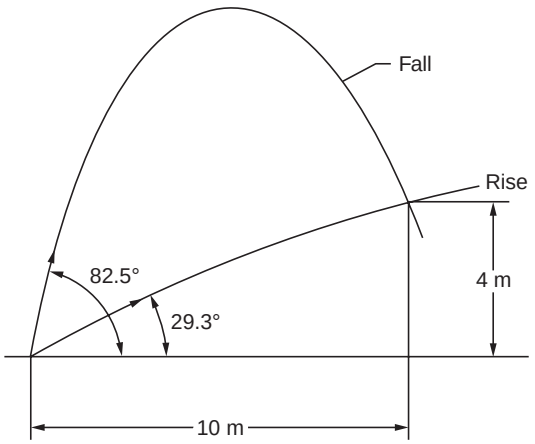


Figure P. 6.21 Jet trejectory

Refer Problem 6.17, eqn. *C*

$$Z = \frac{V_{zo}}{V_{xo}} x - \frac{1}{2} \frac{g x^2}{V_{xo}^2}$$

Substituting in terms of V_o and θ.

$$Z = \frac{V_o \sin\theta}{V_o \cos\theta} x - \frac{1}{2} \frac{g x^2}{V_o^2 \cos^2\theta} = x \tan\theta - \frac{1}{2} \frac{g x^2}{V_o^2} (\sec^2\theta)$$

$$Z = x \tan\theta - \frac{1}{2} \frac{g x^2}{V_o^2} (1 + \tan^2\theta)$$

Substituting the given values, $4 = 10 \tan\theta - \frac{1}{2} \frac{9.81 \times 10^2}{20^2} (1 + \tan^2\theta)$

Hence, $\tan^2\theta - 8.155 \tan\theta + 4.262 = 0$, solving **tan θ = 7.594 or 0.5613**

This corresponds to θ = 82.5° or 29.3°. In the first case it clears the height during the fall. In the second case it clears the height during the rise. See Fig. P.6.21.

Problem 6.22 *From a water tank two identical jets issue at distances H_1 and H_2 from the water level at the top. Both reach the same point at the ground level of the tank. If the* ***distance*** *from the ground level to the jet levels are y_1 and y_2.* ***Show that $H_1y_1 = H_2y_2$.***

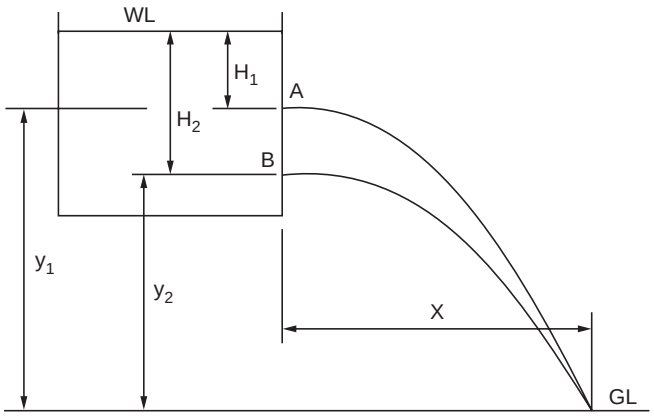


Figure P. 6.22 Problem model

In this case the jets issue out at A and B horizontally and so the position can be taken as the Z_{max} position.

Referring to Problem 6.17, eqn. (*D*)

$$Z_{max} = \frac{V_{zo}^{\ 2}}{2g}, \quad y_1 = \frac{V_{zo1}^{\ 2}}{2g} \quad \text{or} \quad V_{zo1} = \sqrt{2gy_1}$$

Similarly, $$y_2 = \frac{V^2_{zo2}}{2g} \quad \text{or} \quad V_{zo2} = \sqrt{2gy_2} \qquad (A)$$

(V_{zo1} and V_{zo2} are the Z components at point where the jet touches the ground)

$$X_{max} = \frac{V_{zo}V_{xo}}{g} \text{ and so } \frac{V_{zo1}V_{xo1}}{g} = \frac{V_{zo2}V_{xo2}}{g} \qquad (B)$$

$$V_{xo1} = \sqrt{2gH_1}\,, V_{xo2} = \sqrt{2gH_2} \qquad (C)$$

Substituting results (A) and (C) in equation (B), and simplifying,

$$\frac{\sqrt{2gH_1}\,\sqrt{2gy_1}}{g} = \frac{\sqrt{2gH_2}\,\sqrt{2gy_2}}{g} \quad \therefore \quad \mathbf{H_1\,y_1 = H_2\,y_2}$$

Problem 6.23 *A jet of water initially 12 cm dia when directed vertically upwards, reaches a maximum height of 20 m. Assuming the jet remains circular* ***determine the flow rate and area of jet at 10 m height***.

As $V = 0$ at a height of 20 m, Bernoulli equation reduces to

$$\frac{V^2}{2g} = 20,$$

$\therefore \qquad V = (20 \times 9.81 \times 2)^{0.5} = 19.809$ m/s

$$\textbf{Flow rate} = \text{area} \times \text{velocity} = \frac{\pi \times 0.12^2}{4} \times 19.809 = \mathbf{0.224\ m^3/s}$$

When the jet reaches 10 m height, the loss in kinetic energy is equal to the increase in potential energy. Consider this as level 2 and the maximum height as level 1 and ground as datum,

$$P_1 = P_2,\ V_1 = 0,\ Z_2 = Z_1 - 10 = (20 - 10) = 10$$

$$20 = 10 + \frac{V^2{}_{20}}{2g} \quad \therefore \quad \frac{V_2{}^2}{2g} = 10,$$

$\therefore \qquad \mathbf{V_2} = (10 \times 2 \times 9.81)^{0.5} = \mathbf{14\ m/s}$

$$\textbf{Flow rate} = \text{area} \times \text{velocity},\ 0.224 = \frac{\pi \times D^2}{4} \times 14 \quad \therefore \quad \mathbf{D = 0.1427\ m}$$

Problem 6.24 *Water is discharged through a 150 mm dia pipe fitted to the bottom of a tank. A pressure gauge fitted at the bottom of the pipe which is 10 m below the water level shows 0.5 bar. Determine the flow rate. Assume the frictional loss as* $4.5V_2^2/2g$.

Applying Bernoulli equation between the water level, 1 and the bottom of the pipe, 2 and this level as datum

$$\frac{P_1}{\gamma} + \frac{V_1{}^2}{2g} + Z_1 = \frac{P_2}{\gamma} + \frac{V_2{}^2}{2g} + Z_2 + \text{losses}$$

$$0 + 0 + 10 = \frac{0.5 \times 10^5}{9810} + \frac{V_2{}^2}{2 \times 9.81} + 0 + 4.5\ \frac{V_2{}^2}{2 \times 9.81}$$

Solving, $\qquad \mathbf{V_2 = 4.18\ m/s}$

$$\textbf{Flow rate} = \frac{\pi \times 0.15^2}{4} \times 4.18 = \mathbf{0.0739\ m^3/s = 73.9\ l/s.}$$

Problem 6.25 *An open tank of diameter D containing water to depth* h_o *is emptied by a smooth orifice at the bottom. Derive an expression for the time taken to reduce the height to h. Also find the time* t_{max} *for emptying the tank.*

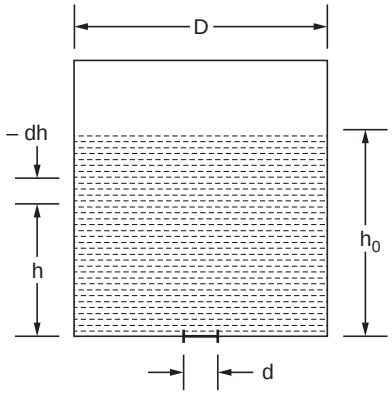


Figure P. 6.25 Problem model

Considering point 1 at the top of the tank and point 2 at the orifice entrance, and point 2 as datum

$$P_{atm} + \frac{V_1{}^2}{2g} + h = \frac{V_2{}^2}{2g} + p_{atm}$$

$\therefore \qquad \dfrac{V_1{}^2}{2g} + h = \dfrac{V_2{}^2}{2g}$

Also $\qquad V_1{}^2 = V_2{}^2 \left[\dfrac{d}{D}\right]^4$

$$\therefore \qquad V_2 = \sqrt{\frac{2gh}{1-(d/D)^4}}$$

Let the level at the time considered be h.

The drop in level dh during time dt is given by (as dh is negative with reference to datum)

$$\frac{dh}{dt} = -\frac{V_2 A_2}{A_1} = -\left(\frac{d}{D}\right)^2 \sqrt{\frac{2gh}{1-\left(\frac{d}{D}\right)^4}}$$

Taking $\left(\frac{d}{D}\right)^2$ inside and rearranging

$$\frac{dh}{dt} = -\sqrt{\frac{2gh}{\left(\frac{D}{d}\right)^4 - 1}}$$

Separating variables and integrating

$$\int_{h_o}^{h} \frac{dh}{\sqrt{h}} = -\sqrt{\frac{2g}{\left(\frac{D}{d}\right)^4 - 1}} \cdot \int_0^t dt$$

$$2\left[\sqrt{h_o} - \sqrt{h}\right] = \sqrt{\frac{2g}{\left(\frac{D}{d}\right)^4 - 1}} \cdot t \qquad (A)$$

$$t = 2(\sqrt{h_o} - \sqrt{h}) \Big/ \sqrt{\frac{2g}{\left(\frac{D}{d}\right)^4 - 1}} = \sqrt{h_o} - \sqrt{h} \Big/ \sqrt{\frac{g/2}{\left(\frac{D}{d}\right)^4 - 1}} \qquad (B)$$

Equation (A) can be rearranged to give

$$\frac{h}{h_0} = \left[1 - \frac{t\sqrt{g/2\,h_o}}{\sqrt{\left(\frac{D}{d}\right)^4 - 1}}\right]^2 \qquad (C)$$

Equation (B) will be useful to find the drop in head during a given time interval.

Consider a numerical problem.

Let **D = 0.5 m, d = 0.025 m, h_o = 0.5 m,**

Time for emptying is calculated as h = 0,

$$t = \sqrt{h_o} \Big/ \sqrt{\frac{g/2}{\left(\frac{D}{d}\right)^4 - 1}}$$

$$= \sqrt{0.5}/\sqrt{\frac{9.81/2}{\left(\frac{0.5}{0.025}\right)^4 - 1}} = 127.7 \text{ seconds.}$$

To find the drop in level in say 100 seconds.

$$\frac{h}{h_o} = \left[1 - \frac{100\sqrt{9.81/2 \times 0.5}}{\sqrt{\left(\frac{0.5}{0.25}\right)^4 - 1}}\right]^2 = 0.0471$$

$\therefore$ Drop in head = 0.5 (1 – 0.0471) = 0.4764 m

In case $d << D$, then $V_2 = \sqrt{2gh}$ when head is h m

$$\frac{dh}{dt} = -\frac{A_2 V_2}{A_1} = -V_2\left(\frac{d}{D}\right)^2 = -\left(\frac{d}{D}\right)^2 \cdot \sqrt{2gh}$$

Separating variables and integrating

$$\int_{h_o}^{h} \frac{dh}{\sqrt{h}} = -\left(\frac{d}{D}\right)^2 \sqrt{2g} \cdot \int_0^t dt$$

$$2\left[\sqrt{h_o} - \sqrt{h}\right] = \left(\frac{d}{D}\right)^2 \sqrt{2g} \cdot t$$

In this case to empty the tank,

$$2\sqrt{0.5} = \left(\frac{0.025}{0.5}\right)^2 \cdot \sqrt{2 \times 9.81} \cdot t.$$

Solving **t = 127.71 s**.

The same answer because the same diameter of the orifice is used. Say d = 0.01 m, then time for employing is 1130 sec.

Problem 6.26 *Two identical jets issuing from a touch as shown in figure reach the ground at a distance of 10 m.* ***Determine the distances indicated as h and H***.

Consider top jet:

x distance travelled in time t is 10 m.

$\therefore$ $V_{xo1}t = 10$ (*A*)

$t = 10/V_{xo1}$

The height drop is as V_{zo} as start is zero,

$\therefore$ $V_{zo1}t = H = \frac{1}{2} gt^2$ (*B*)

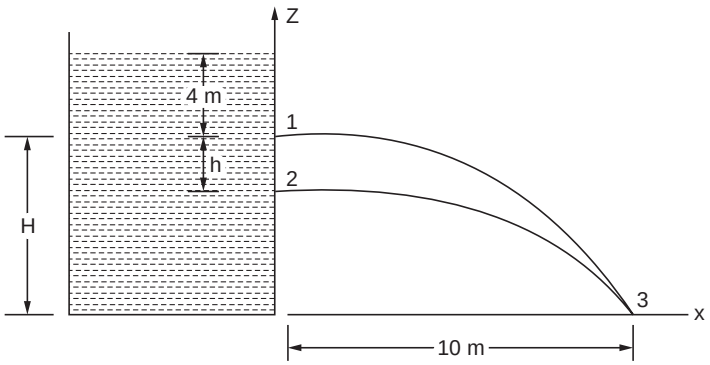


Figure P. 6.26 Problem model

Substituting for t

$$H = \frac{1}{2}\, g\, \frac{100}{V_{xo2}} \quad \therefore \quad V^2_{xo} = \frac{50g}{H}$$

As jet issues from the nozzle it has any x directional velocity V_{xo1}, is present.

$$V^2_{xo1} = 2\, g4 = 8g \qquad (C) \text{ (as head available in 4 m)}$$

Substituting, $\quad 8g = \dfrac{50g}{H}$ or **H = 6.25 m.**

Considering the second jet.

$$V_{xo2}\, t = 10,\ t = \frac{10}{V_{xo2}},$$

The head drop in $(H - h)$ m. As in the previous case $V_{zoc} = 0$ at start

$$H - h = \frac{1}{2}\, gt^2. \text{ Substituting}$$

$$H - h = \frac{1}{2} g\, \frac{1V_o}{V_{xo2}{}^2} = \frac{50g}{V_{xo2}{}^2} \qquad (D)$$

As at start only V_{xo2} is present,

$$V_{xo2}{}^2 = (4 + h)\, g \times 2$$

Substituting in (D)

$$H - h = \frac{50g}{(4+h)\, g \times 2} = \frac{25}{4+h}, \text{ as } \mathbf{H = 6.25\ m.}$$

$$6.25 - h = \frac{25}{4+h}. \text{ This leads to}$$

$$h^2 - 2.25h = 0, \quad \text{or} \quad \mathbf{h = 2.25\ m.}$$

It may be also noted from problem 6.22.

$$H \times 4 = (H - h)\,(4 + h).$$

$$6.25 \times 4 = 4 \times 6.25$$

Hence this condition is also satisfied.

OBJECTIVE QUESTIONS

O Q. 6.1. Fill in the blanks:

1. Kinetic energy of fluid element is due to its ————————.
2. The amount of kinetic energy per kg is given by the expression ———————— the unit used being head of fluid.
3. The kinetic energy in the unit Nm/kg is given by the expression ————————.
4. Potential energy of a fluid element is due to its ————————.
5. Potential energy of a fluid element in head of fluid is given by ————————.
6. Potential energy of a fluid element in Nm/kg is given by ————————.
7. Pressure energy or flow energy of a fluid element is given in head of fluid by the expression ———————.
8. Pressure energy or flow energy of a fluid element in the unit Nm/kg is given by the expression ———————.
9. Internal energy is due to ————————.
10. In the analysis of incompressible fluid flow, internal energy is rarely considered because ———————.
11. Electrical and magnetic energy become important in the flow of ————————.

Answers

(1) motion (2) $V^2/2g$, where V is the velocity (3) $V^2/2g_o$ where g_o is the force conversion constant having a unit of m kg/Ns^2 (4) location in the gravitational field. (5) Z, the elevation from datum (6) Zg/g_o, Z being elevation (7) $(P/\rho)\ (g_o/g) = p/\gamma.g_o$ where γ is specific weight. (8) P/ρ (9) the microscopic activity of atoms/molecules of the matter, exhibited by the temperature (10) temperature change is generally negligible (11) plasma.

O Q. 6.2. Fill in the blanks:

1. Eulers equation is applicable for flow along a ————————.
2. Bernoulli equation is applicable for flows which are ————————.
3. Bernoulli equation states that the total head ————————.
4. Total head in a steady incompressible irrotational flow is the sum of ————————.
5. In steady flow along a horizontal level as the velocity increases the pressure ————————.
6. The pressure along the diverging section of a venturi ————————.
7. Cavitation will occur when the pressure at a point ————————.
8. Draft tube ———————— the available head in the case of reaction turbines.
9. Energy line along the flow ———————— if there are no losses.
10. Hydraulic grade line represents the sum of ———————— along the flow.
11. If a pump supplies energy to the flow the energy line ———————— .
12. If there are frictional losses the energy grade line will ———————— .

Answers

(1) stream line (2) incompressible, steady and irrotational (3) remains constant if there are no irreversibilities (4) dynamic head, pressure head and potential head (5) decreases (6) increases (7) goes below the vapour pressure of the fluid at that temperature. (8) increases (9) will be horizontal parallel to the flow (10) pressure and potential head (11) will increase by a step (12) dip.

O Q. 6.3. Indicate whether the statement is correct or incorrect.

1. Energy line along the direction of flow will dip if there are losses.
2. When a pump supplies energy to a flow stream, the energy line will decreases by a step.
3. For ideal flows the energy line will slope upward along the flow.
4. If velocity increases, the hydraulic grade line will dip along the flow direction.
5. If the differential manometer reading connected to a venturimeter is the same, the flow will be independent of the position or flow direction.
6. For the same reading of the differential manometer connected to a vertical venturimeter, the flow rate will be larger if flow is downwards.
7. A pitot probe connected perpendicular to flow will indicate the total head.
8. A pitot probe facing the flow will indicate the dynamic head.
9. A pitot probe facing the flow will indicate the total head.
10. A pitot-static tube has probes both facing the flow and perpendicular to flow.
11. Flow will take place along hydraulic gradient.
12. Flow will take place along energy gradient.

Answers

Correct – 1, 4, 5, 9, 10, 11 Incorrect – 2, 3, 6, 7, 8, 12

O Q. 6.4. Choose the correct answer:

1. For a free jet the maximum horizontal reach will depend on
   (*a*) the angle of projection only (*b*) the initial velocity only
   (*c*) the fluid flowing in the jet (*d*) the angle of projection and initial velocity.
2. Bernoulli equation is applicable for
   (*a*) steady rotational flow (*b*) steady rotational compressible flow
   (*c*) steady irrotational incompressible flow (*d*) unsteady irrotational incompressible flow
   (*e*) all flows.
3. In a steady flow along a stream line at a location in the flow, the velocity head is 6 m, the pressure head is 3 m, the potential head is 4 m. The height of hydraulic gradient line at this location will be
   (*a*) 13 m (*b*) 9 m (*c*) 10 m (*d*) 7 m
4. In a flow along a varying flow cross section, as the area decreases
   (*a*) the energy line will slope up
   (*b*) the hydraulic gradient line will slope up
   (*c*) the hydraulic gradient line will slope down
   (*d*) the energy line will slope down.
5. In a steady flow of incompressible fluid, as the diameter is doubled, the velocity will
   (*a*) be halved (*b*) be doubled
   (*c*) increase four fold (*d*) decrease four fold.
6. In steady flow in a varying section pipe if the diameter is doubled the kinetic energy will
   (*a*) be doubled (*b*) increase 4 times
   (*c*) increase 8 times (*d*) decrease to one sixteenth.

7. In a source type of flow, the kinetic energy along the radius will vary (constant thickness of fluid along radius)
   (*a*) proportional to radius
   (*b*) directly proportional to the square root of radius
   (*c*) inversely proportional to the square of radius
   (*d*) proportional to the fourth power of radius
8. In a vertical flow of incompressible fluid along a constant pipe section under steady conditions, the pressure along flow direction will
   (*a*) remain constant (*b*) decrease
   (*c*) increase (*d*) increase or decrease depending on the fluid.
9. The differential manometer connected to two points along a pipe line gives a reading of h m. The flow will be
   (*a*) highest if the pipe is horizontal
   (*b*) independent of the slope of pipe and direction of flow
   (*c*) highest if flow is downwards
   (*d*) will depend on the fluid.

Answers

(1) *d* (2) *c* (3) *d* (4) *c* (5) *d* (6) *d* (7) *c* (8) *b* (9) *b*.

O Q. 6.5. Match the sets

1.

Set A	**Set B**
1. Bernoulli equation	(*a*) potential function
2. Continuity equation	(*b*) stream line
3. Eulers equation	(*c*) total head
4. Laplace equation	(*d*) conservation of mass.

2.

Set A	**Set B**
1. potential energy	(*a*) plasma flow
2. kinetic energy	(*b*) temperature
3. internal energy	(*c*) position
4. electrical energy	(*d*) velocity.

Answers

(1) 1*c*, 2*d*, 3*b*, 4*a* (2) 1*c*, 2*d*, 3*b*, 4*a*

EXERCISE PROBLEMS

E 6.1. A pipe inclined at 45° to the horizontal converges from 0.2 m dia to 0.1 m at the top over a length of 2 m. At the lower end the average velocity is $2m/s$. Oil of specific gravity 0.84 flows through the pipe. Determine the pressure difference between the ends, neglecting losses. If a mercury manometer (specific gravity 13.6) is used to measure the pressure, determine the reading of the manometer difference in m of mercury. Oil fills the limbs over mercury in the manometer. **(36.854 N/m^2, 0.201m)**

E 6.2. Oil of specific gravity of 0.9 flows through a venturimeter of diameters 0.4 and 0.2 m. A U-tube mercury manometer shows a head 0.63 m. Calculate the flow rate. **(0.105 m^3/s)**

E 6.3. Water flows from a reservoir 240 m above the tip of a nozzle. The velocity at the nozzle outlet is 66 m/s. The flow rate is 0.13 m^3/s. Calculate (1) the power of the jet. (2) the loss in head due to friction. **(283.14 kW, 17.98 m)**

E 6.4. Water flows in the middle floor tap at 3 m/s. Determine the velocities at the taps in the other two floors shown in Fig. E. 6.4.

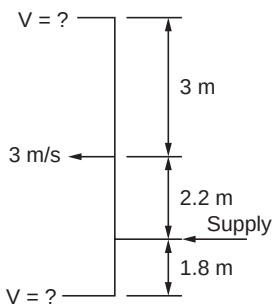


Figure E. 6.4

E 6.5. Oil flows through a horizontal pipe will line which has a diameter of 0.45 m at the start. After some distance the diameter of reduces to 0.3 m at which point the flow divides into pipes of 0.15 m and 0.225 m diameter. The velocity at the beginning is 1.8 m/s. The velocity in the pipe line of 0.225 m dia is 3.6 m/s. If the pressure at the start is 20 m head of oil and the specific gravity of the oil is 0.91 determine the pressure at the fork and also at the end of the two branch pipes. Neglect losses.

(V at fork = 4.05 m/s, $V_{0.15}$ = 8.1 m/s, P_{fork} = 13.33 m, $P_{0.15}$ = 16.821 m, $P_{0.225}$ = 19.5 m)

E. 6.6. A nozzle of 25 mm dia. directs a water jet vertically with a velocity of 12 m/s. Determine the diameter of the jet and the velocity at a height of 6 m. **(38.25 mm, 5.13 m/s)**

E 6.7. A pipe line is 36 m above datum. The pressure and velocity at a section are 410 kN/m^2 and 4.8 m/s. Determine the total energy per kg with reference to the datum. **(774.7 Nm.kg)**

E 6.8. The supply head to a water nozzle is 30 m gauge. The velocity of water leaving the nozzle is 22.5 m/s. Determine the efficiency and power that can be developed if the nozzle diameter is 75mm. **(84.3%, 25.2 kW)**

E 6.9. The suction pipe of a pump slopes at 1 m vertical for 5 m length. If the flow velocity in the pipe is 1.8 m/s and if the pressure in the pipe should not fall by more than 7 m of water, determine the maximum length. **(35.8 m)**

E 6.10. The pressure at the entry to the pipe line of 0.15 m dia. is 8.2 bar and the flow rate at this section is 7.5 m^3/min. The pipe diameter gradually increases to 0.3 m and the levels rises by 3 m above the entrance. Determine the pressure at the location. Neglect losses. **(8.14 bar)**

E 6.11. A tapering pipe is laid at a gradient of 1 in 100 downwards. The length is 300 m. The diameter reduces from 1.2 m to 0.6 m. The flow rate of water is 5500 l/min. The pressure at the upper location is 0.8 bar. Determine the pressure at the lower location. **(0.73 bar)**

E 6.12. A horizontal pipe carrying water is gradually tapering. At one section the diameter is 150 mm and flow velocity is 1.5 m/s. (*i*) If the drop in pressure is 1.104 bar at a reduced section determine the diameter at the section. (*ii*) If the drop in pressure is 5 kN/m^2, what will be the diameter? Neglect losses. **(47.6 mm, 100 mm)**

E 6.13. The diameter of a water jet at nozzle exist is 75 mm. If the diameter at a height of 12 m is 98.7mm, when the jet is directed vertically, determine the height to which the jet will rise. **(18 m)**

E 6.14. Calculate the height to which the jet, issuing at 18.8 m/s will rise when (*i*) The jet is directed vertically (*ii*) when it is directed at 45°. Also find the horizontal distance travelled in this case. **(18 m, 9 m, 9 m)**

E 6.15. A jet directed at 30° reaches a maximum height of 3 m at a horizontal distance of 18 m. Determine the issuing velocity of the jet. **(16.9 m/s)**

E 6.16. Determine the flow rate of a fluid of specific gravity 0.83 upward in the set up as shown in Fig. E. 6.16.

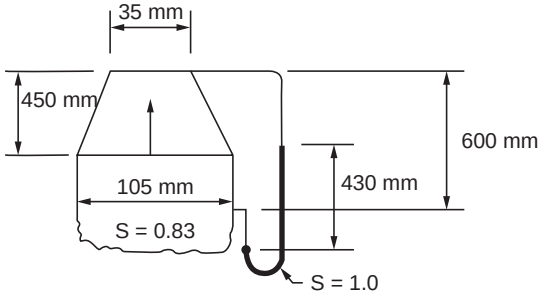


Figure E. 6.16

E 6.17. Determine the flow rate and also the pressure at point 2 in the siphon shown in Fig. E. 6.17. Diameter of the pipe is 2.5 cm. **(4.2 l/s, 45.1 kPa ab)**

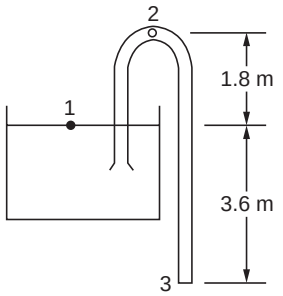


Figure E. 6.17

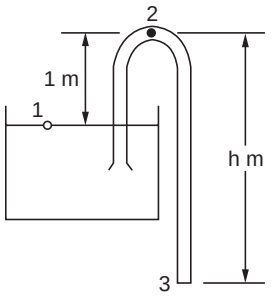


Figure E. 6.18

E 6.18. In the setup of siphon for water flow, shown in Fig. E. 6.18, determine the value of '*h*' and also the pressure at point 2, if velocity at 3 was 11.7 m/s. **(8 m, 22.8 kPa ab)**

E 6.19. A siphon is used to draw water from a tank. The arrangement is shown in Fig. E. 6.19. Calculate the flow rate and pressure at point 2. The frictional loss equals 40 $V^2/2g$. Atmospheric pressure is 10.2 m of water. **(43 l/min, 1.13 m of water)**

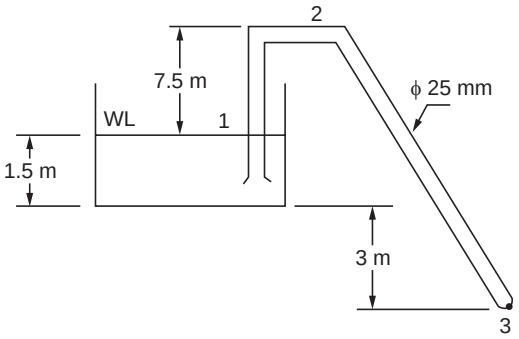


Figure E. 6.19

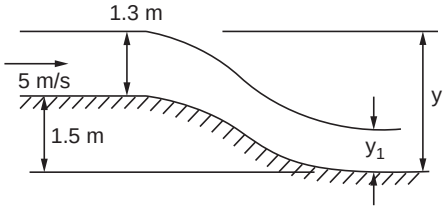


Figure E. 6.20

E 6.20. Water flows through a channel as shown in Fig. E. 6.20. Determine the possible values of depth of water down stream. Neglect losses. Assume uniform velocity of 5 m/s upstream.

E 6.21. Water flows up an inclined duct as shown in Fig. E. 6.21 Determine the possible depth of water upstream.

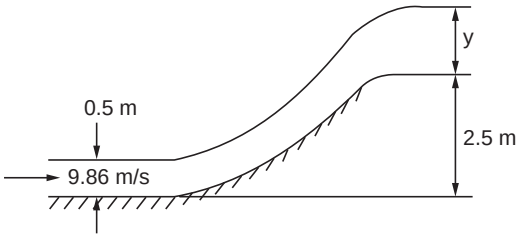


Figure E. 6.21

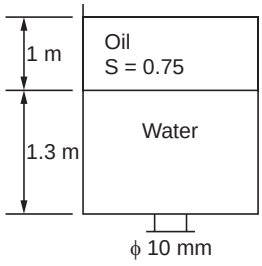


Figure E. 6.22

E 6.22. Neglecting losses, determine the flow rate in the setup shown in Fig. E. 6.22.

E 6.23. Derive an expression for the variation of jet radius r with distance y downwards for a jet directed downwards. The initial radius is R and the head of fluid is H.

E 6.24. For the venturimeter shown in Fig. E. 6.24, determine the flow rate of water.

E 6.25. A horizontal pipe divides into two pipes at angles as shown in Fig. E. 6.25. Determine the necessary forces along and perpendicular to the pipe to hold it in place. Assume that these are no losses.

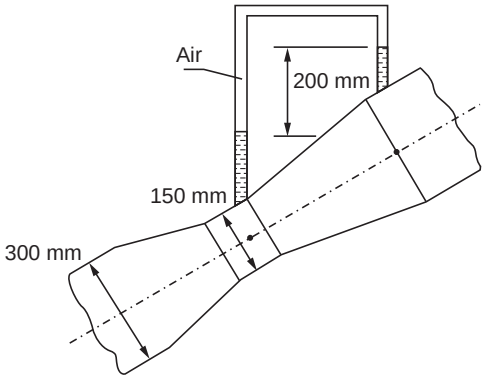


Figure E. 6.24

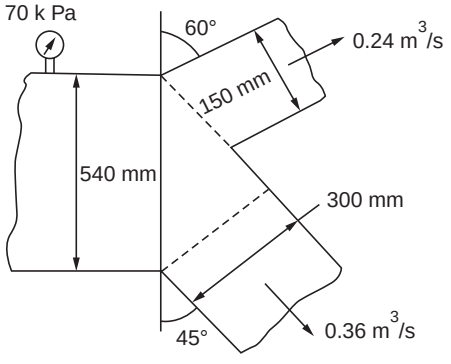


Figure E. 6.25

7 Flow in Closed Conduits (Pipes)

7.0 PARAMETERS INVOLVED IN THE STUDY OF FLOW THROUGH CLOSED CONDUITS

In the previous chapter, the energy level changes along the flow was discussed. The losses due to wall friction in flows was not discussed. In this chapter the determination of drop in pressure in pipe flow systems due to friction is attempted.

Fluids are conveyed (transported) through closed conduits in numerous industrial processes. It is found necessary to design the pipe system to carry a specified quantity of fluid between specified locations with minimum pressure loss. It is also necessary to consider the initial cost of the piping system.

The flow may be laminar with fluid flowing in an orderly way, with layers not mixing macroscopically. The momentum transfer and consequent shear induced is at the molecular level by pure diffusion. Such flow is encountered with very viscous fluids. Blood flow through the arteries and veins is generally laminar. Laminar condition prevails upto a certain velocity in fluids flowing in pipes.

The flow turns turbulent under certain conditions with macroscopic mixing of fluid layers in the flow. At any location the velocity varies about a mean value. Air flow and water flow in pipes are generally turbulent.

The flow is controlled by (*i*) pressure gradient (*ii*) the pipe diameter or hydraulic mean diameter (*iii*) the fluid properties like viscosity and density and (*iv*) the pipe roughness. The velocity distribution in the flow and the state of the flow namely laminar or turbulent also influence the design. Pressure drop for a given flow rate through a duct for a specified fluid is the main quantity to be calculated. The inverse-namely the quantity flow for a specified pressure drop is to be also worked out on occasions.

The basic laws involved in the study of incompressible flow are (*i*) Law of conservation of mass and (*ii*) Newton's laws of motion.

Besides these laws, modified Bernoulli equation is applicable in these flows.

7.1 BOUNDARY LAYER CONCEPT IN THE STUDY OF FLUID FLOW

When fluids flow over surfaces, the molecules near the surface are brought to rest due to the viscosity of the fluid. The adjacent layers also slow down, but to a lower and lower extent. This slowing down is found limited to a thin layer near the surface. The fluid beyond this layer is not affected by the presence of the surface. The fluid layer near the surface in which there is a general slowing down is defined as boundary layer. The velocity of flow in this layer increases from zero at the surface to free stream velocity at the edge of the boundary layer. The development of the boundary layer in flow over a flat plate and the velocity distribution in the layer are shown in Fig. 7.1.1.

Pressure drop in fluid flow is to overcome the viscous shear force which depends on the velocity gradient at the surface. Velocity gradient exists only in the boundary layer. The study thus involves mainly the study of the boundary layer. The boundary conditions are (*i*) at the wall surface, (zero thickness) the velocity is zero. (*ii*) at full thickness the velocity equals the free stream velocity (*iii*) The velocity gradient is zero at the full thickness. Use of the concept is that the main analysis can be limited to this layer.

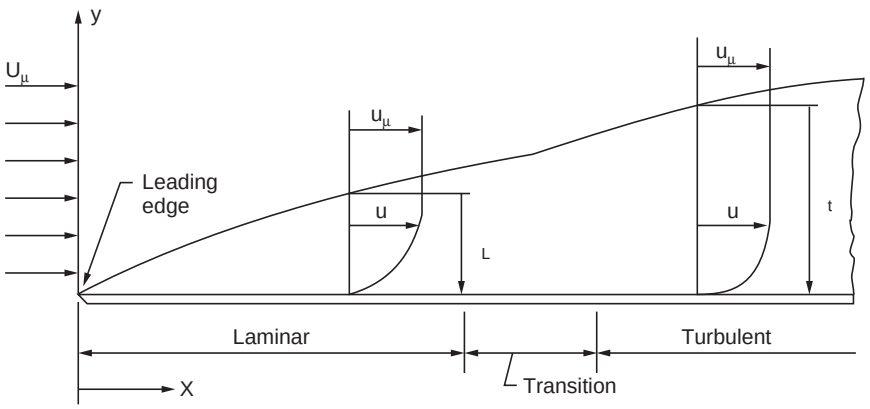


Figure 7.1.1 Boundary Layer Development (flat-plate)

7.2 BOUNDARY LAYER DEVELOPMENT OVER A FLAT PLATE

The situation when a uniform flow meets with a plane surface parallel to the flow is shown in Fig. 7.1.1. At the plane of entry (leading edge) the velocity is uniform and equals free stream velocity. Beyond this point, the fluid near the surface comes to rest and adjacent layers are retarded to a larger and larger depth as the flow proceeds.

The thickness of the boundary layer increases due to the continuous retardation of flow. The flow initially is laminar. There is no intermingling of layers. Momentum transfer is at the molecular level, mainly by diffusion. The viscous forces predominate over inertia forces. Small disturbances are damped out. Beyond a certain distance, the flow in the boundary layer becomes

turbulent with macroscopic mixing of layers. Inertia forces become predominant. This change occurs at a value of Reynolds number (given Re = ux/v, where v is the kinematic viscosity) of about 5×10^5 in the case of flow over flat plates. Reynolds number is the ratio of inertia and viscous forces. In the turbulent region momentum transfer and consequently the shear forces increase at a more rapid rate.

7.3 DEVELOPMENT OF BOUNDARY LAYER IN CLOSED CONDUITS (PIPES)

In this case the boundary layer develops all over the circumference. The initial development of the boundary layer is similar to that over the flat plate. At some distance from the entrance, the boundary layers merge and further changes in velocity distribution becomes impossible. The velocity profile beyond this point remains unchanged. The distance upto this point is known as entry length. It is about 0.04 Re × D. The flow beyond is said to be fully developed. The velocity profiles in the entry region and fully developed region are shown in Fig. 7.3.1*a*. The laminar or turbulent nature of the flow was first investigated by Osborn Reynolds in honour of

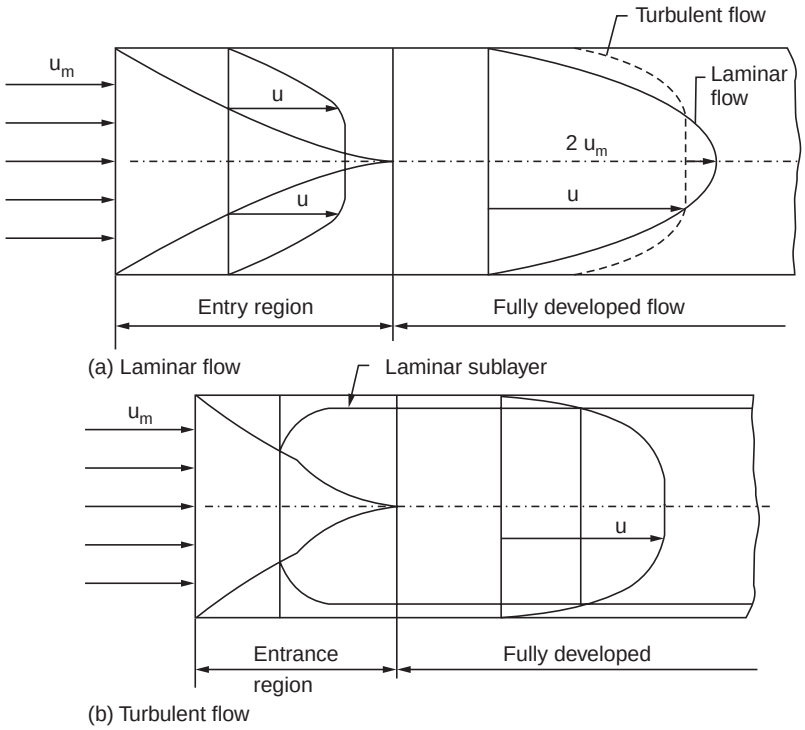


Figure 7.3.1 Boundary layer development (pipe flow)

whom the dimensionless ratio of inertia to viscous forces is named. The flow was observed to be laminar till a Reynolds number value of about 2300. The Reynolds number is calculated on the basis of diameter (ud/v). In pipe flow it is not a function of length. As long as the diameter

is constant, the Reynolds number depends on the velocity for a given flow. Hence the value of velocity determines the nature of flow in pipes for a given fluid. The value of the flow Reynolds number is decided by the diameter and the velocity and hence it is decided at the entry itself. The development of boundary layer in the turbulent range is shown in Fig. 7.3.1*b*. In this case, there is a very short length in which the flow is laminar. This length, *x,* can be calculated using the relation $ux/v = 2000$. After this length the flow in the boundary layer turns turbulent. A very thin laminar sublayer near the wall in which the velocity gradient is linear is present all through. After some length the boundary layers merge and the flow becomes fully developed. The entry length in turbulent flow is about 10 to 60 times the diameter.

The velocity profile in the fully developed flow remains constant and is generally more flat compared to laminar flow in which it is parabolic.

7.4 FEATURES OF LAMINAR AND TURBULENT FLOWS

In laminar region the flow is smooth and regular. The fluid layers do not mix macroscopically (more than a molecule at a time). If a dye is injected into the flow, the dye will travel along a straight line. Laminar flow will be maintained till the value of Reynolds number is less than of the critical value (2300 in conduits and 5×10^5 in flow over plates). In this region the viscous forces are able to damp out any disturbance.

The friction factor, f for pipe flow defined as $4\tau_s/(\rho u^2/2g_o)$ is obtainable as $f = 64/\text{Re}$ where τ_s is the wall shear stress, u is the average velocity and Re is the Reynolds number. In the case of flow through pipes, the average velocity is used to calculate Reynolds number. The dye path is shown in Fig. 7.4.1.

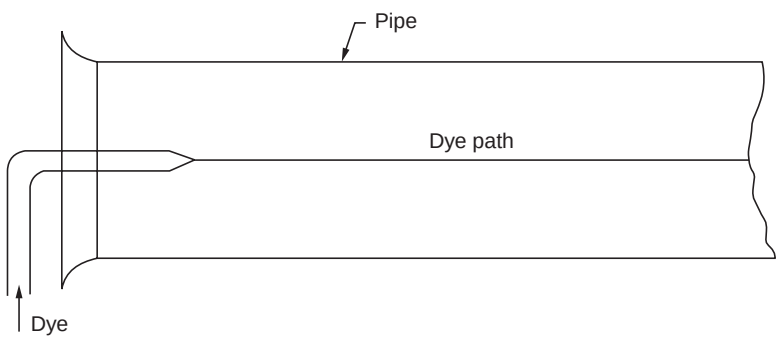


Figure 7.4.1 Reynolds Experiment

In turbulent flow there is considerable mixing between layers. A dye injected into the flow will quickly mix with the fluid. Most of the air and water flow in conduits will be turbulent. Turbulence leads to higher frictional losses leading to higher pressure drop. The friction factor is given by the following empirical relations.

$$f = 0.316/\text{Re}^{0.25} \quad \text{for Re} < 2 \times 10^4 \tag{7.4.1}$$

$$f = 0.186/\text{Re}^{0.2} \quad \text{for Re} > 2 \times 10^4 \tag{7.4.2}$$

These expressions apply for smooth pipes. In rough pipes, the flow may turn turbulent below the critical Reynolds number itself. The friction factor in rough pipe of diameter *D,* with a roughness height of ε, is given by

$$f = 1.325/[\ln \{(\varepsilon/3.7D) + 5.74/\mathrm{Re}^{0.9}\}]^2 \quad (7.4.3)$$

7.5 HYDRAULICALLY "ROUGH" AND "SMOOTH" PIPES

In turbulent flow, a thin layer near the surface is found to be laminar. As no fluid can flow up from the surface causing mixing, the laminar nature of flow near the surface is an acceptable assumption. The thickness of the layer δ_l is estimated as

$$\delta_l = 32.8v/u\sqrt{f} \quad (7.5.1)$$

If the roughness height is ε and if $\delta_1 > 6\varepsilon$, then the pipe is considered as hydraulically smooth. Any disturbance caused by the roughness is within the laminar layer and is smoothed out by the viscous forces. So the pipe is hydraulically smooth. If $\delta_l < 6\varepsilon$, then the pipe is said to be hydraulically rough. The disturbance now extends beyond the laminar layer. Here the inertial forces are predominant. So the disturbance due to the roughness cannot be damped out. Hence the pipe is hydraulically rough.

It may be noted that the relative value of the roughness determines whether the surface is hydraulically rough or smooth.

7.6 CONCEPT OF "HYDRAULIC DIAMETER": (D_h)

The frictional force is observed to depend on the area of contact between the fluid and the surface. For flow in pipes the surface area is not a direct function of the flow. The flow is a direct function of the sectional area which is proportional to the square of a length parameter. The surface area is proportional to the perimeter. So for a given section, the hydraulic diameter which determines the flow characteristics is defined by equation 7.6.1 and is used in the calculation of Reynolds number.

$$D_h = 4A/P \quad (7.6.1)$$

where D_h is the hydraulic diameter, A is the area of flow and P is the perimeter of the section. This definition is applicable for any cross section. For circular section $D_h = D$, as the equals $(4\pi D^2/4\pi D)$. For flow through ducts the length parameter in Reynolds number is the hydraulic diameter. *i.e.,*

$$\mathrm{Re} = D_h \times u/v \quad (7.6.2)$$

Example 7.1 *In model testing, similarity in flow through pipes will exist if Reynolds numbers are equal. Discuss how the factors can be adjusted to obtain equal Reynolds numbers.*

Reynolds number is defined as $Re = uD\rho/\mu$. For two different flows

$$\frac{u_1 D_1 \rho_1}{\mu_1} = \frac{u_2 D_2 \rho_2}{\mu_2} \text{ or } \frac{u_1 D_1}{v_1} = \frac{u_2 D_2}{v_2}$$

As the kinematic viscosities v_1 and v_2 are fluid properties and cannot be changed easily (except by changing the temperature) the situation is achieved by manipulating $u_2 D_2$ and $u_1 D_1$

$$\frac{v_2}{v_1} = \frac{u_2 D_2}{u_1 D_1} \tag{A}$$

this condition should be satisfied for flow similarity in ducts. Reynolds number will increase directly as the velocity, diameter and density. It will vary inversely with the dynamic viscosity of the fluid.

Reynolds number can be expressed also by **Re = G.D/**μ where G is the mass velocity in kg /m^{2}s. So Reynolds number in a given pipe and fluid can be increased by increasing mass velocity. For example if flow similarity between water and air is to be achieved at 20 °C then (using v values in eqn. A)

$$\frac{1.006 \times 10^{-6}}{15.06 \times 10^{-6}} = \frac{\text{Velocity of water} \times \text{diameter in water flow}}{\text{Velocity of air} \times \text{diameter of air flow}}$$

If diameters are the same, the air velocity should be about 15 times the velocity of water for flow similarity. If velocities should be the same, the diameter should be 15 times that for water. For experiments generally both are altered by smaller ratios to keep $u \times D$ constant.

7.7 VELOCITY VARIATION WITH RADIUS FOR FULLY DEVELOPED LAMINAR FLOW IN PIPES

In pipe flow, the velocity at the wall is zero due to viscosity and the value increases as the centre is approached. The variation if established will provide the flow rate as well as an average velocity.

Consider an annular element of fluid in the flow as shown in Fig. 7.7.1*a*. The dimensions are: inside radius = r; outside radius = $r + dr$, length = dx.

Surface area = $2\pi r dx$

Assuming steady fully developed flow, and using the relationship for force balance, the velocity being a function of radius only.

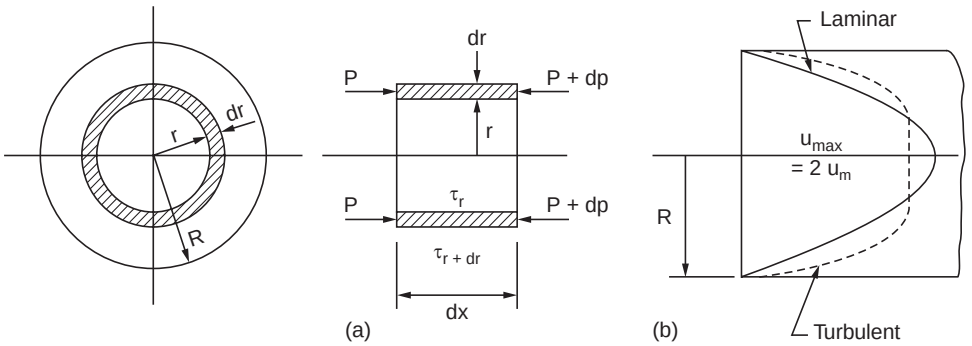


Figure 7.7.1

Net pressure force = $dp\ 2\pi\, rdr$

Net shear force $= \dfrac{d}{dr}\left(\mu \dfrac{du}{dr} 2\pi r dx\right) dr$, Equating the forces and reordering

$$\frac{d}{dr}\left(r \frac{du}{dr}\right) = \frac{1}{\mu} \frac{dp}{dx} r$$

Integrating $$r\frac{du}{dr} = \frac{1}{\mu} \frac{dp}{dx} \frac{r^2}{2} + C, \text{ at } r = 0 \quad \therefore \quad C = 0$$

Integrating again and after simplification,

$$u = \frac{1}{\mu} \frac{dp}{dx} \frac{r^2}{4} + B$$

at $r = R$, $u = 0$ (at the wall)

$$\therefore \qquad B = -\frac{1}{\mu} \frac{dp}{dx} \frac{R^2}{4}$$

$$\therefore \qquad u = -\frac{1}{\mu} \frac{dp}{dx} \frac{R^2}{4}\left[1 - \left(\frac{r}{R}\right)^2\right] \tag{7.7.1}$$

The velocity is maximum at $r = 0$,

$$\therefore \qquad u_{max} = -\frac{1}{\mu} \frac{dp}{dx} \frac{R^2}{4} \tag{7.72}$$

At a given radius, dividing 7.7.1 by (7.7.2), we get 7.7.3, which represents parabolic distribution.

$$\therefore \qquad \frac{u}{u_{max}} = 1 - \left(\frac{r}{R}\right)^2 \tag{7.7.3}$$

If the average velocity is u_{mean} then the flow is given by $Q = \pi R^2\, u_{mean}$ (A)

The flow Q is also given by the integration of small annular flow streams as in the element considered

$$Q = \int_0^R 2\pi u r dr \text{ but } u = u_{max}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

Substituting and integrating between the limits 0 to R, and using equation A

$$Q = \frac{\pi R^2}{u} u_{max} = \pi R^2 u_{mean} \quad \therefore \quad 2\, u_{mean} = u_{max}$$

The average velocity is half of the maximum velocity

$$\therefore \qquad \frac{u}{u_{mean}} = 2\left[1 - \left(\frac{r}{R}\right)^2\right] \tag{7.7.4}$$

In turbulent flow the velocity profile is generally represented by the equation

$$\frac{u}{u_{max}} = \left(1 - \frac{r}{R}\right)^{(1/n)}$$, where n varies with Reynolds number.

The average velocity is 0.79 u_{max} for n = 6 and 0.87 u_{max} for n = 10.

7.8 DARCY–WEISBACH EQUATION FOR CALCULATING PRESSURE DROP

In the design of piping systems the choice falls between the selection of diameter and the pressure drop. The selection of a larger diameter leads to higher initial cost. But the pressure drop is lower in such a case which leads to lower operating cost. So in the process of design of piping systems it becomes necessary to investigate the pressure drop for various diameters of pipe for a given flow rate. Another factor which affects the pressure drop is the pipe roughness. It is easily seen that the pressure drop will depend directly upon the length and inversely upon the diameter. The velocity will also be a factor and in this case the pressure drop will depend in the square of the velocity (refer Bernoulli equation).

Hence we can say that

$$\Delta p \propto \frac{LV^2}{2D} \tag{7.8.1}$$

The proportionality constant is found to depend on other factors. In the process of such determination Darcy defined or friction factor f as

$$f = 4\ \tau_0/(\rho u_m{}^2/2g_0) \tag{7.8.2}$$

This quantity is dimensionless which may be checked.

Extensive investigations have been made to determine the factors influencing the friction factor.

It is established that in laminar flow f depends only on the Reynolds number and it is given by

$$f = \frac{64}{\text{Re}} \tag{7.8.3}$$

In the turbulent region the friction factor is found to depend on Reynolds number for smooth pipes and both on Reynolds number and roughness for rough pipes. Some empirical equations are given in section 7.4 and also under discussions on turbulent flow. The value of friction factor with Reynolds number with roughness as parameter is available in Moody diagram, given in the appendix. Using the definition of Darcy friction factor and conditions of equilibrium, expression for pressure drop in pipes is derived in this section. Consider an elemental length L in the pipe. The pressures at sections 1 and 2 are P_1 and P_2.

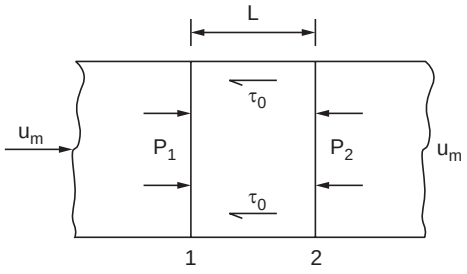


Figure 7.8.1

The other force involved on the element is the wall shear τ_0.

Net pressure force in the element is $(P_1 - P_2)$

Net shear force in the element is $\tau_0 \pi DL$

Force balance for equilibrium yields

$$(P_1 - P_2) \frac{\pi D^2}{4} = \tau_0 \pi DL \tag{7.8.4}$$

From the definition friction factor

$$f = 4 \tau_0 / (\rho u_m^2 / 2 g_0)$$

$$\tau_0 = \frac{f \rho u_m^2}{8 g_0}$$

Substituting and letting $(P_1 - P_2)$ to be ΔP.

$$\Delta P \cdot \frac{\pi D^2}{4} = \frac{f \rho u_m^2}{8 g_0} \cdot \pi DL$$

This reduces to $$\Delta P = \frac{f L u_m^2 \rho}{2 g_0 D} \tag{7.8.5}$$

This equation known as Darcy-Weisbach equation and is generally applicable in most of the pipe flow problems. As mentioned earlier, the value of f is to be obtained either from equations or from Moody diagram. The diameter for circular tubes will be the hydraulic diameter D_h defined earlier in the text.

It is found desirable to express the pressure drop as head of the flowing fluid.

In this case as $$h = \frac{P}{\gamma} = \frac{P g_0}{\rho g}$$

$$\Delta h = h_f = \frac{f L u_m^2}{2 g D} \tag{7.8.6}$$

The velocity term can be replaced in terms of volume flow and the equation obtained is found useful in designs as Q is generally specified in designs.

$$u_m = \frac{4 Q}{\pi D^2}, \; u_m^2 = \frac{16 Q^2}{\pi^2 D^4}$$

Substituting in (7.8.6), we get

$$h_f = \frac{8 f L Q^2}{\pi^2 g D^5} \tag{7.8.7}$$

It is found that $h_f \propto \frac{Q^2}{D^5}$

Another coefficient of friction C_f is defined as $C_f = f/4$

In this case $$h_f = \frac{4C_f\, L\, u_m^{\,2}}{2\, g\, D} \tag{7.8.8}$$

Now a days equation 7.8.5 are more popularly used as value of f is easily available.

7.9 HAGEN–POISEUILLE EQUATION FOR FRICTION DROP

In the case of laminar flow in pipes another equation is available for the calculation of pressure drop. The equation is derived in this section.

Refer to section (7.7) equation (7.7.1)

$$u = -\frac{1}{\mu}\frac{dp}{dL}\frac{R^2}{4}\left[1-\left(\frac{r}{R}\right)^2\right]$$

$\frac{dP}{dL}$ can be approximated to $\Delta P/L$ as the pressure drop is uniform along the length L under steady laminar flow

Using eqn (7.7.2), $u_{max} = -\frac{dP}{dL}\frac{1}{\mu}\cdot\frac{R^2}{4} = 2u_m$

$$\therefore \quad -\frac{dP}{dL} = \frac{8\, u_m\, \mu}{R^2}$$

$$\therefore \quad -\frac{dP}{dL} = \frac{8u_m\mu}{R^2} = \frac{32u_m\mu}{D^2}, \text{ Substituting for } -\frac{dP}{dL} \text{ as } \frac{\Delta P}{L}$$

$$\Delta P = \frac{32\,\mu\, u_m L}{D^2} \tag{7.9.1}$$

This can also be expressed in terms of volume flow rate Q as

$$Q = \frac{\pi D^2}{4}\cdot u_m$$

$$\therefore \quad u_m = 4Q/\pi D^2, \text{ substituting}$$

$$\Delta P = 128\,\mu\, L\, Q/\pi\, D^4 \tag{7.9.2}$$

Converting ΔP as head of fluid

$$h_f = \frac{32\, \upsilon u_m\, L g_0}{g D^2} \tag{7.9.3}$$

This equation is known as Hagen-Poiseuille equation

g_0 is the force conversion factor having a value of unity in the SI system of unit. Also $(\mu / \rho) = \nu$.

Equations 7.9.1, 7.9.2 and 7.9.3 are applicable for laminar flow only whereas Darcy-Weisbach equation (7.8.6) is applicable for all flows

Example 7.2 *Using the Darcy-Weisbach equation and the Hagen Poiseuille equation obtain an expression for friction factor f, in terms of Reynolds number in laminar region.*

The equation are $h_f = \dfrac{fLu_m^2}{2gD}$ and $h_f = \dfrac{32\, u_m\, vL}{gD^2}$

equating and simplifying a very useful relationship is obtained, namely

$$f = \frac{2 \times 32\ \nu}{u_m D} = \frac{64}{\mathrm{Re}}, \text{ as } \left(\frac{u_m D}{v}\right) = \mathrm{Re}$$

In the laminar flow region the friction factor can be determined directly in terms of Reynolds number.

7.10 SIGNIFICANCE OF REYNOLDS NUMBER IN PIPE FLOW

Reynolds number is the ratio of inertia force to viscous force. The inertia force is proportional to the mass flow and velocity *i.e.,* ($\rho u.u$). The viscous force is proportional to $\mu(du/dy)$ or $\mu\, u/D$, dividing

$$\frac{\text{inertia force}}{\text{viscous force}} = \frac{\rho u u D}{\mu u} = \frac{\rho u D}{\mu} = \frac{uD}{v}$$

Viscous force tends to keep the layers moving smoothly one over the other. Inertia forces tend to move the particles away from the layer. When viscous force are sufficiently high so that any disturbance is smoothed down, laminar flow prevails in pipes. When velocity increases, inertia forces increase and particles are pushed upwards out of the smoother path. As long as Reynolds number is below 2,300, laminar flow prevails in pipes. The friction factor in flow is also found to be a function of Reynolds number (in laminar flow, $f = 64/\mathrm{Re}$).

Example 7.3. *Lubricating Oil at a velocity of 1 m/s (average) flows through a pipe of 100 mm ID. Determine* ***whether the flow is laminar or turbulent.*** *Also* ***determine the friction factor*** *and the pressure drop over 10 m length. What should be the velocity for the flow to turn turbulent? Density = 930 kg/m^3 . Dynamic viscosity $\mu = 0.1$ Ns/m^2 (as N/m^2 is call Pascal, μ can be also expressed as Pa.s).*

$$\mathbf{Re} = \frac{uD\,\rho}{\mu} = \frac{1 \times 0.1 \times 930}{1 \times 0.1} = \mathbf{930, \text{ so the flow is laminar}}$$

Friction factor, $\mathbf{f} = 64/930 = \mathbf{0.06882}$

$$\mathbf{h_f} = f\, L\, u_m^2/2gD = (64/930) \times 10 \times 1^2/(2 \times 9.81 \times 0.1)$$
$$= \mathbf{0.351 \text{ m head of oil.}}$$

or $\Delta P = 0.351 \times 0.93 \times 9810 = \mathbf{3200\ N/m^2}$

At transition Re = 2000 (can be taken as 2300 also)

Using (7.9.1) (Hagen-Poiseuille eqn.)

$$\mathbf{\Delta P} = \frac{32 \times \mu \times u_m \times L}{D^2} = \frac{32 \times 0.1 \times 1 \times 10}{0.1^2}$$

$$= \mathbf{3200\ N/m^2.} \text{ (same as by the other equation)}$$

To determine velocity on critical condition

$$2300 = 4\ m \times 0.1 \times 930/0.1$$

$\therefore$ $\mathbf{u_m = 2.47\ m/s.}$

7.11 VELOCITY DISTRIBUTION AND FRICTION FACTOR FOR TURBULENT FLOW IN PIPES

The velocity profile and relation between the mean and maximum velocity are different in the two types of flow. In laminar flow the velocity profile is parabolic and the mean velocity is half of the maximum velocity. Such a relation is more complex in turbulent flow. For example one such available relation is given by

$$\frac{u_m}{u_{max}} = \frac{1}{1 + 1.33\sqrt{f}} \tag{7.11.1}$$

The friction factor f is a complex function of Reynolds number.

A sample velocity variation is given in equation (7.12.2).

$$u = (1 + 1.33\sqrt{f})\ u_m - 2.04\sqrt{f}\ u_m \log\ (R/(R-1)) \tag{7.11.2}.$$

For higher values of f the velocity variation will be well rounded at the centre compared to low values of f.

A new reference velocity called shear velocity is defined as below.

$$u^* = \sqrt{\frac{\tau_0\ g_0}{\rho}} \tag{7.11.3}$$

Several other correlation using the reference velocity are listed below.

$$\frac{u}{u^*} = 5.75 \log \frac{Ru^*}{\nu} + 5.5 \tag{7.11.4}$$

For rough pipes,

$$\frac{u}{u^*} = 5.75 \ln \frac{(R-r)}{\varepsilon} + 8.5 \tag{7.11.5}$$

where ε is the roughness dimension.

The mean velocity u_m is obtained for smooth and rough pipes as

$$\frac{u_m}{u^*} = 5.75 \log \frac{(R-r)\,u^*}{\nu} + 7.5 \tag{7.11.6}$$

and

$$\frac{u_m}{u^*} = 5.75 \log \frac{R}{\varepsilon} + 4.75 \tag{7.11.7}$$

The laminar sublayer thickness is used for defining smooth pipe. The thickness of this layer is given by

$$\delta_t = 11.6\ \nu/u^* \tag{7.11.8}$$

In the case of turbulent show the wall shear force is given by the following equation.

$$\tau_0 = \frac{f}{4} \cdot \frac{\rho u_m^2}{2} \tag{7.11.9}$$

Similar to velocity profile, several correlations are available for friction factor. These correlations together with correlations for velocity profile are useful in numerical methods of solution.

The friction factor for very smooth pipes can be calculated by assuming one seventh power law leading to,

$$f = 0.316/\text{Re}^{0.25} \text{ for Re} < 2 \times 10^4 \tag{7.11.10}$$

For all ranges either of the following relations can be used

$$f = 0.0032 + (0.221/\text{Re}^{0.237}) \tag{7.11.11}$$

$$1/\sqrt{f} = 1.8 \log \text{Re} - 1.5186 \tag{7.11.12}$$

For rough pipes of radius R

$$\frac{1}{\sqrt{f}} = 2 \log \frac{R}{\varepsilon} + 1.74 \tag{7.11.13}$$

Charts connecting f, Re and ε/D are also available and can be used without appreciable error. As in laminar flow the frictional loss of head is given by

$$h_f = f L u_m^2/2 gD_h \tag{7.11.14}$$

Also
$$h_f = \frac{8 fL}{g\pi^2} \frac{Q^2}{D^5} \tag{7.11.15}$$

The value of f is to be determined using the approximate relations or the chart.

7.12 MINOR LOSSES IN PIPE FLOW

Additional frictional losses occur at pipe entry, valves and fittings, sudden decrease or increase in flow area or where direction of flow changes.

The frictional losses other than pipe friction are called minor losses. In a pipe system design, it is necessary to take into account all such losses.

These losses are generally expressed as $h_f = C u_m^2/2g$ where C is constant, the value of which will depend on the situation and is called the loss coefficient. The expression is applicable both for laminar and turbulent flows.

***(i)* Loss of head at entrance:** At the entrance from the reservoir into the pipe, losses take place due to the turbulence created downstream of the entrance. Three types of entrances are known.

(a) Bell mouthed: This is a smooth entrance and turbulence is suppressed to a great extent and $C = 0.04$ for this situation.

(b) Square edged entrance: Though it is desirable to provide a bell mouthed entrance it will not be always practicable. Square edged entrance is used more popularly. The loss coefficient, $C = 0.5$ in this case.

(c) Reentrant inlet: The pipe may sometimes protrude from the wall into the liquid. Such an arrangement is called reentrant inlet. The loss coefficient in this case is about 0.8.

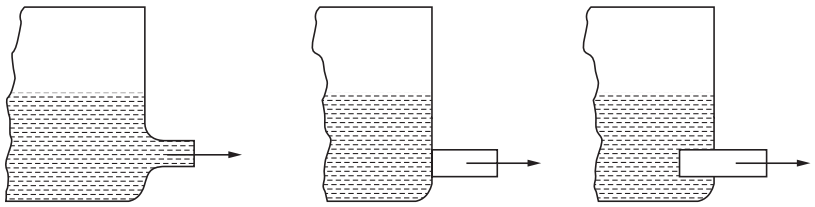


Figure 7.12.1 Types of entrance

***(ii)* Loss of head at submerged discharge:** When a pipe with submerged outlet discharges into a liquid which is still (not moving) whole of the dynamic head $u^2/2g$ will be lost. The loss coefficient is 1.0. The discharge from reaction turbines into the tail race water is an example. The loss is reduced by providing a diverging pipe to reduce the exit velocity.

***(iii)* Sudden contraction:** When the pipe section is suddenly reduced, loss coefficient depends on the diameter ratio. The value is 0.33 for $D_2/D_1 = 0.5$. The values are generally available in a tabular statement connecting D_2/D_1 and loss coefficient. Gradual contraction will reduce the loss. For gradual contraction it varies with the angle of the transition section from 0.05 to 0.08 for angles of 10° to 60°.

***(iv)* Sudden expansion:** Here the sudden expansion creates pockets of eddying turbulence leading to losses. The loss of head h_f is given by

$$\text{Loss of head} = (u_1 - u_2)^2 / 2g. \tag{7.12.1}$$

where u_1 and u_2 are the velocities in the smaller and larger sections. Gradual expansion will reduce the losses.

***(v)* Valves and fittings :** Losses in flow through valves and fittings is expressed in terms of an equivalent length of straight pipe.

For gate valves $L = 8D$, and for globe valves it is 340 D. For 90° bends it is about 30 D.

7.13 EXPRESSION FOR THE LOSS OF HEAD AT SUDDEN EXPANSION IN PIPE FLOW

The situation is shown in Fig 7.13.1.

Using Bernoulli equation and denoting the ideal pressure at section 2 as P_2 (without losses), datum remaining unaltered,

$$\frac{P_2}{\rho g} = \frac{P_1}{\rho g} + \frac{u_1^2}{2g} - \frac{u_2^2}{2g} \quad \text{or} \quad \frac{P_2}{\rho} = \frac{P_1}{\rho} + \frac{u_1^2}{2} - \frac{u_2^2}{2} \tag{1}$$

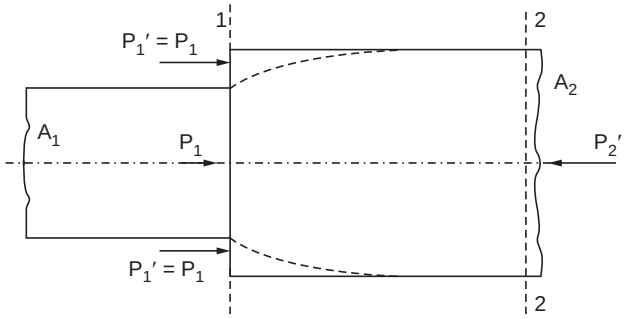


Figure 7.13.1 Sudden Expansion

Applying conservation of momentum principle to the fluid between section 1 and 2, and denoting the actual pressure at section 2 as P_2,

The pressure forces are (here the pressure on the annular section of fluid at 1 is assumed as P_1)

$$(P_1 A_1 - P_2' A_2).$$

The change in momentum is given by

$$(\rho A_2 u_2 u_2 - \rho A_1 u_1 u_1)$$

noting $A_1 u_1 = A_2 u_2$, replacing $A_1 u_1$ by $A_2 u_2$ and equating the net forces on the element to the momentum change,

$$P_1 A_1 - P_2' A_2 = \rho A_2 u_2^2 - \rho A_2 u_2 u_1$$

Dividing by ρ and A_2 allthrough

$$\frac{P_1}{\rho} - \frac{P_2'}{\rho} = u_2^2 - u_1 u_2$$

or
$$\frac{P_2'}{\rho} = \frac{P_1}{\rho} - (u_2^2 - u_1 u_2) \qquad (2)$$

Subtracting on either side of equations 1 and 2 (ideal and real)

$$\frac{P_2 - P_2'}{\rho} = \frac{P_1}{\rho} + \frac{u_1^2}{2} - \frac{u_2^2}{2} - \frac{P_1}{\rho} - (u_2^2 - u_1 u_2)$$

$\therefore$
$$\frac{P_2 - P_2'}{\rho} = \frac{u_1^2}{2} - \frac{u_2^2}{2} + (u_2^2 - u_1 u_2)$$

Multiplying both sides by 2

$$\frac{2(P_2 - P_2')}{\rho} = u_1^2 - u_2^2 + 2u_2^2 - 2u_1 u_2 = (u_1 - u_2)^2$$

Dividing the both sides by g and simplifying

$$\frac{P_2 - P_2'}{\rho g} = \frac{(u_2 - u_2)^2}{2g}$$

But
$$\frac{P_2 - P_2'}{\rho g} = h_f \text{ (head loss)}$$

$\therefore$
$$h_f = \frac{(u_1 - u_2)^2}{2g}.$$

7.14 LOSSES IN ELBOWS, BENDS AND OTHER PIPE FITTINGS

Fittings like valves, elbows etc. introduce frictional losses either by obstruction or due to secondary flows. The losses may be accounted for by a term equivalent length which will depend on the type of fitting or in terms of $(u^2/2g)$ or dynamic head. In the case of bends, the loss is due to the variation of centrifugal force along different stream lines which causes secondary flows. In large bends fitting curved vanes will reduce the loss. The loss will vary with radius of the bend. Globe valves are poorer compared to gate valves with regard to pressure drop.

7.15 ENERGY LINE AND HYDRAULIC GRADE LINE IN CONDUIT FLOW

The plot of the sum of pressure head and dynamic head along the flow path is known as energy line. This refers to the total available energy of the system at the location. The line will dip due to losses. For example in straight constant area pipe the line will slope proportional to the head drop per m length. There will be sudden dips if there are minor losses due to expansions, fittings etc.

Hydraulic grade line is the plot of pressure head along the flow path. Hydraulic grade line will be at a lower level and the difference between the ordinates will equal the dynamic head *i.e.*, $u^2/2g$. This line will dip sharply if velocity increases and will slope upwards if velocity decreases. This line will also dip due to frictional losses. Flow will be governed by hydraulic grade line.

Introduction of a pump in the line will push up both the lines. Specimen plot is given in Fig. 7.15.1 (pump is not indicated in figure).

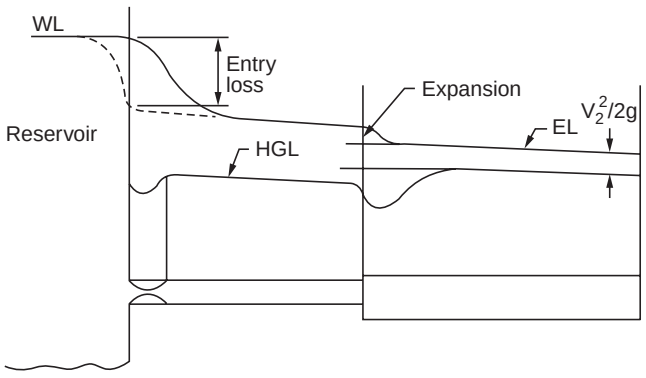


Figure 7.15.1 Energy and Hydraulic grade lines

Example 7.4. *A pump takes in water from a level 5 m below its centre line and delivers it at a height of 30 m above the centre line, the rate of flow being 3 m^3/hr. The diameter of the pipe line allthrough is 50 mm (ID). The fittings introduce losses equal to 10 m length of pipe in addition to the actual length of 45 m of pipe used.* ***Determine the head to be developed by the pump.***

The head to be developed will equal the static head, friction head and the dynamic head.

Static head = 30 + 5 = 35 m, Friction head = $f L u_m^2/2g D$,

$L = 45 + 10 = 55$ m, $D = 0.05$ m,

$u_m = (3/3600)\ 4/\pi \times 0.05^2 = 0.4244$ m/s.

Assuming the temperature as 20 °*C*,

$v = 1.006 \times 10^{-6}$ m^2/s.

$\therefore$ $\text{Re} = u_m D/v = 0.05 \times 0.4244/1.006 \times 10^{-6} = 21093$

$\therefore$ The flow is turbulent

Assuming smooth pipe, $f = 0.316/\text{Re}^{0.25} = 0.316/21093^{0.25} = 0.02622$

(check $f = 0.0032 + 0.221/\text{Re}^{0.237} = 0.0241$), using the value 0.02622,

Frictional loss of head $= 0.02622 \times 55 \times 0.4244^2/2 \times 9.81 \times 0.05 = 0.265$ m

Dynamic head $= u_m^2/2g = \dfrac{0.4244^2}{2 \times 9.81} = 0.0092$ m

$\therefore$ Total head $= 35 + 0.265 + 0.0092 = 35.2562$ m.

Head to be developed by the pump = **35.2562 m head of water.**

Example 7.5 *A pipe 250 mm dia, 4000 m long with f = 0.021 discharges water from a reservoir at a level 5.2 m below the water reservoir level.* ***Determine the rate of discharge.***

The head available should equal the sum of frictional loss and the dynamic head.

Frictional Head = $f L u_m^2/2g D$, Dynamic Head = $u_m^2/2g$

When long pipes are involved, minor losses may often be neglected.

$5.2 = [(0.021 \times 4000 \times u_m^2)/(2 \times 9.81 \times 0.25)] + [u_m^2/(2 \times 9.81)]$

$= 17.176\ u_m^2$

$\therefore$ $u_m = 0.55$ m/s.

$\therefore$ **Flow rate** $= (\pi \times 0.25^2/4) \times 0.55 = 0.027$ m^3/s. or **97.23 m^3/hr.**

7.16 CONCEPT OF EQUIVALENT LENGTH

For calculation of minor losses it is more convenient to express the pressure drop in fittings, expansion–contraction and at entry in terms of a length of pipe which will at that discharge rate lead to the same pressure drop. This length is known as equivalent length, L_e. From the relation knowing *C*, *f* and *D*.

$L_e u_m^2/2g D = C u_m^2/2g \quad \therefore \quad L_e$ can be calculated

As a number of fittings at various positions may be involved causing minor losses in a pipe system, this is a convienent way to estimate minor losses.

7.17 CONCEPT OF EQUIVALENT PIPE OR EQUIVALENT LENGTH

When pipes of different friction factors are connected in series (or in parallel) it is convenient to express the losses in terms of one of the pipes (Refer to 7.11.15). The friction loss h_f for pipe 1 with L_1 and f_1 is given by

$$h_{f1} = 8\, f_1 L_1\, Q^2/g\, \pi^2\, D_1^{\,5}$$

For the same pressure loss and flow rate Q and discharge through another pipe of diameter D_2 with f_2, the equivalent pipe will have a length L_2. Hence

$$h_{f2} = 8\, f_2 L_2\, Q^2/g\, \pi^2\, D_2^{\,5}$$

Cancelling common terms

$$\frac{f_1\, L_1}{D_1^{\,5}} = \frac{f_2\, L_2}{D_2^{\,5}} \quad \text{or} \quad L_2 = L_1\, \frac{f_1}{f_2}\, \frac{D_2^{\,5}}{D_1^{\,5}} \tag{7.17.1}$$

If pipes are in series a common diameter can be chosen and the equivalent length concept can be used conveniently to obtain the solution.

In parallel arrangement, as the pressure loss is the same, then

$$\frac{8\, f_1\, L_1\, Q_1^{\,2}}{g\, \pi^2\, D_1^{\,5}} = \frac{8\, f_2\, L_2\, Q_2^{\,2}}{g\, \pi^2\, D_2^{\,5}}$$

$$\therefore \qquad \frac{Q_2}{Q_1} = \left[\frac{f_1}{f_2}\, \frac{L_1}{L_2} \left(\frac{D_2}{D_1}\right)^5\right]^{0.5} \tag{7.17.2}$$

The idea of using equivalent length thus helps to reduce tediusness in calculations.

Example 7.6. *Three pipes of 400 mm, 350 mm and 300 mm diameter* ***are connected in series*** *between two reservoirs with a difference in level of 12 m. The friction factors are 0.024, 0.021 and 0.019 respectively. The lengths are 200 m, 300 m and 250 m respectively. Determine the flow rate neglecting minor losses.*

This problem can be solved using

$$12 = \frac{8\, f_1\, L_1\, Q^2}{\pi^2\, g\, D_1^{\,5}} + \frac{8\, f_2\, L_2\, Q^2}{\pi^2\, g\, D_2^{\,5}} + \frac{8\, f_3\, L_3\, Q^2}{\pi^2\, g\, D_3^{\,5}}$$

Solving $\qquad Q^2 = 0.04 \quad \therefore \quad \mathbf{Q = 0.2\ m^3/s}$

Using equivalent length concept and choosing 0.4 m pipe as the base.

Refer 7.17.1 $\qquad L_{2e} = 300\ (0.021/0.024) \times (0.4/0.35)^5 = 511.79$ m

$L_{3e} = 250(0.019/0.024) \times (0.4/0.3)^5 = 834.02$ m

Total length $\qquad = 200 + 511.79 + 834.02 = 1545.81$ m

$12 = (8 \times 0.024 \times 1545.81 \times Q^2)/(9.81 \times \pi^2 \times 0.4^5)$

$Q^2 = 0.04$; $\mathbf{Q = 0.2\ m^2/s.}$

Example 7.7. *Two reservoirs are connected by three* ***pipes in parallel*** *with the following details of pipes:*

Pipe No.	*Length, m*	*Diameter, m*	*Friction factor*
1	600	0.25	0.021
2	800	0.30	0.019
3	400	0.35	0.024

The total flow is 24,000 *l*/min. Determine the flow in each pipe and also the level difference between the reservoirs.

Let the flows be designated as Q_1, Q_2, Q_3

Then $Q_1 + Q_2 + Q_3 = 24000/(60 \times 1000) = 0.4\ m^3/s$

Using equation (7.17.2), Considering pipe 1 as base

$$\frac{Q_2}{Q_1} = \left[\frac{f_1}{f_2}\frac{L_1}{L_2}\left(\frac{D_2}{D_1}\right)^5\right]^{0.5} = \left[\frac{0.021}{0.019}\times\frac{600}{800}\times\left(\frac{0.3}{0.25}\right)^5\right]^{0.5} = 1.4362$$

$\therefore \quad Q_2 = 1.4362\ Q_1$

$$\frac{Q_3}{Q_1} = \left[\frac{f_1}{f_3}\frac{L_1}{L_3}\left(\frac{D_3}{D_1}\right)^5\right]^{0.5} = \left[\frac{0.021}{0.024}\times\frac{600}{400}\times\left(\frac{0.35}{0.25}\right)^5\right]^{0.5} = 2.6569$$

$\therefore \quad Q_3 = 2.6569\ Q_1$

Total flow $= 0.4 = Q_1 + 1.4362\ Q_1 + 2.6569\ Q_1 = 5.0931\ Q_1$

$\therefore \quad \mathbf{Q_1 = 0.07854\ m^3/s}$

$\therefore \quad \mathbf{Q_2 = 1.4362\ Q_1 = 0.11280\ m^3/s}$

$\therefore \quad \mathbf{Q_3 = 2.6569\ Q_1 = 0.20867\ m^3/s}$

Total $= 0.4001\ m^3/s$

Head loss or level difference (Ref para 7.11, eqn 7.11.15

Pipe 1 $\mathbf{h_f} = 8\ f\ L\ Q^2/\pi^2\ g\ D^5$

$8 \times 0.021 \times 600 \times 0.07854^2/\ \pi^2 \times 9.81 \times 0.25^5 =$ **6.576 m**

Check with other pipes

Pipe 2, $h_f = 8 \times 0.019 \times 800 \times 0.1128^2/\pi^2 \times 9.81 \times 0.3^5 =$ **6.576 m**

Example 7.8. *Water is drawn from two reservoirs at the same water level through pipe 1 and 2 which join at a common point. $D_1 = 0.4$ m, $L_1 = 2000$ m, $f_1 = 0.024$, $D_2 = 0.35$ m, $L_2 = 1500$ m, $f_2 = 0.021$. The water from the common point is drawn through pipe 3 of 0.55 m dia over a length of 1600 m to the supply location. The total head available is 25.43 m. Determine the flow rate through the system. The value of $f_3 = 0.019$.*

Pipes 1 and 2 meet at a common location. The two reservoir levels are equal. So, the head drops are equal (refer para 7.17, eqn. 7.17.2. Let the flow in pipe 1 be Q_1 and that in pipe 2 be Q_2.

Then $$\frac{Q_2}{Q_1} = \left[\frac{f_1}{f_2}\frac{L_1}{L_2}\left(\frac{D_2}{D_1}\right)^5\right]^{0.5}$$

Here $f_1 = 0.024$, $f_2 = 0.021$, $L_1 = 2000$ m, $L_2 = 1500$ m, $D_1 = 0.4$ m, $D_2 = 0.35$ m

$$\therefore \quad Q_2 = Q_1\left[\frac{0.024}{0.021}\times\frac{2000}{1500}\times\left(\frac{0.35}{0.4}\right)^5\right]^{0.5} = 0.8841\ Q_1$$

$\therefore \quad Q_1 + Q_2 = 1.8841\ Q_1$

This flow goes through pipe 3. The total head drop equals the sum of the drops in pipe 1 and in pipe 3.

$$25.43 = \frac{8 \times 0.024 \times 2000\, Q_1^2}{\pi^2 \times 9.81 \times 0.4^5} + \frac{8 \times 0.019 \times 1600 \times 1.8841^2\, Q_1^2}{\pi^2 \times 9.81 \times 0.55^5}$$

$$= 387.31\, Q_1^2 + 177.17\, Q_1^2 = 564.48\, Q_1^2$$

$\therefore$ $\mathbf{Q_1 = 0.2123\ m^3/s,\ Q_2 = 0.1877\ m^3/s}$

$\mathbf{Q_3} = Q_1 + Q_2 = \mathbf{0.4\ m^3/s}$, check for pressure at common point

$$\mathbf{h_{f1}} = \frac{8 \times 0.024 \times 2000 \times 0.2123^2}{\pi^2 \times 9.81 \times 0.4^5} = \mathbf{17.46\ m}$$

$$\mathbf{h_{f2}} = \frac{8 \times 0.021 \times 1500 \times 0.1877^2}{\pi^2 \times 9.81 \times 0.35^5} = \mathbf{17.46\ m}$$

both are equal as required. Check for drop in the third pipe

$$\mathbf{h_{f3}} = \frac{8 \times 0.019 \times 1600 \times 0.4^2}{\pi^2 \times 9.81 \times 0.55^5} = \mathbf{7.99\ m}$$

Total head = 7.99 + 17.46 = **25.45 m , checks.**

7.18 FLUID POWER TRANSMISSION THROUGH PIPES

High head and medium head hydal plants convey water from a high level to the power house through pressure pipe called penstock pipes. The choices of the pipe diameter depends on the expected efficiency of transmission and also on the economical aspect of the cost of pipe. Higher efficiencies can be obtained by the use of larger diameter pipes, but this will prove to be costly. It is desirable to maximise the power transmitted as compared to an attempt to increase efficiency. Applications are also there in hydraulic drives and control equipments.

7.18.1 Condition for Maximum Power Transmission

Consider that the head available is h and the frictional loss is h_f (neglecting minor losses) left the pipe diameter be D and the flow velocity be u.

Net head available = $h - h_f$. Quantity flow = $\pi D^2 u/4$

$h_f = f L u^2/2gD$, Power = mass flow × net head

Power, $$P = \frac{\pi D^2}{4}\, u\rho\, [h - (fLu^2/2gD)] = \frac{\pi D^2}{4}\, \rho \left[uh - \frac{f L u^3}{2g D} \right]$$

Differetiating P with respect to u, for maximum power,

$$\frac{dP}{du} = \frac{\pi D^2 \rho}{4}\, [h - 3(fLu^2/2gD)] = \frac{\pi D^2 \rho}{4}\, [h - 3h_f]$$

Equating to zero $\mathbf{h_f = h/3}$ (7.18.1)

For maximum power generation frictional loss will equal one third of available head and the corresponding transmission efficiency is 66.67%. If the available rate of flow is known the velocity and then the diameter can be determined or if the diameter is fixed the flow rate can be obtained. The friction factor for the pipe can be fixed as this is nearly constant above a

certain value of Reynolds number. For maximum power when flow rate is specified, pipe diameter is fixed and when diameter is specified the flow rate will be fixed.

Example 7.9 *In a hydroelectric plant the head available is 450 m of water. 25 cm penstock pipe with friction factor of 0.014 is used. Determine* ***the maximum power*** *that can be developed. The length of the pipe line is 3600 m.*

Using equation 7.18.1, $h_f = h/3 = 450/3 = 150$ m

$$150 = (0.014 \times 3600 \times u^2)/(2 \times 9.81 \times 0.25)$$

solving, $u = 3.82$ m/s, flow rate $= (\pi D^2/4) \times u = 0.18755$ m^3/s

Power developed $= Q\rho g \times (h - h_f) = 0.18755 \times 1000 \times 9.81\ (450 - 150) = 551963\ W$

$= \mathbf{551.963\ kW}$

Example 7.10. *Determine for the data in example 7.9 the* ***power transmitted for u = 4.5 m/s and u = 3 m/s.***

(*i*) $u = 4.5$ m/s, $h_f = (0.014 \times 3600 \times 4.5^2)/(2 \times 9.81 \times 0.25) = 208.07$ m

Power $= (\pi \times 0.25^2/4) \times 4.5 \times 1000 \times 9.81\ (450 - 208.07) = 524246\ W =$ **524.25 kW**

(*ii*) $u = 3$ m/s, $h_f = (0.014 \times 3600 \times 3^2)/(2 \times 9.81 \times 0.25) = 92.48$ m

Power $= (\pi \times 0.25^2/4) \times 3 \times 1000 \times 9.81\ (450 - 92.48) = 516493\ W =$ **516.49 kW**

This brings out clearly that the maximum power for a given diameter and head is when the frictional drop equals one third of available head.

Example 7.11 *In a hydrosystem the flow availability was estimated as 86.4 × 10^3 m^3/day. The head of fall was estimated as 600 m. The distance from the dam to the power house considering the topography was estimated as 3000 m. The available pipes have friction factor 0.014.* ***Determine the pipe diameter for transmitting maximum power,*** *and also calculate the* ***velocity and power transmitted.***

Refer Eqn 7.18.1. The frictional drop is equal to one third of available head.

$\therefore$ $h_f = 600/3 = 200$ m

$h_f = fLu^2/2g\,D$, Here both u and D are not specified.

But Q = area × velocity

$\therefore$ $u = 4Q/\pi D^2 \quad \therefore \quad u^2 = 16\,Q^2/\pi^2 D^4$

$\therefore$ $$h_f = \frac{fL\ 16\,Q^2}{2\pi^2\ gD^5} = 200,\ Q = 86.4 \times 10^3/(24 \times 3600) = 1\ \text{m}^3/\text{s}$$

$$200 = (0.014 \times 3000 \times 16 \times 1^2)/(2\pi^2 \times 9.81)\ D^5,$$

$$D^5 = 0.01735,\ \mathbf{D = 0.4445\ m}$$

Velocity $= 4 \times 1/(\pi \times 0.4445^2) =$ **6.444 m/s**

Power $= 1000 \times 9.81 \times 400 =$ **3.924 × 10^6 *W* or 3.924 *MW***

Check for frictional loss

$$h_f = (0.014 \times 3000 \times 6.444^2)/(2 \times 9.81 \times 0.4445) = 200\ \text{m (checks)}$$

7.19 NETWORK OF PIPES

Complex connections of pipes are used in city water supply as well as in industrial systems. Some of these are discussed in the para.

7.19.1 Pipes in Series—Electrical Analogy

Series flow problem can also be solved by use of resistance network. Consider equation 7.11.15. For given pipe specification the equation can be simplified as

$$h_f = 8\, f\, L\, Q^2/\pi^2 g\, D^5 = R\, Q^2$$

Note: The dimension for R is s^2/m^5. For flow in series Q is the same through all pipes. This leads to the relation

$$h_{f1} + h_{f2} + h_{f3} + \ldots\ldots\ldots h_{fn} = h_f = (R_1 + R_2 + R_3 + \ldots. + R_n)\, Q^2$$

The R values for the pipe can be calculated. As the total head is also known Q can be evaluated. The length L should include minor losses in terms of equivalent lengths. The circuit is shown in Fig. 7.19.1.

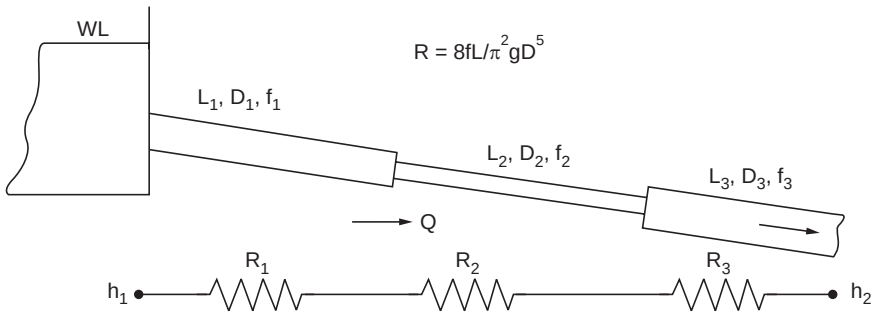


Figure 7.19.1 Equivalent circuit for series flow

Example 7.12. *A reservoir at a level with respect to datum of 16 m supplies water to a ground level reservoir at a level of 4 m. Due to constraints pipes of different diameters were to be used. Determine the flow rate.*

No.	Diameter, m	Length including minor losses, m	f
1	0.30	220	0.02
2	0.35	410	0.018
3	0.45	300	0.013
4	0.40	600	0.015

The resistance values are calculated using Eqn. 7.11.15

Pipe 1. $\mathbf{R_1} = \dfrac{8 \times 0.02 \times 220}{\pi^2 \times 9.81 \times 0.3^5} = \mathbf{149.61}$, Pipe 2. $\mathbf{R_2} = \dfrac{8 \times 0.018 \times 410}{\pi^2 \times 9.81 \times 0.35^5} = \mathbf{116.1}$

Pipe 3. $\mathbf{R_3} = \dfrac{8 \times 0.013 \times 300}{\pi^2 \times 9.81 \times 0.45^5} = \mathbf{17.463}$, Pipe 4. $\mathbf{R_4} = \dfrac{8 \times 0.015 \times 600}{\pi^2 \times 9.81 \times 0.4^5} = \mathbf{72.62}$

$$h = [R_1 + R_2 + R_3 + R_4]\, Q^2$$

$$(16 - 4) = (149.61 + 17.463 + 72.62)\, Q^2 = 355.793\, Q^2$$

$\therefore$ **Q = 0.18365 m³/s**

Check $\Sigma h = \Sigma (R_1 Q_1^2) = 5.046 + 3.916 + 0.589 + 2.449 = 12$ m

7.19.2 Pipes in Parallel

Such a system is shown in Fig. 7.19.2

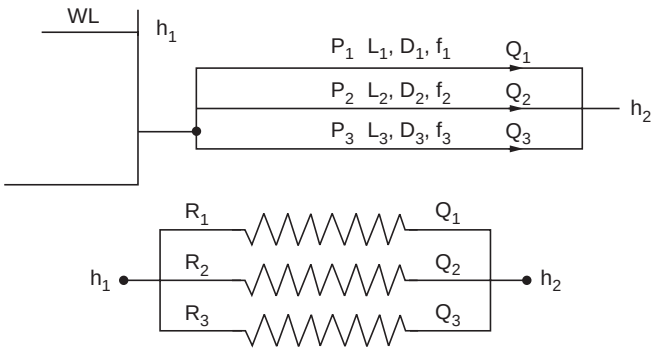


Figure 7.19.2

Case (*i*) The head drop between locations 1 and 2 are specified: The total flow can be determined using

$$h_f = \frac{8 f_1 L_1 Q_1^2}{\pi^2 g D_1^5} = \frac{8 f_2 L_2 Q_2^2}{\pi^2 g D_2^5} = \frac{8 f_3 L_3 Q_3^2}{\pi^2 g D_3^5}$$

As h_f and all other details except flow rates Q_1, Q_2 and Q_3 are specified, these flow rates can be determined.

Total flow $Q = Q_1 + Q_2 + Q_3$

The process can be extended to any number of connections.

Case (*ii*) Total flow and pipe details specified. One of the methods uses the following steps:

1. Assume by proper judgement the flow rate in pipe 1 as Q_1.
2. Determine the frictional loss.
3. Using the value find Q_2 and Q_3.
4. Divide the total Q in the proportion $Q_1 : Q_2 : Q_3$ to obtain the actual flow rates.

Case (*iii*) Electrical analogy is illustrated or in problem Ex. 7.13 and Ex. 7.14.

Example 7.13 *The details of a* ***parallel pipe system*** *for water flow are given below.*

No.	length, m	Diameter, m	Friction factor
1	800	0.2	0.022
2	1200	0.3	0.02
3	900	0.4	0.019

1. If the frictional drop between the junctions is 15 m of water, determine the total flow rate
2. If the total flow rate is 0.66 m^3/s, determine the individual flow and the friction drop.

The system is shown in Fig. Ex. 7.13.

Case (*i*) Let the flows be Q_1, Q_2 and Q_3. Total flow $Q = Q_1 + Q_2 + Q_3$, using equation 7.11.15

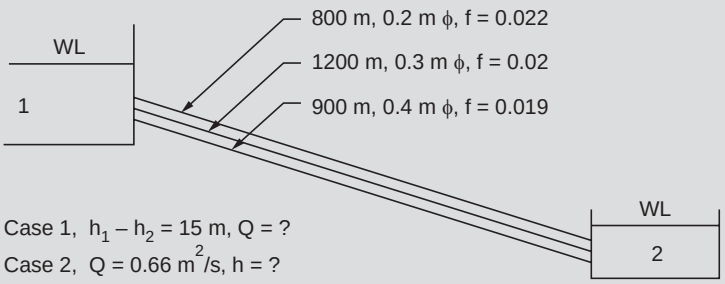


Figure Ex. 7.13

The flow rates are calculated individually with $h_f = 15$ m and totalled.

$$15 = \frac{8 f_1 L_1 Q_1^2}{\pi^2 g D_1^5} = \frac{8 \times 0.022 \times 800 \times Q_1^2}{\pi^2 \times 9.81 \times 0.2^5} \quad \textit{solving } \mathbf{Q_1 = 0.05745\ m^3/s}$$

$$15 = \frac{8 f_2 L_2 Q_2^2}{\pi^2 g D_2^5} = \frac{8 \times 0.02 \times 1200 \times Q_2^2}{\pi^2 \times 9.81 \times 0.3^5} \quad \textit{solving } \mathbf{Q_2 = 0.1355\ m^3/s}$$

$$15 = \frac{8 f_3 L_3 Q_3^2}{\pi^2 g D_3^5} = \frac{8 \times 0.019 \times 900 \times Q_3^2}{\pi^2 \times 9.81 \times 0.4^5} \quad \textit{solving } \mathbf{Q_3 = 0.32971\ m^3/s}$$

$$\mathbf{Q} = Q_1 + Q_2 + Q_3 = \mathbf{0.522\ m^3/s}$$

Case (*ii*) Total flow is 0.66 m^3/s . Already for 15 m head individual flows are available. Adopting method 2 the total flow is divided in the ratio of $Q_1 : Q_2 : Q_3$ as calculated above.

$$Q_1 = \frac{0.66 \times 0.05745}{0.52274} = 0.07254\ m^3/s,$$

$$Q_2 = \frac{0.66 \times 0.13558}{0.52274} = 0.17117\ m^3/s$$

$$Q_3 = \frac{0.66 \times 0.32971}{0.52274} = 0.41629\ m^3/s$$

Calculation for frictional loss.

Pipe 1 $$h_f = \frac{8 \times 0.022 \times 800 \times 0.07254^2}{\pi^2 \times 9.81 \times 0.2^5} = 23.91\ m$$

Pipe 2 $$h_f = \frac{8 \times 0.02 \times 1200 \times 0.17117^2}{\pi^2 \times 9.81 \times 0.3^5} = 23.91\ m$$

Pipe 3 $$h_f = \frac{8 \times 0.019 \times 900 \times 0.41629^2}{\pi^2 \times 9.81 \times 0.4^5} = 23.91 \text{ m}$$

Electrical analogy : For parallel pipe network also electrical analogy can be used. In the case of parallel flow as the pressure drop is the same

$$h_f = R_1 Q_1^2 = R_2 Q_2^2 = R_3 Q_3^2 \text{ ..... or}$$

$$Q_1 = \sqrt{h_f / R_1}\text{ , Total flow equals } Q_1 + Q_2 + Q_3 \text{ .....}$$

$$Q = \sqrt{h_f}\left[\frac{1}{\sqrt{R_1}} + \frac{1}{\sqrt{R_2}} + \text{.....}\right]$$

An equivalent resistance R can be obtained by

$$\frac{1}{\sqrt{R}} = \frac{1}{\sqrt{R_1}} + \frac{1}{\sqrt{R_2}} + \text{ ..... and } Q^2 = h_f/R$$

Example 7.14. *Work out problem 7.13 by analogy method.*

Case 1. $$\boldsymbol{R_1} = \frac{8 \times 0.022 \times 800}{\pi^2 \times 9.81 \times 0.2^5} = \mathbf{4544.48}, \boldsymbol{R_2} = \frac{8 \times 0.02 \times 1200}{\pi^2 \times 9.81 \times 0.3^5} = \mathbf{816.07}$$

$$\boldsymbol{R_3} = \frac{8 \times 0.019 \times 900}{\pi^2 \times 9.81 \times 0.4^5} = \mathbf{137.98}, \frac{1}{\sqrt{R}} = \frac{1}{\sqrt{4544.48}} + \frac{1}{\sqrt{816.07}} + \frac{1}{\sqrt{137.98}}$$

$\therefore$ $$\mathbf{R} = \left(\frac{1}{0.13497}\right)^2 = \mathbf{54.893} \quad \therefore \quad Q^2 = 15/54.893 = 0.2733$$

$\therefore$ $Q = \mathbf{0.52274}$ (checks with the previous case)

Case 2. $Q = 0.66$, $R = 54.893$

$\therefore$ $h_f = R\,Q^2 = 54.893 \times 0.52274^2 = 23.91$ m

$Q_1^2 = h_f/R_1 = 23.91/4544.48$, $Q_1 = 0.07254$ m^3/s

$Q_2^2 = h_f/R_2 = 23.91/816.07$, $Q_2 = 0.17117$ m^3/s

$Q_3^2 = h_f/R_3 = 23.91/137.98$, $Q_3 = 0.41629$ m^3/s

Note: Checks in all cases.

7.19.3 Branching Pipes

The simplest case is a three reservoir system interconnected by three pipes (Ref. Fig. 7.19.3). The conditions to be satisfied are (*i*) The net flow at any junction should be zero due to continuity principle. (*ii*) The Darcy-Weisbach equation should be satisfied for each pipe. If flows are Q_1, Q_2, Q_3 , then the algebraic sum of $Q_1 + Q_2 + Q_3 = 0$. **If one of the flow rate is specified the solution is direct. If none are specified, trial solution becomes necessary.** The flow may be from the higher reservoir to the others or it may be from both high level reservoirs to the low level one. The hydraulic grade line controls the situation. If the head at the junction is above both the lower reservoirs, both of these will receive the flow. If the head

at the junction is below the middle one, the total flow will be received by the lowest level reservoir. This is shown in Fig 7.19.3.

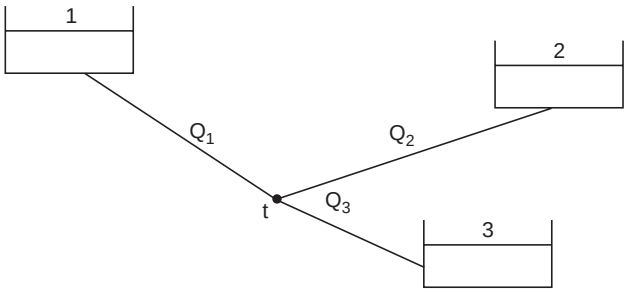


Figure 7.19.3

The method of solution requires iteration.

(*i*) **A value for the head at the junction is assumed and the flow rates are calculated from pipe details.**

(*ii*) **The sum of these (algebraic) should be zero. But at the first attempt, the sum may have a positive value or negative value.**

(*iii*) **If it is positive, inflow to the junction is more. So increase the value of head assumed at the junction.**

(*iv*) **If it is negative, the outflow is more. So reduce the value of head assumed. Such iteration can be also programmed for P.C.**

Example 7.15 *Three reservoirs A, B and C at water levels of 25 m, 12 m and 8 m are connected by a pipe network. 1200 m length pipe of diameter 0.5 m and f = 0.013 draws water from A. 1000 m length pipe of diameter 0.4 m and f = 0.015 draws water from B and joins the pipe end from A. The reservoir C is connected to this junction by 900 m length of pipe 0.6 m diameter with f = 0.011. Determine the flow from / to each reservoir.*

Using the equation 7.11.15

$$h_f = \frac{8\, fLQ^2}{\pi^2\, g\, D^5}, \text{ and writing this } h_f = R\, Q^2$$

$$\mathbf{R_A} = \frac{8 \times 0.013 \times 1200}{\pi^2 \times 9.81 \times 0.5^5} = \mathbf{41.2473},$$

$$\mathbf{R_B} = \frac{8 \times 0.015 \times 1000}{\pi^2 \times 9.81 \times 0.4^5} = \mathbf{121.0354}$$

$$\mathbf{R_C} = \frac{8 \times 0.011 \times 900}{\pi^2 \times 9.81 \times 0.6^5} = \mathbf{10.5196}$$

Considering flow from A, $(25 - Z_J)/41.2473 = Q_A^2$,

where Z_J level at junction J. Similarly for flow from B and C,

$\mathbf{(12 - Z_J)/121.0354 = Q_B^2}$, $\mathbf{(8 - Z_J)/10.5196 = Q_C^2}$

Assumed value of Z_J	Q_A	Q_B	Q_C	$Q_A + Q_B + Q_C$
10	0.6030	0.1286	– 0.4360	0.2556
13.0	0.5394	– 0.0909	– 0.6894	– 0.2409
11.5	0.5721	0.06423	– 0.5768	0.0596
11.8	**0.5657**	**0.0406**	**– 0.6010**	**0.0053**

This is sufficient for the trial. The flow rates in the last column can be used.

Z_J = 11.825 m gives a residue of 0.00019.

Considering the value, the flows are

Q_A = 0.56517 m³/s, Q_B = 0.03802 m³/s and Q_C = – 0.603 m³/s

7.19.4 Pipe Network

More complex network of pipes exist in practice. A sample is shown in Fig. 7.12

For analysis of the system the following conditions are used.

1. **The algebraic sum of the pressure drop around each circuit must be zero.**
2. **The flow into the junction should equal the flow out of the junction.**
3. **For each pipe the proper relation between head loss and discharge** should be **maintained.**

Analytical solution to such a problem is more involved. Methods of successive approximation are used. With the use of computers, it is now possible to solve any number of simultaneous equations rather easily. Use of the above conditions leads to a set of simultaneous equations. This set can be solved using computers.

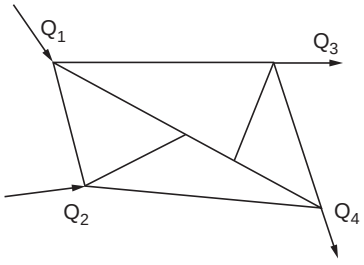


Figure 7.19.4 Pipe network

SOLVED PROBLEMS

Problem. 7.1. *An oil of specific gravity 0.82 and kinematic viscosity 16 × 10⁻⁶ m²/s flows in a smooth pipe of 8 cm diameter at a rate of 2l/s. Determine whether the flow is laminar or turbulent. Also calculate the velocity at the centre line and the velocity at a radius of 2.5 cm. What is head loss for a length of 10 m.* ***What will be the entry length****? Also determine the* ***wall shear****.*

Average flow velocity = volume flow/area = $4 \times 0.002/\pi \times 0.08^2$ = 0.4 m/s

$$\text{Re} = \frac{uD}{v} = \frac{0.4 \times 0.08}{16 \times 10^{-6}} = 2000$$

This value is very close to transition value. However for smooth pipes the flow may be taken as laminar.

Centre line velocity = 2 × average velocity = 0.8 m/s

For velocity at 2.5 cm radius

$$\frac{u}{u_{\max}} = 1 - \left(\frac{r}{R}\right)^2 \quad \therefore \quad \mathbf{u} = 0.8\left[1 - \left(\frac{2.5}{4}\right)^2\right] = \mathbf{0.4875\ m/s}$$

$$\mathbf{f} = 64/\mathrm{Re} = 64/2000 = \mathbf{0.032}$$

$$\mathbf{h_f} = fLu^2/2gd = (0.032 \times 10 \times 0.4^4)/(2 \times 9.81 \times 0.08)$$

$$= \mathbf{0.03262\ m} \text{ of oil}$$

as $$\mathbf{\Delta p} = h_f\gamma = 0.03262 \times 9810 \times 0.82 = \mathbf{262.4}\ \mathrm{N/m^2}$$

$$\mathbf{Entry\ length} = 0.058\ \mathrm{Re}.D. = 0.058 \times 2000 \times 0.08 = \mathbf{9.28\ m}$$

For highly viscous fluid entry length will be long. Wall shear is found from the definition of f.

$$\tau_o = \frac{f}{4}\frac{\rho}{g_o}\frac{u_m^{\,2}}{2} = \frac{0.032}{4} \times \frac{820}{1} \times \frac{0.4^2}{2} = \mathbf{0.5248\ N/m^2}$$

Wall shear can also be found using, $\tau_o = -\rho\, v\dfrac{du}{dr}$

$$u = u_{\max}\left[1 - \frac{r^2}{R^2}\right], \frac{du}{dr} = -\frac{U_{\max}\, 2r}{R^2}, \text{ at } r = R, \frac{du}{dr} = -u_{\max}\frac{2}{R}$$

Substituting, $\tau_o = 820 \times 16 \times 10^{-6} \times 0.8 \times 2/0.04 = 0.5248\ \mathrm{N/m^2}$.

Problem 7.2. *A circular and a square pipe are of equal sectional area. For the same flow rate, determine which section will lead to a higher value of Reynolds number.*

$\mathrm{Re} = uD_h/v$, For the same section and same flow rate of a specified fluid, $R_e \propto D_h$ hydraulic Diameter.

Circular Pipe : $D_h = D$

Square Pipe of side a : $D_h = 4a^2/4a = a$

as areas are equal, $a^2 = \pi D^2/4, \quad \therefore \quad a = 0.886\ D$

The hydraulic diameter of a square section of the same area is lower by about 11.4%. So the Reynolds number in this case will be lower by about 11.4% and hence for the same flow rate f will be higher for the square section.

Problem 7.3. *The kinematic viscosity of water at 30°C is 0.832 × 10^{-6} m^2/s.* ***Determine the maximum flow rate*** *through a 10 cm dia pipe for the flow to be laminar. Assume smooth pipe. Also determine the head loss/m at this flow condition.*

The condition is that Reynolds number should be about 2000.

$$2000 = (0.1 \times u)/(0.832 \times 10^{-6})$$

$\therefore$ $\mathbf{u = 0.01664\ m/s}$.

The flow rate will be $= (\pi \times 0.1^2/4) \times 0.01664 =$ **1.307×10^{-4} m³/s = 0.1307 l/s.**

$f = 64/2000 = 0.032$

Head of water, $h_f = 0.032 \times 1 \times 0.01664^2/(2 \times 9.81 \times 0.1) = 4.516 \times 10^{-6}$ **m/m.**

Note: The flow turns turbulent even at a low flow velocity as the kinematic viscosity is low.

Problem 7.4. *Air at 1 atm and 30 °C flows through a pipe of 30 cm dia. The kinematic viscosity at this condition is 16×10^{-6} m²/s. The density is 1.165 kg/m³. Determine the maximum average velocity for the flow to remain laminar. What will be the* ***volume and mass flow rates*** *at this condition? Also determine the head loss / m due to friction.*

The condition is that Reynolds number should equal 2000.

$\therefore \quad 2000 = (u_m \times 0.3)/16 \times 10^{-6} \quad \therefore \quad u_m = 0.107$ m/s

Volume flow rate $= u\,A = 0.107 \times \pi \times 0.3^2/4 =$ **7.54×10^{-3} m³/s** or **7.54 l/s**

Mass flow = volume flow × density $= 7.54 \times 10^{-3} \times 1.165$

$=$ **8.784×10^{-3} kg/s**

$f = 64/\text{Re} = 64/2000 = 0.032$

$h_f = (0.032 \times 0.107^2 \times 1)/(2 \times 9.81 \times 0.3)$

$= 62.2 \times 10^{-6}$ m/m (head of air)

Problem 7.5. *Oil with a kinematic viscosity of 241×10^{-6} m²/s and density of 945 kg/m³ flows through a pipe of 5 cm dia. and 300 m length with a velocity of 2 m/s. Determine* **the *pump power***, *assuming an overall pump efficiency of 45%, to overcome friction.*

Re $= uD/v = 2 \times 0.05/241 \times 10^{-6} = 415$. So the flow is laminar.

$h_f = (64/415) \times [(2^2 \times 300)/(2 \times 9.81 \times 0.05)] = 188.67$ m head of oil

Mass flow $= (\pi \times 0.05^2/4)\ 2 \times 945$ kg/s $= 3.711$ kg/s

Power required $= mg\,H/\eta = 3.711 \times 9.81 \times 188.67/0.45\ W$

$= 15{,}263\ W$ or **15.263 kW.**

Problem 7.6. *If, in problem P.7.5. the power available was 10 kW, what will be the pumping rate?*

Power available to overcome friction P = power × pump efficiency

$= 10 \times 0.45 = 4.5$ kW or 4500 W

Mass flow $= (\pi D^2/4)\ u\rho$

Frictional loss in head of fluid $= (64/\text{Re}) \times (u^2L/2gD)$

$= (64v/uD) \times (u^2L/2gD)$

$\therefore$ Power = mass flow × g × frictional head

$= (\pi D^2/4)\ u\ \rho\ g\ (64vu^2\ L/u\ D\ 2g\ D) = 8\ \pi\ \rho\ v\ Lu^2$

$4500 = 8 \times \pi \times 945 \times 241 \times 10^{-6} \times 300\ u^2$

$\therefore$ **u = 1.619 m/s**

Flow rate $= (\pi \times 0.05^2/4) \times 1.619 \times 945 =$ **3kg/s**

Note: Check for the flow to be laminar. Re $= 1.619 \times 0.05/241 \times 10^{-6} = 336$

Problem 7.7. *Oil of specific gravity 0.92 flows at a rate of 4.5 litres/s through a pipe of 5 cm dia, the pressure drop over 100 m horizontal length being 15 N/cm^2.* ***Determine the dynamic viscosity of the oil****.*

Using the equation 7.9.2 – Hagen-Poiseuille eqn. $\Delta p = 128\ \mu LQ/\pi D^4$

$$\mu = \Delta p\ .\ \pi.D^4/128LQ$$

$$= 15 \times 10^4 \times \pi \times 0.05^4/128 \times 100 \times 0.0045 = \mathbf{0.05113\ Ns/m^2}\ \text{(Pa.s)}$$

(**Note:** N/cm^2 $\rightarrow 10^4$ N/m^2, litre = 0.001 m^3)

Reynolds number $= uD\ \rho/\mu,\ u = Q \times 4/\pi D^2$

$\therefore$ $\text{Re} = (4Q/\pi\ D^2) \times (D\ \rho/\mu) = (0.0045 \times 920 \times 4)/(\pi \times 0.05 \times 0.05113)$

$= 2061.6$

$\therefore$ Flow is laminar but just on the verge of turning turbulent

(Note: Re = 4Q/π Dv)

Problem 7.8. *In a capillary viscometer the tube is of 2 mm dia and 0.5 m length. If 60 cm^3 of liquid is collected during 10 min with a constant pressure difference of 5000 N/m^2,* ***determine the viscosity of the oil.***

Using $\Delta p = 128\ \mu L\ Q/\pi D^4$ (Hagen Poiseuille equation 7.9.2)

$\mu = \Delta p.\pi D^4/128\ L\ Q$ where Q is the discharge in m^3 per second.

Discharge = 60×10^{-6} m^3/600 sec = 10^{-7} m^3/s

$\therefore$ $\mu = 5000 \times \pi \times 0.002^4/128 \times 0.5 \times 10^{-7} = \mathbf{0.0393\ Ns/m^2}$ **(or Pa.s)**

Problem 7.9. *If an oil of viscosity of 0.05 Ns/m^2 is used in the experiment of problem P.7.8* ***calculate how long it will take to collect 60 cc****. Assume that the other conditions remain unaltered.*

$$\Delta p = 128\ \mu L\ Q/\pi D^4$$

$\therefore$ $Q = \Delta p \times \pi D^4/128\ \mu L$ where Q is in m^3/s

$$Q = 5000 \times \pi \times 0.002^4/128 \times 0.05 \times 0.5 = 7.854 \times 10^{-8}\ \text{m}^3/\text{s}$$

or 7.854×10^{-2} cc/s

$\therefore$ Time for 60 cc = $60/7.854 \times 10^{-2}$ s = 763.94 s or **12.73 min**

Problem 7.10. *Oil of viscosity 0.1 Ns/m^2 is to flow through an inclined pipe by gravity. The pipe diameter is 25 mm and the density of the oil is 930 kg/m^3. If the flow rate is to be 0.25 l/s* ***determine the pipe inclination*** *with horizontal.*

The inclination of the pipe should be such that the drop in head should equal the friction drop along the length or

$$h_f = L \sin\theta,\ \Delta h = fLu^2/2gD,\ f = 64/\text{Re},\ h = \Delta p/\gamma$$

Using Darcy–Weisbach equation and substituting for f in terms of Re

$$h_f = \frac{64}{\text{Re}}\frac{u^2L}{2gD} = L \sin\theta \text{ or } \sin\theta = \frac{64u^2}{2\,\text{Re}\,gD}$$

$$u = 4Q/\pi D^2 = 4 \times 0.25/(\pi \times 0.025^2)\ 1000 = 0.5093\ \text{m/s}$$

$\therefore$ $$\text{Re} = \frac{uD\rho}{\mu} = 0.5093 \times 0.025 \times 930/0.1 = 118.41$$

∴ Flow is laminar

$$\sin\theta = (64 \times 0.5093^2)/(2 \times 118.41 \times 9.81 \times 0.025) = 0.28582$$

∴ **$\theta = 16.6°$ with horizontal**

Problem 7.11. *In a double pipe heat exchanger (to obtain chilled water) water at 10° C flows in the annular area between 30 mm OD inside pipe and the 50 mm ID outer pipe. The kinematic viscosity at this temperature is 1.4×10^{-6} m^2/s.* ***Determine the maximum flow rate if the flow should be laminar.***

The sectional area for flow = $(\pi/4)\,(D^2 - d^2)$ where D = out side dia, d = inside dia. of the annular area. Wetted perimeter = $\pi\,(D + d)$

∴ $\mathbf{D_h} = 4 \times (\pi/4)\,(D^2 - d^2)/\pi(D + d) = D - d = 0.05 - 0.03 =$ **0.02 m**

For laminar conditions Re should be less than 2000.

$$\text{Re} = 2000 = (0.02 \times u)/(1.4 \times 10^{-6}) \quad \therefore \quad u = 0.14 \text{ m/s}$$

∴ **flow rate** $= (\pi/4)\,(0.05^2 - 0.03^2)\,0.14$

$$= \mathbf{1.76 \times 10^{-4}\ m^3/s \text{ or } 0.176\ l/s \text{ or } 633.3\ l/hr.}$$

The friction factor and friction drop in head and power required for a flow rate etc can be determined as in problem P. 7.5. taking care to use D_h in place of D.

Problem 7.12. *Water flows in an experimental 50 mm square pipe at a temperature of 10°C. The flow velocity is 0.012 m/s.* ***Determine the head drop over a length of 10 m.*** *Compare the same with circular section of the same area, $v = 1.4 \times 10^{-6}$ m^2/s.*

As the **section is square**, the hydraulic diameter is to be used.

$$D_h = 4 \text{ area/perimeter} = 4a^2/4a = a = 0.05 \text{ m}$$

$$\text{Re} = D_h\, u/v = 0.05 \times 0.012/1.4 \times 10^{-6} = 428.6$$

∴ The flow is laminar.

$$f = 64/\text{Re} = 64/428.6$$

$$h_f = fLu^2/2g\,D_h = (64/428.6) \times 10 \times 0.012^2/(2 \times 9.81 \times 0.05)$$

$$= \mathbf{2.19 \times 10^{-4}\ m \text{ head of water.}}$$

Circular section: $\pi D^2/4 = 0.05^2 \quad \therefore \quad D = 0.05642$ m

$$Re = 0.05642 \times 0.012/1.4 \times 10^{-6} = 483.6, \text{ laminar,}$$

$$f = 64/483.6 = 0.1323$$

$$\mathbf{h_f} = 0.1323 \times 10 \times 0.012^2/2 \times 9.81 \times 0.05642$$

$$= \mathbf{1.722 \times 10^{-4}\ m} \quad \therefore \quad \text{Lower by: } 21.4\%$$

Problem 7.13. *If in the place of square a rectangular section of 100 mm × 25 mm is used for the data of P. 7.12 determine the head drop over a length of 10 m.*

Hydraulic diameter $= 4A/P = 4 \times 0.025 \times 0.1/2(0.1 + 0.025) = 0.04$ m

$$\text{Re} = D_h\, u/v = 0.4 \times 0.012/1.4 \times 10^{-6} = 342.86$$

Frictional drop in head $= f.L.u^2/2g\,D_h = (64/342.86) \times 10 \times 0.012^2/2 \times 9.81 \times 0.04$

$$= \mathbf{3.425 \times 10^{-4}\ m \text{ head of water}}$$

For the same flow area as compared to 1.722×10^{-4} m head of water for the circular section there is an **increase of 100% in friction drop** for the rectangular section.

Problem 7.14. *Water flows out from a storage tank through a pipe of 50 mm dia. at a rate of 9.82 l/s.* ***Determine the loss of head at entrance*** *if it is (i)* ***bell mounted*** *(ii)* ***square edged*** *and (iii)* ***reentrant****.*

Refer section 7.12

(*i*) In this case the loss coefficient is 0.04

$$u = 4Q/\pi D^2 = 4 \times 9.82/(1000 \times \pi \times 0.05^2) = 5 \text{ m/s}$$

$\therefore$ $h_f = 0.04 \times 5^2/2 \times 9.81 =$ **0.051 m.**

(*ii*) In this case loss coeffcient is 0.5. $\therefore$ $\mathbf{h_f} = 0.5 \times 5^2/2 \times 9.81 =$ **0.637 m**

(*iii*) The loss coefficient in this case = 0.8 $\therefore$ $\mathbf{h_f} = 0.8 \times 5^2/2 \times 9.81 =$ **1.019 m**

Problem 7.15. *Water flowing in a pipe of 500 mm dia suddenly passes into a pipe of 750 mm dia. Determine the loss of head if the initial velocity was 2 m/s.*

Ref. eqn 7.12.1. In this case,

$$h_f = (u_2 - u_1)^2/2g,\ u_1 = 2 \text{ m/s},\ u_2 = 2 \times (0.5/0.75)^2 = 0.889 \text{ m/s}.$$

$\therefore$ $\mathbf{h_f} = (2 - 0.889)^2/2 \times 9.81 =$ **0.0629 m.**

Problem 7.16. *A 30 cm pipe with friction factor f = 0.024 carries water to a turbine at the rate of 0.25 m^3/s over a distance of 160 m. The difference in levels between the water inlet and turbine inlet is 36 m.* ***Determine the efficiency of transmission****. The turbine outlet delivery is submerged into the tailrace and the velocity at the exit is 0.4 times the velocity in the pipe.*

$$\text{The efficiency of transmission} = \frac{\text{Available head for conversion to work}}{\text{Difference in datum}}$$

The losses in this case are the friction head and the dynamic head at exit.

Flow rate = 0.25 m^3/s,

$\therefore$ $u_m = 0.25 \times 4/\pi \times 0.3^2 = 3.54$ m/s.

Friction head $= fLu^2/2gD = [0.024 \times 160 \times 3.54^2/(2 \times 9.81 \times 0.3)] = 8.176$ m

Dynamic head: Exit velocity $= 0.4 \times 3.54$ *m/s*.

$\therefore$ Dynamic head $= (0.4 \times 3.54)^2/2 \times 9.81 = 0.102$ m

Total losses $= 8.176 + 0.102 = 8.28$ m

Efficiency is high but the power delivered is not maximum.

$\therefore$ **Efficiency of transmission** $= (36 - 8.28)/36 =$ **0.77 or 77%**

Problem 7.17. *The flow in a pipe of 100 mm dia with Reynolds number value of 10^5 is found to have a friction factor f = 0.032.* ***Determine the thickness of laminar sublayer****. Also indicate whether the pipe is* ***hyraulicaly smooth or not if*** *the roughness height is 0.4 mm.*

Ref. section 7.5.

$$\delta_l = 32.8\ v/u_m \sqrt{f}\ , Re = u_m\ D/v = 10^5$$

$\therefore$ $v/u_m = D/10^5$, substituting for v/u_m

$\therefore$ $\delta_l = (32.8 \times D)/(10^5 \times \sqrt{f}\ ) = 32.8 \times 0.1/10^5\ \sqrt{0.032}$

$= 1.83 \times 10^{-4}$ m = 0.183 mm

$\varepsilon =$ 0.4 mm 0.3 ε = 0.12 mm, 6ε = 1.098 mm

The sublayer thickness is larger than 0.3ε but less than 6ε.

The pipe cannot be classified definitely as smooth or rough.

Problem 7.18. *Petrol of sp. gravity 0.7 and kinematic viscosity of 0.417 × 10^{-6} m^2/s flows through a smooth pipe of 250 mm ID. The pipe is 800 m long. The pressure difference between the ends is 0.95 bar.* ***Determine the flow rate.***

In this case the determination off involves the velocity as the Reynolds number depends on velocity. The flow rate depends on velocity. A trial solution is necessary. So a value of f = 0.02 is first assumed.

Pressure difference = 0.95 bar or 0.95×10^5 N/m^2. Converting the same to head of fluid, $0.95 \times 10^5/700 \times 9.81$ = 13.834 m of petrol column.

$$13.834 = (fLu^2/2gD) + (u^2/2g)$$
$$= [(0.02 \times 800 \times u^2)/(2 \times 9.81 \times 0.25)] + u^2/2 \times 9.81$$
$$= (3.26 + 0.051)u^2$$
$$\therefore \quad u = 2.045 \text{ m/s.}$$

Now $\quad \text{Re} = uD/v = 2.045 \times 0.25/0.417 \times 10^{-6} = 1.226 \times 10^6$

Ref. section 7.11, eqn 7.11.12, $f = 0.0032 + (0.221/\text{Re}^{0.237}) = 0.01117$

or $\quad 1/\sqrt{f} = 1.8. \log \text{Re} - 1.5186 \qquad \therefore \quad f = 0.01122$

so the value 0.02 is on the higher side. Now using the value 0.01117,

$$13.834 = [0.01117 \times 800 \times u^2)/(2 \times 9.81 \times 0.25)] + [u^2/(2 \times 9.81)]$$
$$= 1.8727\, u^2$$
$$\therefore \quad u = 2.7185 \text{ m/s}, \quad \text{Re} = 2.7185 \times 0.25/0.417 \times 10^{-6} = 1.63 \times 10^6$$
$$f = 0.1065.$$

This is nearer the assumed value and further refinements can be made by repeating the procedure.

Flow rate $\quad = 2.7185 \times \pi \times 0.25^2/4 = 0.1334 \text{ m}^3\text{/s} = 93.4 \text{ kg/s}$

Problem 7.19. ***Determine the diameter of the pipe*** *(smooth) required to convey 150 l of kerosene over a length 1000 m with the loss of head by friction limited to 10 m of kerosene. Density = 810 kg/m^3, kinematic viscosity = 2.37 × 10^{-6} m^2/s*

In this problem also as in P. 7.18, trial is necessary. Assume f = 0.012

Neglecting dynamic head, As $u = Q/A$,

$$10 = [(0.012 \times 1000)/(2 \times 9.81 \times D)] \times [(0.15 \times 4)/(\pi D^2)]^2$$
$$\therefore \quad u = (4 \times 0.15)/\pi D^2, \text{ Simplifying}$$
$$D^5 = [(0.012 \times 1000)/(2 \times 9.81 \times 10)] \times [(0.15^2 \times 4^2)/\pi^2]$$
$$= 2.231 \times 10^{-3}$$
$$\therefore \quad \mathbf{D = 0.295\ m} \text{ and } \mathbf{u = 2.195\ m/s}$$

$$\text{Re} = 0.295 \times 2.195/2.37 \times 10^{-6} = 0.273 \times 10^6$$

Refer eqn. 7.11.11, $f = 0.0032 + (0.221/\text{Re}^{0.0237}) = 0.0146$

Assuming **f = 0.014,** to repeat the procedure

$$10 = [(0.014 \times 10000)/(2 \times 9.81 \times D)] \times [(0.15 \times 4)/\pi D^2]^2$$

$\therefore$ $$D^5 = [(0.014 \times 1000)/(2 \times 9.82 \times 10)]\ [(0.15^2 \times 4^2)/\pi^2] = 2.6 \times 10^{-3}$$

$$\mathbf{D = 0.304\ m,} \quad u = 0.15 \times 4/\pi \times 0.304^2 = 2.065 \text{ m/s}$$

$$\text{Re} = 2.065 \times 0.304/2.37 \times 10^{-6} = 0.265 \times 10^6$$

$$f = 0.0032 + 0.221/(0.265 \times 10^6)^{0.237} = 0.01466$$

The answer can be refined further using this value of f and reworking on the same lines.

Problem 7.20. *Two pipes of 0.35 m and 0.25 m dia and length 2000 m and 1500 m with f values 0.021 and 0.018 connected is series carry water from a reservoir to a supply system, the head available being 8 m.* ***Determine the flow quantity neglecting minor losses.***

The head available should be equal to the sum of the frictional losses in the two pipes. Neglecting loss at sudden contraction

$$\delta = [(0.021 \times 2000 \times u_1^2)/(2 \times 9.81 \times 0.35)] + [(0.018 \times 1500 \times u_2^2)/(2 \times 9.81 \times 0.25)]$$

From continuity equation, we get

$$[(\pi \times 0.35^2)/4] \times u_1 = [(\pi \times 0.25^2)/4]u_2$$

$\therefore$ $$u_2 = (0.35/0.25)^2\ u_1 \quad \text{or} \quad u_2^2 = (0.35/0.25)^4\ u_1^2$$

Substituting, and simplifying and solving,

$$u_1 = 0.542 \text{ m/s},\ u_2 = 1.062 \text{ m/s}$$

flow rate $= (0.542 \times \pi \times 0.35^2)/4 =$ **0.0521 m³/s or 187.7 m³/hr**

check the frictional drop:

$$h_f = [(0.021 \times 2000 \times 0.542^2)/(2 \times 9.81 \times 0.35)] + [(0.018 \times 1500 \times 1.062^2)/(2 \times 9.81 \times 0.25)]$$

$$h_f = 1.8 + 6.2 = 8 \text{ m.}$$

Problem 7.21. *A 300 mm dia pipe carries kerosene at a rate of 200 l/s. The roughness is 0.2 mm.* ***Determine the frictional drop*** *over 100 m length of pipe.*

using equation (7.11.13). $\dfrac{1}{\sqrt{f}} = 2 \log \dfrac{R}{\varepsilon} + 1.74 = 2 \log \dfrac{0.15}{0.2 \times 10^{-3}} + 1.74 = 7.49$

$\therefore$ $f = 0.01782$ m $\qquad u = 4 \times 0.2/\pi \times 0.3^2 = 2.829$ m/s

$$h_f = \frac{0.01782 \times 1000 \times u^2}{2 \times 9.81 \times 0.3} + \frac{u^2}{2 \times 9.81} = 3.0785u^2$$

substituting the value of u, $\mathbf{h_f = 24.65\ m}$ **(head of kerosene)**

Problem 7.22. *Water is drawn from a reservoir through a pipe of diameter D and a constant friction factor f. Along the length water is drawn off at the rate of Km³/s per unit length and the length is L. There is no flow at the end. Derive an expression for the loss of head.*

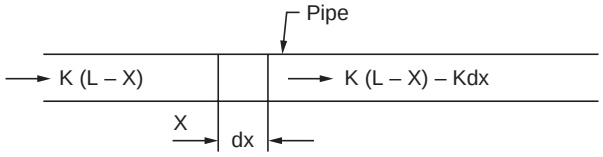


Figure P. 7.22

Consider a length dx at location x, using the equation, the drop dh over length dx is

$$h_f = \frac{fLu^2}{2gD} \quad \therefore \quad dh = \frac{fdx}{D}\frac{u^2}{2g} \tag{1}$$

At this location, the flow rate Q can be obtained as

$Q = K(L - x)$, as total flow is KL and draw off upto x is Kx.

$$u = \frac{4Q}{\pi D^2},\ u^2 = \frac{16Q^2}{\pi^2 D^4} = \frac{16K^2\,(L-x)^2}{\pi^2 D^4}$$

Substituting in eqn. (1)

$$dh = \frac{fdx}{2g}\frac{16K^2\,(L-x)^2}{\pi^2 D^5} = \frac{8fK^2}{g\pi^2\,D^5}\,(L-x)^2\,dx$$

Integrating from $x = 0$ to L

$$h_2 - h_1 = h_f = \frac{8\,fK^2}{g\pi^2\,D^5}\int_0^L [L^2 - 2Lx + x^2]dx = \frac{8\,fK^2}{g\pi^2 D^5}\left[L^3 - L^3 + \frac{L^3}{3}\right]$$

$$\therefore \qquad h_f = \frac{8\,fL^3\,K^2}{3g\pi^2\,D^5}$$ (Note: K has a unit m^3/sm)

for the following data,

$f = 0.024$, $K = 7.5$ *l/hr/m* $= 2.085 \times 10^{-6}$ $m^3/s/m$,

$D = 0.1$ m, $L = 4.8 \times 10^3$ m,

$$\mathbf{h_f} = \frac{8 \times 0.024 \times (4.8 \times 10^3)^3\,(2.085 \times 10^{-6})^2}{3 \times 9.81 \times \pi^2 \times 0.1^5} = \mathbf{31.73\ m}$$

The head drop between lengths L_1 and L_2 can be determined by difference *i.e.,* $(h_{f2} - h_{f1})$

Problem 7.23. *A pipe line 200 mm dia. and 4000 m long connects two reservoirs with a difference in level of 60 m. Water is drawn at 1500 m point at a rate of 50 l/s. Friction coefficient f = 0.024.* ***Determine the flow rates*** *in the two sections. Neglect minor losses*

$$h_f = \frac{fLu^2}{2gD},\ u = \frac{4Q}{\pi D^2},\ u^2 = \frac{16Q^2}{\pi^2 D^2},\ h_f = \frac{8\,fLQ^2}{\pi^2\,gD^5}$$

Considering the two sections, (total drop)

$$60 = \frac{8 \times 0.024 \times 1500 \times Q_1^{\,2}}{\pi^2 g \times 0.2^5} + \frac{8 \times 0.024 \times 2500 \times Q_2^{\,2}}{\pi^2 g \times 0.25}$$

$$= 9295.5\ Q_1^{\,2} + 15492.54\ Q_2^{\,2}$$

but $Q_2^2 = (Q_1 - 0.05)^2$, Substituting and simplifying

$$60 = 9295.52\, Q_1^2 + 1549.25(Q_1^2 + 0.05^2 - 2Q_1 \times 0.05)$$

or $$24788.05\, Q_1^2 - 1549.25\, Q_1 - 21.268 = 0$$

$$Q_1 = \frac{1549.25 \pm [(-1549.25)^2 + 4 \times 21.268 \times 24788.05]^{0.5}}{2 \times 24788.05}$$

$$= \frac{1549.25 \pm 2123.45}{2 \times 24788.05} = \mathbf{0.074082\ m^3/s,}$$

$\therefore$ $\mathbf{Q_2 = 0.024082\ m^3/s}$

The other solution is negative.

Check: For the first section

$$h_f = \frac{8 \times 0.024 \times 1500 \times 0.074082^2}{\pi^2 \times 9.81 \times 0.2^5} = 51.015 \text{ m}$$

For the second section

$$h_f = \frac{8 \times 0.024 \times 2500 \times 0.024082^2}{\pi^2 \times 9.81 \times 0.2^5} = 8.985 \text{ m}, \quad \text{Total head} = 60 \text{ m}$$

Problem 7.24. *Two adjacent city centres B and D receive water from separate sources A and C. The water level in A is 4 m above that in C. Reservoir A supplies city centre B by 0.4 m diameter pipe of 3000 m length with a level difference of 10 m. City centre D's is supplied by reservoir C through a 4000 m long pipe of 0.45 m diameter, with a level difference of 15 m. After sometime it is found that centre B has excess water while centre D is staraved. So it is proposed to interconnect these lines and draw 100 l/s from the line A to B. The junction on AB is at a distance of 2000 m from A. The junction CD is at 3000 m from C.* ***Determine the original supply rates and supply rates with interconnection*** *to centres B and D. Also determine the diameter of the interconnecting pipe, if the length is 1500 m Friction factor, f = 0.01 in all cases.*

The arrangement is shown in Fig. P. 7.24

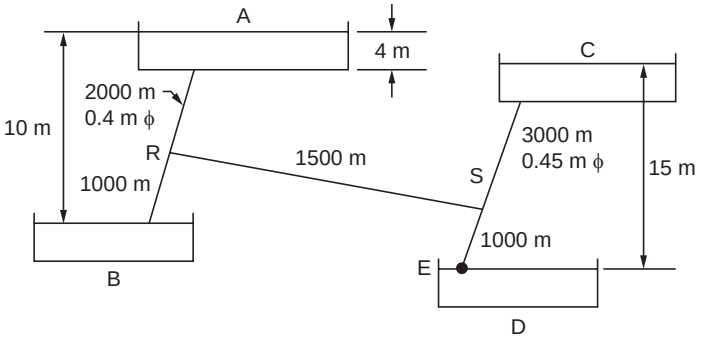


Figure P. 7.24

(*i*) Without interconnection : using equation 7.11.15

$$h_f = \frac{8\, fLQ^2}{\pi^2\, gD^5}$$

Drop in Line *AB* is 10 m

$$10 = \frac{8 \times 0.01 \times 3000 \times Q^2}{\pi^2 \times 9.81 \times 0.4^5}$$

$\therefore$ $\mathbf{Q_{AB} = 0.20325\ m^3/s}$ or $\mathbf{203.25\ l/s}$

Drop in line *CD* is 15 m

$$15 = \frac{8 \times 0.01 \times 4000 \times Q^2}{\pi^2 \times 9.81 \times 0.45^5}$$

$\therefore$ $\mathbf{Q_{CD} = 0.2894\ m^3/s}$ **or** $\mathbf{289.4\ l/s}$

After interconnection: line *AB*:

Let the flow up to *R* be *Q* and then in *RB* $(Q - 0.1)$

Total frictional loss $= 10 = \dfrac{8 \times 0.01 \times 2000 \times Q^2}{\pi^2 \times 9.81 \times 0.4^{0.5}} + \dfrac{8 \times 0.01 \times 1000(Q - 0.1)^2}{\pi^2 \times 9.81 \times 0.4^5}$

This reduces to $3Q^2 - 0.2Q - 0.11393 = 0$. **Solving Q = 0.231 m³/s or 231 l/s**

Now the centre *B* will receive 131 *l/s* (previous 203 *l/s*)

Line *CD*: Let the flow upto *S* be *Q* and then $(Q + 0.1)$ upto *D*

Total head loss $= 15 = \dfrac{8 \times 0.01 \times 3000 \times Q^2}{\pi^2 \times 9.81 \times 0.45^{0.5}} + \dfrac{8 \times 0.01 \times 1000(Q + 0.1)^2}{\pi^2 \times 9.81 \times 0.45^5}$

This reduces to $4Q^2 + 0.2Q - 0.32499 = 0$

Solving $\mathbf{Q = 0.261\ m^3/s}$ **or** $\mathbf{261\ l/s}$

Now the city center *C* will receive 361 *l/s* (previous 289.4 *l/s*)

To determine the diameter of the connecting pipe *RS*:

Head drop from *A* to *R*

$$h_{f1} = \frac{8 \times 0.01 \times 2000 \times 0.231^2}{\pi^2 \times 9.81 \times 0.4^5} = 8.61 \text{ m}$$

Head drop from *S* to *E*

$$h_{f2} = \frac{8 \times 0.01 \times 1000 \times 0.361^2}{\pi^2 \times 9.81 \times 0.45^5} = 5.84 \text{ m}$$

Head drop from *A* to *E* = 4 + 15 = 19 m

$\therefore$ Head available between

$$RS = 19 - 8.61 - 5.84 = 4.55 \text{ m}$$

Considering Pipe *RS*

$$4.55 = \frac{8 \times 0.01 \times 1500 \times 0.1^2}{\pi^2 \times 9.81 \times D^5}$$ **Solving D = 0.307 m.**

OBJECTIVE QUESTIONS

O Q. 7.1. Fill in the blanks:

1. In fully developed laminar flow in pipes the velocity distribution with radius is ————————.
2. In fully developed laminar flow in pipes the shear stress variation with radius is ————————.
3. A dye injected into laminar stream will travel ————————.
4. Momentum transfer in laminar flow is at ———————— level.
5. For laminar flow to prevail in a duct the value of Reynolds number should be ————————.
6. The friction factor is defined as ————————.
7. The friction factor in laminar flow is pipes in given by ————————.
8. In laminar flow in a pipe the shear stress is maximum at ————————.
9. Reynolds number is the ratio between ———————— and ———————— forces.
10. The energy line represents ————————.
11. The hydraulic line represents ————————.

Answers

(1) Parabolic (2) Linear (3) along a line without mixing (4) Microscopic, molecular (5) less than 2000. (6) 4 $\tau_0\, g_0/(\rho u_m{}^2/2)$, (7) 64/Re (8) The wall (9) inertia, viscous (10) Total energy at each location (11) The pressure energy at the location.

O Q. 7.2. Fill in the blanks

1. In laminar flow is through a pipe the average velocity is ———————— of the maximum velocity.
2. In flow through pipes, the flow ———————— change from laminar to turbulent condition.
3. Hydraulic diameter is defined as ————————
4. Hydraulic radius is defined as ————————
5. Entrance length is defined as ————————
6. In laminar flow the entrance length is approximately ————————
7. A pipe is hydraulically smooth when 6 ε is ————————
8. A pipe is hydraulically rough when ————————
9. Chart relating friction factor, Reynolds number and pipe roughness is known as ————————
10. For flow in non circular pipes, the length parameter used in Reynolds number calculation is ————————
11. In a network of pipes the algebraic sum of the frictional losses around any circuit will be ————————
12. In a network of pipes at any node the algebraic sum of flows will be ————————

Answers

(1) One half of (2) does not (3) 4A/P, A-Area P-Perimeter (4) *A/P* (5) the length above which the velocity profile becomes constant (6) 0.58 Re D (7) less than δ_i (8) $\delta_1 < 6\,\varepsilon$, (9) Moody diagram (10) Hydraulic diameter (11) zero (12) zero

O. Q. 7.3. Fill in the blanks

1. The entrance head loss for square edged entrance is ————————.
2. The head loss due to sudden expansion is ————————.
3. The frictional loss in globe valve is ———————— compared to that in gate valve.
4. To reduce losses in large bends ———————— can be used.

5. Minor losses are losses due to ————————.
6. The very thin layer adjacent to the wall in turbulent flow is callecl ————————.
7. The velocity variation is ———————— in laminar sublayer.
8. Equivalent length of pipe fitting is ————————.
9. Equivalent length of a pipe as referred to another pipe is ————————.
10. For the same flow area and flow rate, a square section will give a ———————— pressure drop.
11. For a given available head, maximum power will be transmitted when the frictional loss of head equals ———————— of the total head.

Answers

(1) $0.5u^2/2g$ (2) $(u_1 - u_2)^2/2g$ (3) larger (4) vanes (5) changes in section and fittings (6) laminar sublayer (7) linear (8) pipe length with the same frictional drop (9) the length to produce the same frictional drop for the same flow (10) higher (11) 1/3

O Q. 7.4. Choose the correct answer

1. Reynolds number is given by
   (*a*) $\rho u\mu/D$, (*b*) $\mu D\rho/u$, (*c*) $\rho u/\mu D$ (*d*) $\rho uD/\mu$.
2. Reynolds number signifies the ratio of
   (*a*) gravity forces top viscous forces (*b*) inertial forces to viscous forces
   (*c*) inertia forces to gravity forces (*d*) buoyant forces to inertia forces.
3. In pipe flow the critical Reynolds number is about
   (*a*) 640 (*b*) 5×10^5 (*c*) 2000 (*d*) 64000
4. The entry length in pipe flow will be higher for
   (*a*) highly viscous fluids (*b*) low viscosity fluid
   (*c*) high velocity of flow (*d*) small diameters
5. A pipe is said to be hydraulically rough if the laminar sublayer thickness δ_L as compared to physical roughness ε is
   (*a*) $\delta_L > \varepsilon$ (*b*) $\delta_L < \varepsilon$ (*c*) $\delta_L > 6\varepsilon$ (*d*) $\delta_L < 6\varepsilon$.
6. With constant flow rate if the diameter is doubled in laminar flow in pipes, the frictional drop will change by a factor of
   (*a*) 2 (*b*) 0.5 (*c*) 1/32 (*d*) 1/16.
7. In laminar fully developed flow in a pipe the ratio, average velocity/maximum velocity is
   (*a*) 2 (*b*) 0.5 (*c*) $\sqrt{2}$ (*d*) $1/\sqrt{2}$.
8. In fully developed turbulent flow, if the diameter is halved without changing the flow rate, the frictional drop will change by the factor
   (*a*) 32 times (*b*) 16 times (*c*) 8 times (*d*) 4 times.
9. In laminar pipe flow for a given flow rate Q, the power required to overcome friction will be proportional to
   (*a*) Q (*b*) Q^2 (*c*) $\sqrt{Q}$ (*d*) $Q^{3/2}$.
10. In turbulent flow in a pipe with flow rate Q the power required to overcome frictional losses is proportional to
    (*a*) Q (*b*) Q^2 (*c*) Q^3 (*d*) Q^4
11. The shear stress at the wall of a 16 cm dia pipe in laminar flow is 36 N/m^2. The shear stress at a radius of 4 cm in N/m^2 is
    (*a*) 9 (*b*) 18 (*c*) 6 (*d*) 72

12. The velocity along the centre line in laminar flow through a pipe of 8 cm dia is 2 m/s. The velocity at a radius of 2 cm in m/s is
    (*a*) 1 (*b*) 1.5 (*c*) 1.414 (*d*) 1/1.414.
13. The friction factor in pipe flow at near critical conditions is around
    (*a*) 0.064 (*b*) 0.025 (*c*) 0.64 (*d*) 0.032.
14. The friction factor in laminar flow in a pipe was measured as 0.05. The Reynolds number should be around
    (*a*) 1280 (*b*) 1000 (*c*) 640 (*d*) 2000.
15. An oil with a density of 800 kg/m^3 and dynamic viscosity of 8×10^{-3} Ns/m^2 flows through a smooth pipe of 100 mm dia. The maximum velocity for laminar flow to prevail will be (in m/s.)
    (*a*) 0.4 (*b*) 0.2 (*c*) 1.0 (*d*) 4.0.
16. The velocity profile in turbulent flow is
    (*a*) parabolic (*b*) logarithmic
    (*c*) 2nd degree polynomial (*d*) 4th degree polynomial.

Answers

1. (*d*) 2. (*b*) 3. (*c*) 4. (*a*) 5. (*d*) 6. (*d*) 7. (*b*) 8. (*a*) 9. (*b*) 10. (*c*) 11. (*b*) 12. (*b*) 13. (*d*) 14. (*a*) (15). (*b*) 16. (*b*).

O Q. 7.5. Indicate whether the statements are correct or incorrect

1. Friction factor will be higher in laminar flow.
2. In the household pipe system the flow of water is laminar.
3. Blood flow in blood vessels is turbulent.
4. Pipes of equal diameter and equal roughness will have the same friction factor.
5. The development of flow over a flat plate and entry section of a pipe will be similar.
6. The velocity profile after the entry section in a pipe will remain the same.
7. For the same total sectional area two equal diameter pipes will deliver more than a single pipe for the same pressure drop.
8. In turbulent flow for the same diameter and friction factor the pressure drop for a given length will vary as the square of flow rate.
9. If velocity is doubled the pressure drop will reduce to half the value.
10. A bell mouthed inlet is desirable.
11. For the same sectional area and flow rate, square section will have a lower Reynolds Number.
12. For the same sectional area and flow rate square section will lead to lower frictional drop.

Answers

Correct 1, 5, 6, 8, 10, 11. Incorrect: 2, 3, 4, 7, 9, 12

O Q.7.6. Fill in the blanks with "increases", "decreases" for "remains constant".

1. In flow over a flat plate, the boundary layer thickness ———— with distance.
2. In flow through pipes, after the entry length the velocity profile ————.
3. As Reynolds number decreases in laminar flow in pipes, the friction factor ————.
4. For a specified flow of fluid in a pipe, (*i.e.,* with fixed mass velocity) the Reynolds number will ———— as the dynamic viscosity of the fluid increases.
5. For the same flow rate and friction factor, the pressure drop in turbulent flow will ———— as the diameter increases.

6. For a specified roughness, beyond a certain Reynolds number the friction factor ————————
7. Minor losses will ———————— as velocity increases.
8. When a pipe of smaller diameter is converted to an equivalent pipe of larger diameter, the length ————————
9. When a pipe with a lower friction factor is converted to an equivalent pipe with higher friction factor, the length ————————
10. When two pipes in parallel flow are replaced by a single pipe with its area equal to the sum of the areas of the two pipes, the flow rate will ————————

Answers

Increases: 1, 3, 7, 8, 10 Decreases : 4, 5, 9

Remains constant: 2, 6

O Q. 7.7. Match the pairs:

(1)

(*A*) Laminar flow	(1) Logarithmic velocity profile
(*B*) Turbulent flow	(2) Parabolic velocity profile
(*C*) Laminar pipe flow	(3) Molecular level mixing
(*D*) Turbulent pipe flow	(4) Macroscopic mixing

(2) In pipe flow,

(*A*) Sudden expansion	(1) $10u^2/2g$
(*B*) u/u_{max}	(2) $0.5u^2/2g$
(*C*) Square entrance	(3) $1 - (r/R)^2$
(*D*) Globe valve	(4) $\Delta u^2/2g$

Answers

(1) *A* – 3, *B* – 4, *C* – 2, *D* – 1 (2) *A* – 4, *B* – 3, *C* – 2, *D* – 1

EXERCISE PROBLEMS

E 7.1. An oil with specific gravity of 0.85 flows in a pipe of 100 mm dia., the flow rate being 0.5 l/s. The kinematic viscosity at this condition is 1.8×10^{-5} m^2/s. Determine whether flow is laminar or turbulent. **(Re = 54, Laminar)**

E 7.2. Show that the velocity profile in laminar flow through a circular pipe is parabolic. Find the average velocity in terms of maximum velocity.

E 7.3. Show that the wall shear in laminar flow through a pipe $\tau_0 = \frac{f}{4}\frac{\rho u^2}{2g}$.

E 7.4. Show that in laminar flow through a pipe $f = 64/\text{Re}$.

E 7.5. For the data in problem E 7.1, determine

(*a*) centre line velocity, (*b*) velocity at $r = 2$ cm,

(*c*) friction factor, (*d*) wall shear and

(*e*) head loss/m. **(0.127 m/s, 0.107 m/s, 0.18, 0.077 N/m^2, 0.37×10^{-3} m/m)**

E 7.6. Oil of specific gravity 0.9 and dynamic viscosity 5.992×10^{-3} Ns/m^2 flows in a pipe of 15 cm dia. at a rate of 0.06 m^3/s. The head loss over a length of 120 m was found to be 16 m. Determine the value of friction factor. Is the flow laminar or turbulent?

(0.034, Re = 11475, turbulent)

E 7.7. Water at 20 °C flows through a 50 cm dia. pipe with roughness ε/R = 0.00018. If the energy gradient (h_f/L) is 0.006, determine the flow rate $v = 1 \times 10^{-6}$ m^2/s. **(0.42 m^3/s)**

E 7.8. Three pipes in series connect two water reservoirs with a level difference of 10 m. The pipe diameters are 30 cm, 20 cm and 25 cm, the lengths being 300 m, 150 m and 250 m respectively. The friction factors are 0.019, 0.021 and 0.02 respectively. Neglecting minor losses, determine the flow rate. **(0.083 m^3/s)**

E 7.9. Pipe lines as shown in Fig. E. 7.9 provide water supply from a reservoir. Determine the flow rates in lines *C*, *A* and *B*. **(0.055 m^3/s, 0.042, 0.013 m^3/s)**

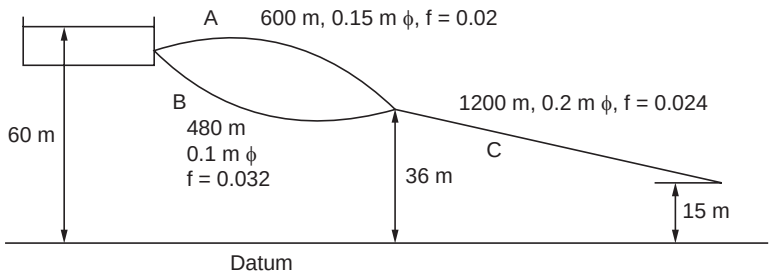


Figure E. 7.9

E 7.10. A pipeline 1.2 m dia. and 720 m length having a friction factor of 0.04 connects two reservoirs with a difference in level of 6 m. The pipe line rises to a level of 3 m above the level of the upper reservoir at a distance of 240 m. Determine the flow rate and the pressure at the highest point. **(2.51 m^3/s, – 5.6 cm of water)**

E 7.11. A riveted steel pipe of 300 mm dia. carries water over a length of 300 m, the head available being 6 m. Determine the flow rate if the roughness height is 3 mm. **(0.124 m^3/s)**

E 7.12. Determine the diameter of the pipe to convey 250 l/s of oil over 3000 m length with a loss of 25 m. $v = 1 \times 10^{-5}$ m^2/s. **(413 mm)**

E. 7.13. Show from basics that in sudden contraction, the loss of head equals $(V_2 - V_1)^2/2g$. In the case of formation of vena contracta show that the loss equals $[(1/C_c) - 1]^2/2g$, where $C_c = A_c/A_2$.

E 7.14. Show using the expression in E 7.13, the pressure difference across a sudden contraction is given by $(P_1 - P_2)/\gamma = \dfrac{V_2^2}{2g}\left[1 + \left\{\left(\dfrac{1}{C_c}\right) - 1\right\}^2\right] - \dfrac{V_2^2}{2g}$.

E 7.15. A pipe carries 56 *l/s* of water and there is a sudden change in diameter, the coefficient of contraction has a value of 0.62. Determine in the following cases the loss of head (Hint, see problem E 9.13 and 14).

(*i*) Contraction from 0.2 m to 0.15 m.

(*ii*) Contraction from 0.3 m to 0.5 m.

(*iii*) Contraction from 0.45 m to 0.15 m.

Calculate the pressure difference between sections.

(0.192 m in all cases, 0.54 m, 0.673 m, 0.699 m)

E 7.16. Derive the following expression the loss of head due to sudden expansion. $h_f = (V_1 - V_2^2)/2g$. Show that it can be reduced to the form $k\ V_1^2/2g$ where $k = [1 - (A_1/A_2)]^2$.

E 7.17. A pipe line of total length 3000 m is made up of two diameters, 200 mm for the first run and 150 mm for the second run, connects two reservoirs. The first run ends at a level 1.5 m below the level of the higher reservoir and the total difference in levels is 13.5 m. The friction coefficient for both sections is 0.02 m. Determine the maximum length of the run so that the pressure at this point does not go more than 3 m below atmosphere. Also calculate the flow rate. Neglect minor losses. **(2034 m, 0.0207 m³/s)**

E 7.18. Water is conveyed by a pipe line of 1.2 m dia. and 720 m length from a reservoir whose level is 6 m above the level of down stream reservoir. The pipe line due to the terrain has to be laid such that its level is 3 m above that of the first reservoir level at a distance of 240 m from the entry. Determine the pressure at this point and also the flow rate. $f = 0.04$

(2.51 m³/s, 5.6 m of water)

E 7.19. A pipe line of two sections, the first of 50 mm dia. and 15 m length and the second of 75 mm dia and 24 m length connected in series empties a reservoir at the rate of 168 l/min. The entry is sharp edged. The enlargement is sudden. Discharge is to atmosphere. The values of friction factor are 0.0192 and 0.0232 for the pipes. Determine the difference in height between the reservoir level and the discharge point. **(0.852 m)**

E 7.20. Supply is drawn uniformly at the rate of 7.5 l/hr per m length along the length of 4800 m pipe. Assuming $f = 0.024$, For a head loss of m 31 m determine the diameter of the pipe to provide the flow. **(0.1 m)**

E 7.21. Two reservoirs are connected in parallel by two pipes. One pipe is 5000 m long and the diameter is 100 mm. $f_1 = 0.02$ and the other pipe is 4000 m long and its diameter is 120 mm. The friction factor is 0.025. If the flow rate through the first pipe is 60 *l/s*, determine the flow rate through the second pipe. **(94.65 l/s)**

E 7.22. Determine the discharge through a pipe system described below connecting two reservoirs with a difference in level of 6 m. A single pipe of 0.6 m dia. of 3000 m length takes off from the higher reservoir and feeds to a junction from which, two pipes of 0.3 m dia. and 3000 m length each feed the water in parallel to the lower reservoir. $f = 0.04$. **(72.3 l/s)**

E 7.23. If in the problem E 7.22, one of the pipes downstream is closed for maintenance, determine the flow rate. **(37.76 l/s)**

E 7.24. Water is transported from reservoir *A* to reservoirs *B* and *C* by pipe line system shown in Fig. E. 7.24. Determine the flow to reservoirs *B* and *C*. $f = 0.04$. **(Q_B = 28.3 l/s, Q_c = 90.7 l/s)**

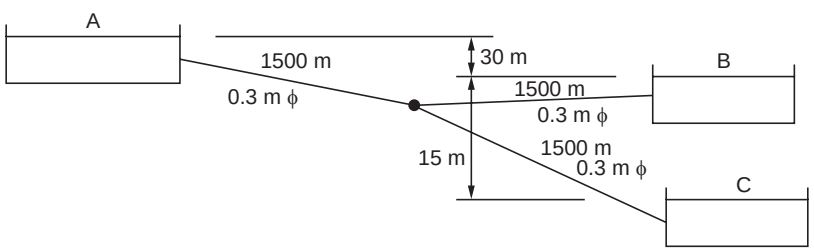


Figure E. 7.24

E 7.25. Water is drawn from a reservoir through two pipes of 900 mm dia. and 1500 m length to a common junction from where a pipe of 1200 mm diameter and 2400 m length carries it to the lower reservoir. If the difference in water levels is 10 m, determine the flow rate. $f = 0.02$, neglect other losses. **(0.973 m³/s)**

E 7.26. If in the problem E 7.25, one inlet branch is shut off for maintenance, calculate the flow rate. **(1.307 m^3/s)**

E 7.27. Two reservoirs with a difference in level of 6 m are connected by a pipe system. 3000 m length of 0.6 m dia. pipe takes off from the reservoir. At this point 36 l/s water is drawn off and the diameter is reduced to 0.3 m for the next 3000 m. Determine the flow rate in the first branch. f = 0.04. **(72.16 l/s)**

E 7.28. A ridge interposes between two reservoirs whose level difference is 30 m. The distance upto the ridge is 300 m. The total length of the pipe is 3000 m. The pipe diameter is 600 mm. Determine the maximum height of the ridge that the line can cross if the pressure at this point should not go below 3 m of water (absolute). Also determine the flow rate. f = 0.03. **(5 m, 0.56 m^3/s)**

E 7.29. A fire hose of 75 mm dia. and 180 m length ends in a nozzle of 25 mm dia. The discharge coefficient of the nozzle is 0.94. The tip of the nozzle is 9 m above pump outlet. Calculate the head to be developed by the pump for a flow rate of 480 l/min, f = 0.048. **(43.425 m)**

E 7.30. A smooth concrete duct of square section of side 1.5 m is 40 m long. Determine the head loss required for a flow rate of 9 m^3/s. f= 0.02. Express the head loss as slope. **(0.44 m, 1.1/100)**

E 7.31. Two reservoirs with a level difference of 40 m are connected by a 3 km long, 0.9 m dia. pipe with friction factor f = 0.021. Determine the flow rate. If the middle 1 km pipe is replaced by two pipes of 0.64 m dia. determine the flow rate. If one of the pipes in the middle section is blocked, calculate the flow rate. **(2.13m^3/s, 2.008 m^3/s, 1.347 m^3/s)**

8 Dimensional Analysis

8.0 INTRODUCTION

Fluid flow is influenced by several parameters like, the geometry, fluid properties and fluid velocity. In the previous chapters analytical methods used in fluid flow studies were discussed. In the study of flow of real fluids analytical methods alone are found insufficient. Experimental methods and results have contributed heavily for the development of fluid mechanics. The solution of realistic problems usually involves both anlytical and experimental studies. Experiments are used to validate analytical results as well as generalize and extend their applications. Depending either solely on analytical methods or experiments for the design of systems is found to lead to inadequate performance and high cost.

Experimental work is rather costly and time consuming, particularly when more than three parameters are involved. Hence it is necessary to plan the experiments so that most information is obtained from fewest experiments. Dimensional analysis is found to be a very useful tool in achieving this objective. The mathematical method of dimensional analysis comes to our help in this situation. The number of parameters can be reduced generally to three by grouping relevant variables to form dimensionless parameters. In addition these groups facilitate the presentation of the results of the experiments effectively and also to generalize the results so that these can be applied to similar situations.

Flow through pipes can be considered as an example. Viscosity, density, flow velocity and diameter are found to influence the flow. If the effect of each of these parameters on flow is separately studied the number of experiments will be large. Also these results cannot be generalized and its usefulness will be limited. When the number of these variables are combined to form a dimensionless group like ($u\ D\ \rho/\mu$) few experiments will be sufficient to obtain useful information. This parameter can be varied by varying one of the variables which will be the easier one to vary, for example velocity u. The results will be applicable for various combinations of these parameters and so the results can be generalized and extended to new situations. The results will be applicable also for different fluids and different diameters provided the value of the group remains the same. Example 8.1 illustrates the advantage dimensional analysis in experiment planning. The use of the results of dimensional analysis is the basis for similitude and modal studies. The topic is discussed in the next chapter.

Example 8.1. *The drag force F on a stationary sphere in flow is found to depend on diameter D, velocity u, fluid density ρ and viscosity μ. Assuming that to study the influence of a parameter 10 experimental points are necessary, estimate the* ***total experimental points needed*** *to obtain complete information. Indicate how the number of experiments can be reduced.*

To obtain a curve *F vs u*, for fixed values of ρ, μ and *D*, experiments needed = 10.

To study the effect of ρ these 10 experiments should be repeated 10 times with 10 values of ρ the total now being 10^2.

The 10^2 experiments have to repeated 10 times each for different values of μ.

Total experiments for *u*, ρ and μ = 10^3.

To study the effect of variation of diameter all the experiments have to be repeated 10 times each. Hence total **experiments required = 10^4.**

These parameters can be combined to obtain two dimensionless parameters,

$$\frac{F}{\rho u^2 D^2} = f\left(\frac{\rho\, uD}{\mu}\right)$$

(The method to obtain such grouping is the main aim of this chapter)

Now only 10 experiments are needed to obtain a comprehensive information about the effect of these five parameters.

Experiments can be conducted for obtaining this information by varying the parameter ($uD\rho/\mu$) and determining the values for $F/\rho u^2 D^2$. Note : It will be almost impossible to find fluids with 10 different densities and 10 different viscosities.

8.1 METHODS OF DETERMINATION OF DIMENSIONLESS GROUPS

1. Intuitive method: This method relies on basic understanding of the phenomenon and then identifying competing quantities like types of forces or lengths etc. and obtaining ratios of similar quantities.

Some examples are: Viscous force *vs* inertia force, viscous force *vs* gravity force or roughness dimension *vs* diameter. This is a difficult exercise and considerable experience is required in this case.

2. Rayleigh method: A functional power relation is assumed between the parameters and then the values of indices are solved for to obtain the grouping. For example in the problem in example 1 one can write

$$(\pi_1, \pi_2) = F^a\, \rho^b D^c \mu^d U^e$$

The values of *a, b, c, d*, and *e* are obtained by comparing the dimensions on both sides the dimensions on the L.H.S. being zero as π terms are dimensionless. This is also tedious and considerable expertise is needed to form these groups as the number of unknowns will be more than the number of available equations. This method is also called "indicial" method.

3. Buckingham Pi theorem method: The application of this theorem provides a fairly easy method to identify dimensionless parameters (numbers). However identification of the influencing parameters is the job of an expert rather than that of a novice. This method is illustrated extensively throughout this chapter.

8.2 THE PRINCIPLE OF DIMENSIONAL HOMOGENEITY

The principle is basic for the correctness of any equation. It states "If an equation truly expresses a proper relationship between variables in a physical phenomenon, then each of the additive terms will have the same dimensions or these should be dimensionally homogeneous."

For example, if an equation of the following form expresses a relationship between variables in a process, then each of the additive term should have the same dimensions. In the expression, $A + B = C/D$, A, B and (C/D) each should have the same dimension. This principle is used in dimensional analysis to form dimensionless groups. Equations which are dimensionally homogeneous can be used without restrictions about the units adopted. Another application of this principle is the checking of the equations derived.

Note : Some empirical equations used in fluid mechanics may appear to be non homogeneous. In such cases, the numeric constants are dimensional. The value of the constants in such equations will vary with the system of units used.

8.3 BUCKINGHAM PI THEOREM

The statement of the theorem is as follows : If a relation among n parameters exists in the form

$$f(q_1, q_2, \ldots\ldots q_n) = 0$$

then the n parameters can be grouped into $n - m$ independent dimensionless ratios or π parameters, expressed in the form

$$g(\pi_1, \pi_2 \ldots\ldots \pi_{n-m}) = 0 \qquad (8.3.1)$$

or $$\pi_1 = g_1 (\pi_2, \pi_3 \ldots\ldots \pi_{n-m})$$

where m is the number of dimensions required to specify the dimensions of all the parameters, $q_1, q_2, \ldots q_n$. It is also possible to form new dimensionless π parameters as a discrete function of the $(n - m)$ parameters. For example if there are four dimensionless parameters π_1, π_2, π_3 and π_4 it is possible to obtain π_5, π_6 etc. as

$$\pi_5 = \frac{\pi_1}{\pi_3 \pi_4} \quad \text{or} \quad \pi_6 = \frac{\pi_1^{0.5}}{\pi_2^{2/3}}$$

The limitation of this exercise is that the exact functional relationship in equation 8.3.1 cannot be obtained from the analysis. The functional relationship is generally arrived at through the use of experimental results.

8.3.1 Determination of π Groups

Irrespective of the method used the following steps will systematise the procedure.

Step 1. List all the parameters that influence the phenomenon concerned. This has to be very carefully done. If some parameters are left out, π terms may be formed but experiments then will indicate these as inadequate to describe the phenomenon. If unsure the parameter can be added. Later experiments will show that the π term with the doubtful

parameters as useful or otherwise. Hence a careful choice of the parameters will help in solving the problem with least effort. Usually three type of parameters may be identified in fluid flow namely fluid properties, geometry and flow parameters like velocity and pressure.

Step 2. Select a set of primary dimensions, (mass, length and time), (force, length and time), (mass, length, time and temperature) are some of the sets used popularly.

Step 3. List the dimensions of all parameters in terms of the chosen set of primary dimensions. Table 8.3.1. Lists the dimensions of various parameters involved.

Table 8.3.1. Units and Dimensions of Variables

Variable	Unit (SI)	Dimension	
		MLT θ system	**FLT θ system**
Mass	kg	M	FT^2/L
Length	m	L	L
Time	s	T	T
Force	N	ML/T^2	F
Temperature	deg C or K	θ	θ
Area	m^2	L^2	L^2
Volume	m^3	L^3	L^3
Volume flow rate	m^3/s	L^3/T	L^3/T
Mass flow rate	kg/s	M/T	FT/L
Velocity	m/s	L/T	L/T
Angular velocity	Rad/s	1/T	1/T
Force	N	ML/T^2	F
Pressure, stress, Bulk modulus	N/m^2	M/LT^2	F/L^2
Moment	Nm	ML^2/T^2	FL
Work, Energy	J, Nm	ML^2/T^2	FL
Power	W, J/s	ML^2/T^3	FL/T
Density	kg/m^3	M/L^3	FT^2/L^4
Dynamic viscosity	kg/ms, Ns/m^2	M/LT	FT/L^2
Kinematic viscosity	m^2/s	L^2/T	L^2/T
Surface tension	N/m	M/T^2	F/L
Specific heat	J/kg K	L^2/T^2 θ	$L^2/T^2\theta$
Thermal conductivity	W/mK	ML/T^3 θ	F/Tθ
Convective heat transfer coefficient	W/m^2 K	M/T^3 θ	F/LTθ
Expansion coefficient	(m/m)/K	1/T	1/T

Step 4. Select from the list of parameters a set of repeating parameters equal to the number of primary dimensions. Some guidelines are necessary for the choice. (*i*) the chosen set should contain all the dimensions (*ii*) two parameters with same dimensions should not be chosen. say L, L^2, L^3, (*iii*) the dependent parameter to be determined should not be chosen.

Step 5. Set up a dimensional equation with the repeating set and one of the remaining parameters, in turn to obtain $n - m$ such equations, to determine π terms numbering $n - m$. The form of the equation is,

$$\pi_1 = q_{m+1} \cdot q_1^{\,a} \cdot q_2^{\,b} \cdot q_3^{\,c} \ldots\ldots q_m^{\,d}$$

As the LHS term is dimensionless, an equation for each dimension in terms of a, b, c, d can be obtained. The solution of these set of equations will give the values of a, b, c and d. Thus the π term will be defined.

Step 6. Check whether π terms obtained are dimensionless. This step is essential before proceeding with experiments to determine the functional relationship between the π terms.

Example 8.2. *The pressure drop ΔP per unit length in flow through a smooth circular pipe is found to depend on (i) the flow velocity, u (ii) diameter of the pipe, D (iii) density of the fluid ρ, and (iv) the dynamic viscosity μ.*

(*a*) *Using π theorem method, evaluate the dimensionless parameters for the flow.*

(*b*) *Using Rayleigh method (power index) evaluate the dimensionless parameters.*

Choosing the set mass, time and length as primary dimensions, the dimensions of the parameters are tabulated.

S.No.	Parameter	Unit used	Dimension
1	Pressure drop/m, ΔP	($N/m^2/m$ ($N = kgm/s^2$)	M/L^2T^2
2	Diameter, D	m	L
3	Velocity, u	m/s	L/T
4	Density, ρ	kg/m^3	M/L^3
5	Dynamic viscosity, μ	kg/ms	M/LT

There are five parameters and three dimensions. Hence two π terms can be obtained. As ΔP is the dependent variable D, ρ and μ are chosen as repeating variables.

Let $\pi_1 = \Delta P\, D^a \rho^b u^c$, Substituting dimensions,

$$M^0L^0T^0 = \frac{M}{L^2T^2} L^a \frac{M^b}{L^{3b}} \frac{L^c}{T^c}$$

Using the principle of dimensional homogeneity, and in turn comparing indices of mass, length and time.

$$1 + b = 0 \quad \therefore \quad b = -1, \quad -2 + a - 3b + c = 0 \quad \therefore \quad a + c = -1$$

$$-2 - c = 0 \quad \therefore \quad c = -2, \quad \text{Hence } a = 1.$$

Substituting the value of indices we obtain

$\therefore \qquad \pi_1 = \Delta PD/\rho u^2;$

This represents the ratio of pressure force and inertia force.

Check the dimension :

$$\frac{M}{L^2T^2} L \frac{L^3}{M} \frac{T^2}{L^2} = M^0L^0T^0$$

Let $\pi_2 = \mu D^a \rho^b u^c$, substituting dimensions and considering the indices of *M, L* and *T,*

$$M^0L^0T^0 = \frac{M}{LT} L^a \frac{M^b}{L^{3b}} \frac{L^c}{T^c}$$

$$1 + b = 0 \quad \text{or} \quad b = -1, \quad -1 + a - 3b + c = 0, \quad a + c = -2, -1 - c = 0, \quad c = -1 \quad a = -1$$

Substituting the value of indices,

$\therefore \qquad \pi_2 = \mu/u\rho D$

check, $$\frac{M}{LT} \frac{T}{L} \frac{L^3}{M} \frac{1}{L} = M^0L^0T^0$$

This term may be recognised as inverse of Reynolds number. So π_2 can be modified as $\pi_2 = \rho uD/\mu$ also $\pi_2 = (uD/v)$. The significance of this π term is that it is the ratio of inertia force to viscous force. In case *D, u* and μ had been choosen as the repeating, variables, $\pi_1 = \Delta PD^2/u\ \mu$ and $\pi_2 = \rho Du/\mu$. The parameter π_1/π_2 will give the dimensionless term. $\Delta P\ D/\rho u^2$. In this case π_1 represents the ratio pressure force/viscous force. This flow phenomenon is influenced by the three forces namely pressure force, viscous force and inertia force.

Rayleigh method: (Also called method of Indices). The following functional relationship is formed first. There can be two p terms as there are five variables and three dimensions.

$D P^a D^b r^c m^d u^e = (p_1\ p_2)$, Substituting dimensions,

$$\frac{M^a}{L^{2a}\ T^{2a}} L^b \frac{M^c}{L^{3c}} \frac{M^d}{L^dT^d} \frac{L^e}{T^e} = L^0\ M^0\ T^0$$

Considering indices of *M, L* and *T*, three equations are obtained as below

$$a + c + d = 0, -2a + b - 3c - d + e = 0, -2a + d - e = 0$$

There are five unknowns and three equations. Hence some assumptions are necessary based on the nature of the phenomenon. As D*P*, the dependent variable can be considered to appear only once. We can assume $a = 1$. Similarly, studying the forces, m appears only in the viscous force. So we can assume $d = 1$. Solving $a = 1$, $d = 1$, $b = 0$, $c = -2$, $e = -3$, $(p_1\ p_2) = DPm/r^2\ u^3$. Multiply and divide by *D*, then $p_1 = DPD/ru^2$ and $p_2 = m/ruD$. Same as was obtained by p theorem method. This method requires more expertise and understanding of the basics of the phenomenon.

Example 8.3. *The pressure drop ΔP in flow of incompressible fluid through rough pipes is found to depend on the length l, average velocity u, fluid density,* ρ, dynamic viscosity μ, *diameter D and average roughness height e.* ***Determine the dimensionless groups*** *to correlate the flow parameters.*

The variables with units and dimensions are listed below.

S.No.	Variable	Unit	Dimension
1	ΔP	N/m^2	M/LT^2
2	l	L	L
3	u	m/s	L/T
4	ρ	kg/m^3	M/L^3
5	μ	kg/ms	M/LT
6	D	L	L
7	e	L	L

There are seven parameters and three dimensions. So four π terms can be identified. Selecting u, D and ρ as repeating variables, (as these sets are separate equations, no problem will arise in using indices a, b and c in all cases).

Let $\pi_1 = \Delta P\, u^a D^b \rho^c$, $\pi_2 = L\, u^a D^b\, \rho^c$, $\pi_3 = \mu\, u^a\, D^b\, \rho^c$, $\pi_4 = e\, u^a D^b \rho^c$

Consider π_1, $$M^0L^0T^0 = \frac{M}{LT^2}\frac{L^a}{T^a}L^b\frac{M^c}{L^{3c}}$$

Equating the indices of M, L and T,

$$1 + c = 0,\ c = -1,\ -1 + a + b - 3c = 0,\ -2 - a = 0,\quad a = -2,\ b = 0.$$

Substituting the value of indices we get

$\therefore$ $$\boldsymbol{\pi_1 = \Delta P/\rho u^2}$$

Consider π_2, $$M^0L^0T^0 = L\,\frac{L^a}{T^a}L^b\frac{M^c}{L^{3c}}$$

Equating indices of M, L and T, $c = 0$, $1 + a + b - 3c = 0$, $a = 0$, $\therefore$ $b = -1$, $\therefore$ $\boldsymbol{\pi_2 = L/D}$

Consider π_3 $$M^0L^0T^0 = \frac{M}{LT}\frac{L^a}{T^a}L^b\frac{M^c}{L^{3c}}$$

Comparing the indices of μm, L and T,

gives $1 + c = 0$ or $c = -1$, $-1 + a + b - 3c = 0$, $-1 - a = 0$ or $a = -1$, $\therefore$ $b = -1$

$\therefore$ $$\boldsymbol{\pi_3 = \mu/\rho Du \quad \text{or} \quad \rho uD/\mu}$$

Consider π_4, $$M^0L^0T^0 = L\,\frac{L^a}{T^a}L^b\frac{M^c}{L^{3c}}$$

This gives, $c = 0$, $1 + a + b - 3c = 0$, $-a = 0$, $b = -1$ $\therefore$ $\boldsymbol{\pi_4 = e/D}$

These π terms may be checked for dimensionless nature.

The relationship can be expressed as $$\frac{\Delta P}{\rho u^2} = f\left[\frac{L}{D}, \frac{e}{D}, \frac{\rho uD}{\mu}\right]$$

8.4 IMPORTANT DIMENSIONLESS PARAMETERS

Some of the important dimensionless groups used in fluid mechanics are listed in Table 8.4.1. indicating significance and area of application of each.

Table 8.4.1 Important Dimensionless Parameters

Name	Description	Significance	Applications
Reynolds Number, Re	$\rho uD/\mu$ or uD/v	Inertia force/ Viscous force	All types of fluid dynamics problems
Froude Number Fr	$u/(gl)^{0.5}$ or u^2/gl	Inertia force/ Gravity force	Flow with free surface (open channel and ships)
Euler Number Eu	$P/\rho u^2$	Pressure force/ Inertia force	Flow driven by pressure
Cauchy Number Ca	$\rho u^2/E_v$ (E_v– bulk modulous)	Inertia force/ Compressibility force	compressible flow
Mach Number M	u/c, c–Velocity of sound	Inertia force/ Compressibility force	Compressible flow
Strouhal Number St	$\omega l/u$, ω–Frequency of oscillation	Local inertia Force/ Convective inertia force	Unsteady flow with frequency of oscillation
Weber Number We	$\rho u^2 l/\sigma$, σ = Surface tension	Inertia force/ Surface tension force	Problems influenced by surface tension free surface flow
Lift coefficient C_L	$L/(1/2\ \rho A u^2)$ L = lift force	Lift force/ Dynamic force	Aerodynamics

8.5 CORRELATION OF EXPERIMENTAL DATA

Dimensional analysis can only lead to the identification of relevant dimensionless groups. The exact functional relations between them can be established only by experiments. The degree of difficulty involved in experimentation will depend on the number of π terms.

8.5.1 Problems with One Pi Term

In this case a direct functional relationship will be obtained but a constant c has to be determined by experiments. The relationship will be of the form $\pi_1 = c$. This is illustrated in example 8.4.

Example 8.4. *The drag force acting on a spherical particle of diameter D falling slowly through a viscous fluid at velocity u is found to be influenced by the diameter D, velocity of fall u, and the viscosity* μ. *Using the method of dimensional analysis obtain a relationship between the variables.*

The parameters are listed below using *M, L, T* dimension set.

S.No.	Parameter	Symbol	Unit	Dimension
1	Drag Force	F	N or kgm/s^2	ML/T^2
2	Diameter	D	m	L
3	Velocity	u	m/s	L/T
4	Viscosity	μ	kg/ms	M/LT

There are four parameters and three dimensions. Hence only one π term will result.

$\pi_1 = F\, D^a\, u^b\, \mu^c$, Substituting dimensions,

$$M^0L^0T^0 = \frac{ML}{T^2}\, L^a\, \frac{L^b}{T^b}\, \frac{M^c}{L^cT^c}\,, \qquad \text{Equating indices of } M, L \text{ and } T$$

$$0 = 1 + c,\ c = -1,\ 1 + a + b - c = 0,\ 2 + b + c = 0,\ b = -1,\ c = -1$$

$$\boldsymbol{\pi_1 = F/uD\mu} \qquad \therefore \quad \boldsymbol{F/uD\mu = \text{constant} = c}$$

or $F = cuD\mu$ or drag force varies directly with velocity, diameter and viscosity. A single test will provide the value of the constant. However, to obtain a reliable value for c, the experiments may have to be repeated changing the values of the parameters.

In this case an approximate solution was obtained theoretically for c as 3π. Hence drag force F in free fall is given by $F = 3\pi\mu uD$. This can be established by experiments.

This relation is known as Stokes law valid for small values of Reynolds Number (Re << 1). This can be used to study the settling of dust in still air. Inclusion of additional variable, namely density will lead to another π term.

8.5.2 Problems with Two Pi Terms

In example 8.2 two π terms were identified. If the dimensional analysis is valid then a single universal relationship can be obtained. Experiments should be conducted by varying one of the group say π_1 and from the measurement the values of the other group π_2 is calculated. A suitable graph (or a computer program) can lead to the functional relationship between the π terms. Linear semilog or log/log plots may have to be used to obtain such a relationship. The valid range should be between the two extreme values used in the experiment. Extrapolation may lead to erroneous conclusions. This is illusration by example 8.5.

Example 8.5. *In order to determine the pressure drop in pipe flow per m length an experiment was conducted using flow of water at 20°C through a 20 mm smooth pipe of length 5 m. The variation of pressure drop observed with variation of velocity is tabulated below.The density of water = 1000 kg/m³. Viscosity = 1.006 × 10⁻³ kg/ms.*

Velocity, m/s	0.3	0.6	0.9	1.5	2.0	3	5
Pressure drop, N	404	1361	2766	6763	11189	22748	55614

Determine the functional relationship between the dimensionless parameters ($D\,\Delta P/\rho u^2$) and ($\rho uD/\mu$).

Using the data the two π parameters together with log values are calculated and tabulated below.

u	0.3	0.6	0.9	1.5	2.0	3	5
$D\Delta P/\rho u^2$	0.01798	0.01512	0.01366	0.01202	0.01119	0.01011	0.00890
$\rho uD/\mu$	5964	11928	17894	29821	39761	59642	99400
logRe	3.78	4.08	4.25	4.48	4.6	4.78	4.997
$\log(D\Delta P/\rho u)$	– 1.745	– 1.821	– 1.865	– 1.92	– 1.951	– 1.995	– 2.051

A plot of the data is shown in Fig. Ex. 8.5 (*a*). The correlation appears to be good. Scatter may indicate either experimental error or omission of an influencing parameter. As the direct plot is a curve., fitting an equation can not be done from the graph. A log log plot results in a straight line, as shown in the Fig. 8.5 (*b*). To fit an equation the following procedure is used.

The slope is obtained by taking the last values:

$$= \{-2.051 - (-1.745)\}/(4.997 - 3.78) = -0.2508$$

When extrapolating we can write, the slope using the same $-2.051 - (x)/(5 - 0) = -0.2508$

This gives $x = -0.797$.

This corresponds to the value of 0.16. Hence we can write,

$$\frac{D\Delta P}{\rho u^2} = 0.16 \left(\frac{PuD}{\mu}\right)^{-0.2508} = 0.16 \times \mathrm{Re}^{-0.2508}$$

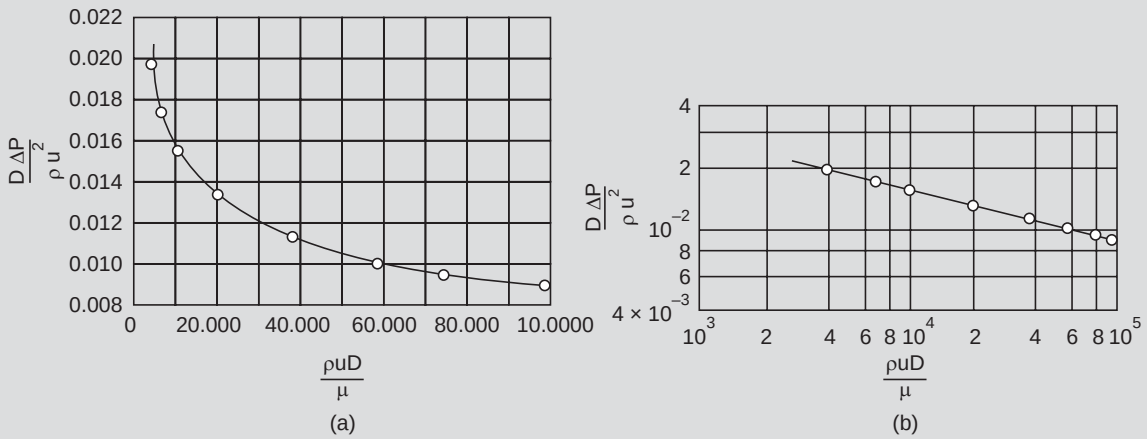


Figure Ex. 8.5

8.5.3 Problems with Three Dimensionless Parameters

In this case experiments should be conducted for different constant values of π_3, varying π_1 and calculating the corresponding values of π_2. Such a set of experiments will result in curves of the form shown in Fig. 8.5.3.

These curves can also be converted to show the variation of π_1 with π_3 at constant values of π_2 by taking sections at various values of π_2. By suitable mathematical techniques correlation of the form below can be obtained.

$$\pi_2 = c\pi_1^{n_1}\ \pi_2^{n_2}$$

When there are more than three π terms, two of these should be combined and the numbers reduced to three. The procedure as described above can then be used to obtain the functional relationship.

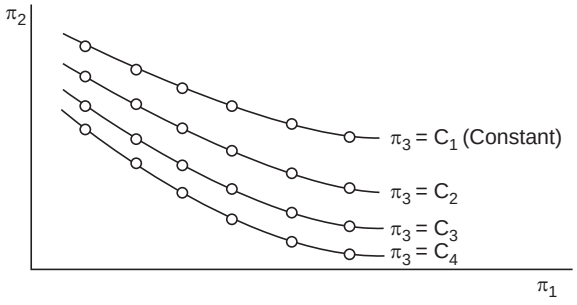


Figure 8.5.1

SOLVED PROBLEMS

Problem 8.1. *The pressure drop ΔP in flow through pipes per unit length is found to depend on the average velocity μ, diameter D, density of the fluid ρ, and viscosity μ. Using FLT set of dimensions evaluate the dimensionless parameters correlating this phenomenon.*

The dimensions of the influencing parameters are tabulated below choosing FLT set.

S.No.	Variables	Unit	Dimensions
1	Pressure drop per unit length, $\Delta P/l$	$(N/m^2)/m$	F/L^3
2	Diameter, D	m	L
3	Velocity, u	m/s	L/T
4	Density, ρ	kg/m^3	FT^2/L^4
5	Viscosity, μ	Ns/m^2	FT/L^2

As there are five variables and three dimensions, two π terms can be obtained.

Using *D, u* and ρ as repeating parameters,

Let $\pi_1 = \Delta P d^a u^b \rho^c$ or $F^0L^0T^0 = \frac{F}{L^3} L^a \frac{L^b}{T^b} \frac{F^c\, T^{2c}}{L^{4c}}$

Comparing the indices of *M, L* and *T* solving for *a, b* and *c*,

$$1 + c = 0, -3 + a + b - 4c = 0, -b + 2c = 0$$

$\therefore$ $c = -1, b = -2, a = 1$

Substituting the value of indices

$\therefore$ $\boldsymbol{\pi_1 = D\Delta P/\rho u^2}$

Let, $\pi_2 = \mu D^a\, u^b\, \rho^c$, or $F^0\, L^0\, T^0 = \frac{F}{L^2} L^a \frac{L^b}{T^b} \frac{F^c\, T^{2c}}{L^{4c}}$

Comparing the value of indices for *M, L* and *T*

$\therefore$ $1 + c = 0, -2 + a + b - 4c = 0, 1 - b + 2c = 0$

Solving, $a = -1, b = -1, c = -1$ substituting the values of *a*, *b*, *c*, *d*

$\therefore$ $\boldsymbol{\pi_2 = \mu/\rho uD}$ or $\boldsymbol{\rho uD/\mu}$

$\therefore$ $$\frac{D\Delta P}{\rho u^2} = f\left[\frac{\rho uD}{\mu}\right]$$

The result is the same as in example 8.2. The dimension set choosen should not affect the final correlation.

Problem 8.2. *The drag force on a smooth sphere is found to be affected by the velocity of flow, u, the dimaeter D of the sphere and the fluid properties density* ρ *and viscosity* μ. *Using dimensional analysis obtain the dimensionless groups to correlate the parameters.*

The dimensions of the influencing variables are listed below, using M, L, T set.

S.No.	Variables	Unit	Dimensions
1	Drag force, *F*	*N*, (kgm/s^2)	*ML*/*T*^{2}
2	Diameter, *D*	m	*L*
3	Velocity, *u*	m/s	*L*/*T*
4	Density, ρ	kg/m^3	M/L^3
5	Viscosity, μ	kg/ms	*M*/*LT*

There are five variables and three dimensions. So two π terms can be obtained. Choosing *D, u* and ρ as repeating variables,

Let $\pi_1 = F\, D^a\, u^b\, \rho^c$, or $M^0L^0T^0 = \frac{ML}{T^2} L^a \frac{L^b}{T^b} \frac{M^c}{L^{3c}}$

Comparing the values of indices for M, L and T

$$1 + c = 1, \quad \therefore \quad c = -1, 1 + a + b - 3c = 0, -2 - b = 0$$

$$\therefore \qquad b = -2, a = -2$$

Substituting the values of a, b, c

$$\therefore \qquad \boldsymbol{\pi_1 = F/\rho u^2 D^2}$$

Let $$\pi_2 = \mu D^a u^b \rho^c \quad \text{or} \quad M^0L^0T^0 = \frac{M}{LT} L^a \frac{L^b}{T^b} \frac{M^c}{L^{3c}}$$

Comparing the values of indices of M, L and T

$$\therefore \qquad 1 + c = 0, -1 + a + b - 3c = 0, -1 - b = 0 \quad \therefore \quad c = -1, b = -1, a = -1$$

Susbtituting the values of a, b, c.

$$\therefore \qquad \boldsymbol{\pi_2 = \mu/\rho uD \quad or \quad \rho uD/\mu}$$

$$\therefore \qquad \frac{F}{\rho u^2 D^2} = f\left[\frac{\rho uD}{\mu}\right]; \quad \text{Check for dimensions of } \pi_1 \text{ and } \pi_2.$$

$$\pi_1 = \frac{ML}{T^2} \frac{L^3}{M} \frac{T^2}{L^2} \frac{1}{L^2} = M^0L^0T^0 \quad \text{or} \quad \pi_2 = \frac{M}{L^3} \frac{L}{T} L \frac{LT}{M} = M^0L^0T^0$$

Note: the significance of the π term. $F/\rho u^2D^2 \rightarrow F/\rho u\, Du \rightarrow F/mu \rightarrow$ Drag force/inertia force.

Problem 8.3. *The thrust force, F generated by a propeller is found to depend on the folllowing parameters: diameter D, forward velocity u, density* ρ, *viscosity* μ *and rotational speed N. Determine the dimensionless parameters to correlate the phenomenon.*

The influencing parameters with dimensions are listed below using *MLT* set.

S.No.	Parameters	Unit	Dimensions
1	Thrust force, F	N	ML/T^2
2	Diameter, D	m	L
3	Forward velocity, u	m/s	L/T
4	Density, ρ	kg/m^3	M/L^3
5	Viscosity, μ	kg/ms	M/LT
6	Rotational speed, N	1/s	$1/T$

There are 6 variables and three dimensions. So three π terms can be obtained. Choosing D, u and ρ as repeating variables,

Let $$\pi_1 = F\, u^a D^b \rho^c, \quad \text{or} \quad M^0L^0T^0 = \frac{ML}{T^2} \frac{L^a}{T^a} L^b \frac{M^c}{L^{3c}}$$

Comparing indices of M, L and T

$\therefore \quad 1 + c = 0, 1 + a + b - 3c = 0, -2 - a = 0$

$\therefore \quad a = -2, b = -2, c = -1$

Substituting the values of a, b, and c

$\therefore \quad \boldsymbol{\pi_1 = F/u^2D^2\rho}$, (Thrust force/Inertia force)

Let $\quad \pi_2 = \mu\, u^a D^b \rho^c \quad \text{or} \quad M^0L^0T^0 = \dfrac{M}{LT}\dfrac{L^a}{T^a}L^b\dfrac{M^c}{L^{3c}}$

Comparing the indices M, L and T

$\therefore \quad 1 + c = 0, -1 + a + b - 3c = 0, -1 - a = 0$

$\therefore \quad a = -1, b = -1, c = -1$

Substituting the values of a, b and c

$\therefore \quad \boldsymbol{\pi_2 = \mu/\rho uD \quad or \quad \rho uD/\mu}$ (Inertia force/Viscous force)

Let $\quad \pi_3 = N\, u^a\, D^b\, \rho^c, \quad \text{or} \quad M^0L^0T^0 = \dfrac{1}{T}\dfrac{L^a}{T^a}L^b\dfrac{M^c}{L^{3c}}$

Comparing the indices of M, L and T

$c = 0, a + b - 3c = 0, -1 - a = 0, \quad \therefore \quad a = -1, b = 1$

Susbtituting the values of a, b and c

$\therefore \quad \boldsymbol{\pi_3 = ND/u}$ (Rotational speed/Forward speed)

$\therefore \quad F/u^2D^2\,\rho = f\left[\dfrac{uD\rho}{\mu}, \dfrac{ND}{u}\right]$

Problem 8.4. *At higher speeds where compressibility effects are to be taken into account the performance of a propeller in terms of force exerted is influenced by the diameter, forward speed, rotational speed, density, viscosity and bulk modulus of the fluid. Evaluate the dimensionless parameters for the system.*

The influencing parameters and dimensions are tabulated below, using M, L, T set.

S.No.	Parameters	Unit	Dimensions
1	Force, F	N/m^2	M/LT^2
2	Diameter, D	m	L
3	Forward velocity, u	m/s	L/T
4	Rotational speed, N	l/s	$1/T$
5	Density, ρ	kg/m^3	M/L^3
6	Viscosity, μ	kg/ms	M/LT
7	Bulk Modulus, E	$(m^3/m^3)N/m^2$	M/LT^2

There are seven variables and three dimensions, So four π terms are possible. Selecting D, u and ρ as repeating parameters,

Let $\pi_1 = F\,\rho^a u^b d^c$, or $M^0L^0T^0 = \dfrac{M}{LT^2}\dfrac{M^a}{L^{3a}}\dfrac{L^b}{T^b}\,L^c$

The general procedure is to compare the indices of *M, L* and T on both sides and from equations.

$$1 + a = 0, -1 - 3a + b + c = 0, -2 - b = 0$$

$\therefore$ $c = 0, b = -2, a = -1$

$\therefore$ $\boldsymbol{\pi_1 = F/\rho u^2} \rightarrow$ (force exerted/inertia force)/m^2

Let $\pi_2 = N\rho^a u^b D^c$, or $M^0L^0T^0 = \dfrac{1}{T}\dfrac{M^a}{L^{3a}}\dfrac{L^b}{T^b}\,L^c$

$\therefore$ $a = 0, -3a + b + c = 0, -1 - b = 0$

$\therefore$ $a = 0, b = -1, c = 1.$

$\boldsymbol{\pi_2 = ND/u}$ (or rotational speed/forward speed)

Let $\pi_3 = \mu\rho^a\, u^b D^c$, or $M^0L^0T^0 = \dfrac{M}{LT}\dfrac{M^a}{L^{3a}}\dfrac{L^b}{T^b}\,L^c$

$\therefore$ $1 + a = 0, -1 - 3a + b + c = 0, -1 - b = 0$

$\therefore$ $a = -1, b = -2, c = -1,$

$\therefore$ $\boldsymbol{\pi_3 = \mu/\rho uD}$ **or** $\boldsymbol{\rho uD/\mu}$ (Reynolds number)

Let $\pi_4 = E\rho^a u^b D^c$, or $M^0L^0T^0 = \dfrac{M}{LT^2}\dfrac{M^a}{L^{3a}}\dfrac{L^b}{T^b}\,L^c$

$\therefore$ $1 + a = 0, -1 - 3a + b + c = 0, -2 - b = 0$

$\therefore$ $a = -1, b = -2, c = 0$

$\boldsymbol{\pi_4 = E/\rho u^2}$ (Compressibility force/inertia force)

$\therefore$ $$\frac{F}{\rho u^2} = f\left[\frac{ND}{u}, \frac{\rho\, uD}{\mu}, \frac{E}{\rho u^2}\right]$$

Problem 8.5. *Using dimensional analysis, obtain a correlation for the frictional torque due to rotation of a disc in a viscous fluid. The parameters influencing the torque can be identified as the diameter, rotational speed, viscosity and density of the fluid.*

The influencing parameters with dimensions are listed below, using *M, L, T* set.

S.No.	Parameters	Unit	Dimensions
1	Torque, τ	Nm	ML^2/T^2
2	Diameter, *D*	m	*L*
3	Rotational speed, *N*	*l*/s	$1/T$
4	Density, ρ	kg/m^3	M/L^3
5	Viscosity, μ	kg/ms	M/LT

There are five variables and three dimensions. So two π parameters can be identified.

Considering D, N and ρ as repeating variables.

Let $\pi_1 = \tau D^a N^b \rho^c$ or $M^0L^0T^0 = \frac{ML^2}{T^2} L^a \frac{1}{T^b} \frac{M^c}{L^{3c}}$

$\therefore$ $1 + c = 0, 2 + a - 3c = 0, \quad -2 - b = 0 \quad \therefore \quad c = -1, b = -2, a = -5,$

$\therefore$ $\boldsymbol{\pi_1 = \tau/\rho N^2 D^5}$

Let $\pi_2 = \mu D^a N^b \rho^c$ or $M^0L^0T^0 = \frac{M}{LT} L^a \frac{1}{T^b} \frac{M^c}{L^{3c}}$

$\therefore$ $1 + c = 0, -1 + a - 3c = 0, -1 - b = 0 \quad \therefore \quad c = -1, b = -1, a = -2$

$\therefore$ $\boldsymbol{\pi_2 = \mu/\rho D^2 N}$, (Another form of Reynolds number, as DN $\rightarrow u$)

$\therefore$ $\frac{\tau}{\rho N^2 D^5} = f\left[\frac{\mu}{\rho D^2 N}\right]$ Check for the dimensions of π_1 and π_2

Note: Rotational speed can also be expressed as angular velocity, ω. In that case N will be replaced by ω as the dimension of both these variables is $1/T$.

Problem 8.6. *A rectangular plate of height, a and width, b is held perpendicular to the flow of a fluid. The drag force on the plate is influenced by the dimensions a and b, the velocity u, and the fluid properties, density* ρ *and viscosity* μ. *Obtain a correlation for the drag force in terms of dimensionless parameters.*

The parameters with dimensions are listed adopting M, L, T set of dimensions.

S.No.	Parameters	Unit	Dimensions
1	Drag force, F	N	ML/T^2
2	Width, b	m	L
3	Height, a	m	L
4	Velocity, u	m/s	L/T
5	density, ρ	kg/m^3	M/L^3
6	Viscosity, μ	kg/ms	M/LT

There are 6 parameters and three dimensions. Hence three π terms can be obtained. Selecting b, u and ρ as repeating variables.

Let $\pi_1 = F b^a u^b \rho^c$ or $M^0L^0T^0 = \frac{ML}{T^2} L^a \frac{L^b}{T^b} \frac{M^c}{L^{3c}}$

$\therefore$ $1 + c = 0, 1 + a + b - 3c = 0, -2 - b = 0$

$\therefore$ $c = -1, b = -2, a = -2$

$\therefore$ $\boldsymbol{\pi_1 = F/\rho u^2 b^2}$

Let $\pi_2 = a\, b^a u^b \rho^c$ or $M^0L^0T^0 = LL^a \dfrac{L^b}{T^b} \dfrac{M^c}{L^{3c}}$

$c = 0,\ 1 + a + b - 3c = 0,\ -b = 0,$

$\therefore \quad a = -1 \quad \therefore \quad \boldsymbol{\pi_2 = a/b}$

Let $\boldsymbol{\pi_3 = \mu b^a u^b \rho^c}$ or $M^0L^0T^0 = \dfrac{M}{LT} L^a \dfrac{L^b}{T^b} \dfrac{M^c}{L^{3c}}$

$\therefore \quad 1 + c = 0,\ -1 + a + b - 3c = 0,\ -1 - b = 0.$

$\therefore \quad b = -1,\ c = -1,\ b = -1$

$\therefore \quad \boldsymbol{\pi_3 = \mu/\rho ub \quad or \quad \pi_3 = \rho ub/\mu}$

$\therefore \quad \dfrac{F}{\rho u^2 b^2} = f\left[\dfrac{a}{b}, \dfrac{\rho\, ub}{\mu}\right]$

π_3 is Reynolds number based on length b. π_1 is (drag force/unit area)/inertia force.

Problem 8.7. *In film lubricated journal bearings, the frictional torque is found to depend on the speed of rotation, viscosity of the oil, the load on the projected area and the diameter. Evaluate dimensionless parameters for application to such bearings in general.*

The variables with dimensions are listed below, adopting *MLT* set.

S.No.	Variable	Unit	Dimensions
1	Frictional Torque, τ	Nm	ML^2/T^2
2	Speed, N	1/s	$1/L$
3	Load per unit area, P	N/m^2	M/LT^2
4	Diameter, D	m	L
5	Viscosity, μ	kg/ms	M/LT

There are five parameters and three dimensions. Hence two π parameters can be found. Considering N, D and μ as repeating variables,

Let $\pi_1 = \tau\, N^a D^b \mu^c$ or $M^0L^0T^0 = \dfrac{ML^2}{T^2} \dfrac{1}{T^a} L^b \dfrac{M^c}{L^cT^c}$

$\therefore \quad 1 + c = 0,\ 2 + b - c = 0,\ -2 - a - c = 0 \quad \therefore \quad c = -1,\ a = -1,\ b = -3$

$\therefore \quad \boldsymbol{\pi_1 = \tau/N\mu D^3}$ Also $\pi = \tau/\mu u D$ (τ–Torque)

Let $\pi_2 = PN^a D^b \mu^c$ or $M^0L^0T^0 = \dfrac{M}{LT^2} \dfrac{1}{T^a} L^b \dfrac{M^c}{L^cT^c}$

$\therefore \quad 1 + c = 0,\ -1 + b - c = 0,\ -2,\ -a - c = 0$

$\therefore \quad c = -1,\ a = -1,\ b = 0$

$\therefore \quad \boldsymbol{\pi_2 = P/N\mu,} \quad \therefore \quad \dfrac{\tau}{N\mu D^3} = f\left[\dfrac{P}{N\mu}\right]$

Note : $P/N\mu$ is also Reynolds number, try to verify.

Problem 8.8. *Obtain a relation using dimensional analysis, for the resistance to uniform motion of a partially submerged body (like a ship) in a viscous compressible fluid.*

The resistance can be considered to be influenced by skin friction forces, buoyant forces and compressibility of the fluid.

The variables identified as affecting the situation are listed below using *MLT* set.

S.No.	Variable	Unit	Dimensions
1	Resistance to motion, R	N/m^2	M/LT^2
2	Forward velocity, u	m/s	L/T
3	Length of the body, l	m	L
4	Density of the fluid, ρ	kg/ms	M/L^3
5	Viscosity, of the fluid, μ	kg/ms	M/LT
6	Gravity, g	m/s^2	L/T^2
7	Bulk modulus, E_v	N/m^2	M/LT^2

There are seven parameters and three dimensions. So four π, terms are possible. Considering velocity, density and length as repeating variables.

Let $\pi_1 = Ru^a\rho^b l^c$ or $M^0L^0T^0 = \frac{M}{LT^2}\frac{L^a}{T^a}\frac{M^b}{L^{3b}}L^c$

$\therefore$ $1 + b = 0, -1 + a - 3b + c = 0, \quad -2 - a = 0$

$\therefore$ $a = -2, b = -1$ and $c = 0$

$\therefore$ $\boldsymbol{\pi_1 = R/\rho u^2}$, Euler number.

Let $\pi_2 = \mu\, u^a\, \rho^b\, l^c$ or $M^0L^0T^0 = \frac{M}{LT}\frac{L^a}{T^a}\frac{M^b}{L^{3b}}L^c$

$\therefore$ $1 + b = 0, -1 + a - 3b + c = 0, -2 - a = 0$

$\therefore$ $a = -1, b = -1$ and $c = -1$

$\therefore$ $\boldsymbol{\pi_2 = \mu/u\rho l}$.

Let $\pi_3 = gu^a\rho^b l^c$ or $M^0L^0T^0 = \frac{L}{T^2}\frac{L^a}{T^a}\frac{M^b}{L^{3b}}L^c$

$\therefore$ $b = 0, 1 + a - 3b + c = 0, -2 - a = 0$

$\therefore$ $a = -2, b = 0$ and $c = 1$

$\therefore$ $\boldsymbol{\pi_3 = gl/u^2}$ $\rightarrow$ can also be expressed as $u/(gL)^{0.5}$ (Froude number.)

Let $\pi_4 = E_v u^a\rho^b l^c$ or $M^0L^0T^0 = \frac{M}{LT^2}\frac{L^a}{T^a}\frac{M^b}{L^{3b}}L^c$

$\therefore$ $1 + b = 0, -1 + a - 3b + c = 0, -2 - a = 0$

$\therefore$ $a = -2, b = -1$ and $c = 0$

$\therefore$ $\boldsymbol{\pi_4 = E_v/\rho u^2}$

$$\frac{R}{\rho u^2} = f\left[\frac{\mu}{ul\rho}, \frac{gl}{u^2}, \frac{E_v}{\rho u^2}\right] \quad \text{or}$$

Euler number = f (Reynolds number, Froude number and Mach number)

In the case of incompressible flow, this will reduce to

$$\frac{R}{\rho u^2} = f\left[\frac{ul\rho}{\mu}, \frac{u}{(gl)^{0.5}}\right] = f(\text{Re, Fr})$$

Problem 8.9. *The velocity of propagation of pressure wave, c through a fluid is assumed to depend on the fluid density ρ and bulk modulus of the fluid E_v. Using dimensional analysis obtain an expression for c in terms of ρ and E_v.*

This is a case were there will be a direct relationship between the variables or one π term.

Note: The definition of the bulk modulus is $dp/(dv/v)$, the dimension being that of pressure, M/LT^2, Writing $c = f(\rho, E_v)$

Let, $$\pi_1 = c\ \rho^a E_v^{\ b} \quad \text{or} \quad M^0L^0T^0 = \frac{L}{T}\frac{M^a}{L^a}\frac{M^b}{L^b\ T^{2b}}$$

$\therefore$ $$a + b = 0,\ 1 - 3a - b = 0,\ -1 - 2b = 0, \quad \therefore \quad b = -0.5,\ a = 0.5$$

$\therefore$ $$\boldsymbol{\pi_1 = c(\rho/E_v)^{0.5}}, \quad \text{or} \quad c = \text{const} \times (E_v/\rho)^{0.5}$$

Problem 8.10. *Obtain a correlation for the coefficient of discharge through a small orifice, using the method of dimensional analysis.*

The following list of parameters can be identified as affecting the coefficient of discharge

S.No.	Parameters	Unit	Dimensions
1	Diameter, D	m	L
2	Head, H	m	L
3	Gravity, g	m/s^2	L/T^2
4	Density of the fluid, ρ	kg/m^3	M/L^3
5	Roughness height, k	m	L
6	Surface tension, σ	N/m	M/T^2
7	Viscosity, μ	kg/ms^2	M/LT

There are seven variables and three dimensions. So four π terms can be identified. Considering ρ, g and H as repeating variables

Let $$\pi_1 = D\ \rho^a g^b H^c \quad \text{or} \quad M^0L^0T^0 = L\ \frac{M^a}{L^{3a}}\frac{L^b}{T^{2b}}\ L^c$$

$\therefore$ $$a = 0,\ 1 - 3a + b + c = 0,\ -2b = 0,$$

$\therefore$ $$c = -1 \quad \therefore \quad \boldsymbol{\pi_1 = D/H \quad \text{or} \quad H/D}$$

Let $\pi_2 = k\rho^a g^b H^c$ or $M^0L^0T^0 = L\,\frac{M^a}{L^{3a}}\,\frac{L^b}{T^{2b}}\,L^c$

$\therefore$ $a = 0,\ 1 - 3a + b + c = 0,\ -2b = 0,$

$\therefore$ $a = 0,\ b = 0,\ c = -1 \quad \therefore \quad \boldsymbol{\pi_2 = k/H}$

Let $\pi_3 = \sigma\rho^a g^b H^c$ or $M^0L^0T^0 = \frac{M}{T^2}\,\frac{M^a}{L^{3a}}\,\frac{L^b}{T^{2b}}\,L^c$

$\therefore$ $a + 1 = 0,\ -3a + b + c = 0,\ -2 - 2b = 0,$

$\therefore$ $a = -1,\ b = -1,\ c = -2$

$\therefore$ $\boldsymbol{\pi_3 = \sigma/\rho g H^2}$

Let $\pi_4 = \mu\rho^a g^b H^c$ or $M^0L^0T^0 = \frac{M}{LT}\,\frac{M^a}{L^{3a}}\,\frac{L^b}{T^{2b}}\,L_c$

$\therefore$ $a + 1 = 0,\ -1 - 3a + b + c = 0,\ -1 - 2b = 0,$

$\therefore$ $a = -1,\ b = -1/2,\ c = -1.5.$

$\therefore$ $\boldsymbol{\pi_4 = \mu/(\rho H\sqrt{gH})}$. As C_d is dimensionless

$$C_d = f\left[\frac{D}{H}, \frac{k}{H}, (\sigma/\rho\, gH^2), \frac{\mu}{(\rho H\sqrt{gH})}\right]$$

Check the dimensions of these π terms.

Problem 8.11. *The volume flow rate of a gas through a sharp edged orifice is found to be influenced by the pressure drop, orifice diameter and density and kinematic viscosity of the gas. Using the method of dimensional analysis obtain an expression for the flow rate.*

The variables and dimensions are listed below, adopting MLT system

S.No.	Variable	Unit	Dimensions
1	Volume flow rate, Q	m^3/s	L^3/T
2	Pressure drop, ΔP	N/m^2	M/LT^2
3	Diameter, D	m	L
4	Density, ρ	kg/m^3	M/L^3
5	Kinematic viscosity, v	m^2/s	L^2/T

There are five parameters and three dimensions. So two π terms can be obtained. Choosing ΔP, D and ρ as repeating variables,

Let $\pi_1 = Q\,\Delta P^a D^b \rho^c$ or $M^0L^0T^0 = \frac{L^3}{T}\,\frac{M^a}{L^aT^{2a}}\,L^b\,\frac{M^c}{L^{3c}}$

$\therefore$ $a + c = 0,\ 3 - a + b - 3c = 0,\ -1 - 2a = 0, \quad \therefore \quad a = -(1/2),\ c = 1/2,\ b = -2$

$\therefore$ $\boldsymbol{\pi_1 = (Q/D^2)\ (\rho/\Delta P)^{1/2}}$

Let $\pi_2 = v\Delta P^a D^b \rho^c$ or $M^0L^0T^0 = \dfrac{L^2}{T}\dfrac{M^a}{L^aT^{2a}}L^b\dfrac{M^c}{L^{3c}}$

$\therefore$ $a + c = 0,\ 2 - a + b - 3c = 0,\ -1 - 2a = 0,$

$\therefore$ $a = -(1/2),\ c = (1/2),\ b = -1$

$\therefore$ $\boldsymbol{\pi_2 = (v/D)\ (\rho/\Delta P)^{1/2}}$ or $\dfrac{Q}{D^2}\left(\dfrac{\rho}{\Delta P}\right)^{1/2} = f\left[\dfrac{v}{D}\left(\dfrac{\rho}{\Delta P}\right)^{1/2}\right]$

Note : π_2 can be also identified as Reynolds number. Try to verify.

Problem 8.12. *In flow through a sudden contraction in a circular duct the head loss h is found to depend on the inlet velocity u, diameters D and d and the fluid properties density* ρ *and viscosity* μ *and gravitational acceleration, g. Determine dimensionless parameters to correlate experimental results.*

The influencing variables with dimensions are tabulated below with *MLT* set.

S.No.	Variable	Unit	Dimensions
1	Loss of head, h	m	L
2	Inlet diameter, D	m	L
3	Outlet diameter, d	m	L
4	Velocity, u	m/s	L/T
5	Density, ρ	kg/m^3	M/L^3
6	Viscosity, μ	kg/ms	M/LT
7	Gravitational acceleration, g	m/s^2	L/T^2

There are seven variables and three dimensions. Hence four π parameters can be found. Considering D, ρ and u as repeating variables,

Let $\pi_1 = h\ D^a\ \rho^b\ u^c$ or $M^0L^0T^0 = LL^a\dfrac{M^b}{L^{3b}}\dfrac{L^c}{T^c}$

$\therefore$ $b = 0,\ 1 + a - 3b + c = 0,\ c = 0$ $\therefore$ $a = -1$ $\therefore$ $\boldsymbol{\pi_1 = h/D}$

Let $\pi_2 = d\ D^a\ \rho^b\ u^c$ or $M^0L^0T^0 = LL^a\dfrac{M^b}{L^{3b}}\dfrac{L^c}{T^c}$

$\therefore$ $b = 0,\ c = 0,\ 1 + a - 3b + c = 0,\ a = -1$ $\therefore$ $\pi_2 = \mathbf{d/D}$

Let $\pi_3 = \mu\ D^a\ \rho^b\ u^c$ or $M^0L^0T^0 = \dfrac{M}{LT}L^a\dfrac{M^b}{L^{3b}}\dfrac{L^c}{T^c}$

$\therefore$ $b + 1 = 0,\ -1 + a - 3b + c = 0,\ -1 - c = 0,$

$\therefore$ $b = -1,\ c = -1,\ a = -1$

$\therefore$ $\boldsymbol{\pi_3 = \mu/D\rho u}$ **or** $\boldsymbol{\rho Du/\mu}$

Let $\pi_4 = g\ D^a\ \rho^b\ u^c$ or $M^0L^0T^0 = \dfrac{L}{T^2}L^a\dfrac{M^b}{L^{3b}}\dfrac{L^c}{T^c}$

$\therefore \quad b = 0, 1 + a - 3b + c = 0, -2 - c = 0$

$\therefore \quad c = -2, a = 1 \quad \therefore \quad \boldsymbol{\pi_4 = gD/u^2}$

$\therefore \quad \dfrac{h}{D} = f\left[\dfrac{d}{D}, \dfrac{\rho Du}{\mu}, \dfrac{gD}{u^2}\right]$

Note : gD/u^2 is the ratio of Potential energy to Kinetic energy.

Problem 8.13. *The volume flow rate, Q over a V-notch depends on fluid properties namely density* ρ, *kinematic viscosity v, and surface tension* σ. *It is also influenced by the angle of the notch, head of fluid over the vertex, and acceleration due to gravity. Determine the dimensionless parameters which can correlate the variables.*

As θ, the notch angle is a dimensionless parameter, the other parameters are listed below with dimensions, adopting *MLT* set.

S.No.	Variable	Unit	Dimension
1	Density, ρ	kg/m^3	M/L^3
2	kinematic vicosity, v	m^2/s	L^2/T
3	Surface tension, σ	N/m	M/T^2
4	Head of fluid, h	m	L
5	Gravitational acceleration, g	m/s^2	L/T^2
6	Flow rate, Q	m^3/s	L^3/T

There are six parameters and three dimensions. So three π terms can be identified. Considering ρ, g and h as repeating variables.

Let $\quad \pi_1 = Q\,\rho^a\, g^b\, h^c \quad \text{or} \quad M^0L^0T^0 = \dfrac{L^3}{T}\,\dfrac{M^a}{L^{3a}}\,\dfrac{L^b}{T^{2b}}\,L^c$

$\therefore \quad a = 0, 3 - 3a + b + c = 0, -1 - 2b = 0$

$\therefore \quad b = -0.5, c = 2.5 \quad \therefore \quad \boldsymbol{\pi_1 = Q/g^{1/2}\,h^{5/2}}$

Let $\quad \pi_2 = v\rho^a\, g^b\, h^c \quad \text{or} \quad M^0L^0T^0 = \dfrac{L^2}{T}\,\dfrac{M^a}{L^{3a}}\,\dfrac{L^b}{T^{2b}}\,L^c$

$\therefore \quad a = 0, 2 - 3a + b + c = 0, -1 - 2b = 0$

$\therefore \quad b = -0.5, c = (-1.5) \quad \therefore \quad \boldsymbol{\pi_2 = v/g^{1/2}\,h^{3/2}}$

Let $\quad \pi_3 = \sigma\,\rho^a\, g^b\, h^c \quad \text{or} \quad M^0L^0T^0 = \dfrac{M}{T^2}\,\dfrac{M^a}{L^{3a}}\,\dfrac{L^b}{T^{2b}}\,L^c$

$\therefore \quad 1 + a = 0, -3a + b + c = 0, -2 - 2b = 0 \quad \therefore \quad a = -1, b = -1, c = -2$

$\therefore \quad \boldsymbol{\pi_3 = \sigma/\rho gh^2} \quad \therefore \quad Q = g^{1/2}\,h^{5/2}\,f\left[\dfrac{v}{g^{1/2}\,h^{3/2}}, \dfrac{\sigma}{\rho\,gh^2}, \theta\right]$

Note : In case surface tension is not considered, π_3 will not exist. π_2 can be identified as Reynolds number.

Problem 8.14. *The capillary rise h is found to be influenced by the tube diameter D, density* ρ, *gravitational acceleration g and surface tension* σ. *Determine the dimensionless parameters for the correlation of experimental results.*

The variables are listed below adopting *MLT* set of dimensions.

S.No.	Variable	Unit	Dimension
1	Diameter, D	m	L
2	Density, ρ	kg/m^3	M/L^3
3	Gravitational acceleration, g	m/s^2	L/T^2
4	Surface tension, σ	N/m	M/T^2
5	Capillary rise, h	m	L

There are five parameters and three dimensions and so two π parameters can be identified. Considering D, ρ and g as repeating variables,

Let $\pi_1 = h\, D^a\, \rho^b\, g^c$ or $M^0L^0T^0 = LL^a\, \dfrac{M^b}{L^{3b}}\, \dfrac{L^c}{T^{2c}}$

$\therefore$ $b = 0,\ 1 + a - 3b + c = 0,\ -2c = 0$

$\therefore$ $a = -1,\ b = 0,\ c = 0 \quad \therefore \quad \boldsymbol{\pi_1 = h/D}$

Let $\pi_2 = \sigma\, D^a \rho^b\, g^c$ or $M^0L^0T^0 = \dfrac{M}{T^2}\, L^a\, \dfrac{M^b}{L^{3b}}\, \dfrac{L^c}{T^{2c}}$

$\therefore$ $1 + b = 0,\ a - 3b + c = 0,\ -2 - 2c = 0 \quad \therefore \quad b = -1,\ c = -1,$ and $a = -2$

$\therefore$ $\boldsymbol{\pi_2 = \sigma/D^2\, g\rho,}$ $g\rho$ can also be considered as specific weight γ

$$\frac{h}{D} = f\left[\frac{\sigma}{D^2\gamma}\right],$$

Note : π_2 can be identified as 1/Weber number.

Problems 8.15. *Show that the power P, developed by a hydraulic turbine can be correlated by the dimensionless parameters* $P/\rho N^3D^5$ *and* N^2D^2/gh, *where* ρ *is the density of water and N is the rotational speed, D is the runner diameter, h is the head and g is acceleration due to gravity.*

The parameters with dimensions are tabulated below using *MLT* set.

S.No.	Variable	Unit	Dimension
1	Power, P	W	ML^2/T^3
2	Density, ρ	kg/m^3	M/L^3
3	Speed, N	1/s	$1/T$
4	Diameter, D	m	L
5	Head, h	m	L
6	Gravitational acceleration, g	m/s^2	L/T^2

There are six parameters and three dimensions. So three π terms can be found. Choosing ρ, D and N as repeating variables,

Let $\pi_1 = P\ \rho^a\ D^b\ N^c$ or $M^0L^0T^0 = \dfrac{ML^2}{T^3}\dfrac{M^a}{L^{3a}}L^b\dfrac{1}{T^c}$

$\therefore$ $1 + a = 0,\ 2 - 3a + b = 0,\ -3 - c = 0 \quad \therefore \quad a = -1,\ c = -3,\ b = -5$

$\therefore$ $\boldsymbol{\pi_1 = P/\rho N^3\ D^5}$ **(Power coefficient)**

Let $\pi_2 = h\ \rho^a\ D^b\ N^c$ or $M^0L^0T^0 = L\ \dfrac{M^a}{L^{3a}}L^b\dfrac{1}{T^c}$

$\therefore$ $a = 0,\ 1 - 3a + b = 0,\ c = 0, \quad \therefore \quad b = -1 \quad \therefore \quad \boldsymbol{\pi_2 = h/D.}$

Let $\pi_3 = g\ \rho^a\ D^b\ N^c$ or $M^0L^0T^0 = \dfrac{L}{T^2}\dfrac{M^a}{L^{3a}}L^b\dfrac{1}{T^c}$

$\therefore$ $a = 0,\ 1 - 3a + b = 0,\ -2 - c = 0$

$\therefore$ $c = -2,\ b = -1 \quad \therefore \quad \boldsymbol{\pi_3 = g/DN^2}$

$\boldsymbol{\pi_2 \times \pi_3 = gh/D^2N^2}$ **(Head coefficient)**

$\therefore$ $$\frac{P}{\rho N^3\ D^5} = f\left[\frac{gh}{D^2N^2}\right]$$

In this expression the first term is called power coefficient and the second one is called head coefficient. These are used in model testing of turbo machines.

Problem 8.16. *The power developed by hydraulic machines is found to depend on the head h, flow rate Q, density* ρ, *speed N, runner diameter D, and acceleration due to gravity, g. Obtain suitable dimensionless parameters to correlate experimental results.*

The parameters with dimensions are listed below, adopting *MLT* set of dimensions.

S.No.	Variable	Unit	Dimension
1	Power, P	W	ML^2/T^3
2	Head, h	m	L
3	Flow rate, Q	m^3/s	L^3/T
4	Density, ρ	kg/m^3	M/L^3
5	Speed, N	1/s	$1/T$
6	Diameter, D	m	L
7	Acceleration due to gravity, g	m/s^2	L/T^2

There are seven variables and three dimensions and hence four π terms can be formed. Taking ρ, D and g as repeating variables.

Let $\pi_1 = P\ \rho^a\ D^b\ g^c$ or $M^0L^0T^0 = \dfrac{ML^2}{T^3}\dfrac{M^a}{L^{3a}}L^b\dfrac{L^c}{T^{2c}}$

$\therefore \quad 1 + a = 0, 2 - 3a + b + c = 0, -3 - 2c = 0$

$\therefore \quad a = -1, c = -3/2 \quad \therefore \quad b = -7/2$

$\therefore \quad \boldsymbol{\pi_1 = P/\rho D^{7/2} g^{3/2}}$

Let $\quad \pi_2 = h\ \rho^a\ D^b\ g^c \quad$ or $\quad M^0L^0T^0 = L\ \dfrac{M^a}{L^{3a}}\ L^b\ \dfrac{L^c}{T^{2c}}$

$\therefore \quad a = 0, 1 - 3a + b + c = 0, -2c = 0$

$\therefore \quad a = 0, b = -1, c = 0 \quad \therefore \quad \boldsymbol{\pi_2 = h/D}$

Let $\quad \pi_3 = Q\ \rho^a\ D^b\ g^c \quad$ or $\quad M^0L^0T^0 = \dfrac{L^3}{T}\ \dfrac{M^a}{L^{3a}}\ L^b\ \dfrac{L^c}{T^{2c}}$

$\therefore \quad a = 0, 3 - 3a + b + c = 0, -1 - 2c = 0$

$\therefore \quad a = 0, c = -1/2, b = -5/2$

$\therefore \quad \boldsymbol{\pi_3 = Q/g^{1/2} D^{5/2}}$

Let $\quad \pi_4 = N\ \rho^a\ D^b\ g^c \quad$ or $\quad M^0L^0T^0 = \dfrac{1}{T}\ \dfrac{M^a}{L^{3a}}\ L^b\ \dfrac{L^c}{T^{2c}}$

$\therefore \quad a = 0, -3a + b + c = 0, -1 - 2c = 0 \quad \therefore \quad a = 0, c = -1/2, b = 1/2$

$\therefore \quad \boldsymbol{\pi_4 = ND^{1/2}/g^{1/2}}$

The coefficients popularly used in model testing are given below. These can be obtained from the above four π terms.

1. Head coefficient $\dfrac{gh}{N^2D^2} = \dfrac{\pi_2}{\pi_4{}^2} = \dfrac{hg}{DN^2\ D} = \dfrac{gh}{N^2D^2}$

2. Flow coefficient $\dfrac{Q}{ND^3} = \dfrac{\pi_3}{\pi_4} = \dfrac{Qg^{1/2}}{g^{1/2}\ D^{5/2}\ ND^{1/2}} = \dfrac{Q}{N\ D^3}$

3. Power coefficient $\dfrac{P}{\rho N^3\ D^5} = \dfrac{\pi_1}{\pi_4{}^3} = \dfrac{Pg^{3/2}}{\rho D^{7/2}\ g^{3/2}\ N^3\ D^{3/2}} = \dfrac{P}{\rho\ N^3D^5}$

4. Specific speed based on Q, for pumps, N_{sp}

$$\frac{N\sqrt{Q}}{(gh)^{3/4}} = \frac{(\text{flow coeff})^{1/2}}{(\text{head coeff})^{3/4}} = \frac{Q^{1/2}}{N^{1/2}D^{3/2}}\ \frac{N^{3/2}D^{3/2}}{(gh)^{3/4}} = \frac{N\sqrt{Q}}{(gh)^{3/4}}$$

(dimensional specific speed $N\sqrt{Q}/h^{3/4}$ is commonly used as mostly water is the fluid used)

5. Specific speed based on power, for Turbines

$$N_{st} = \frac{N\sqrt{P}}{\rho^{1/2}\ (gh)^{5/4}} = \frac{(\text{power coefficient})^{1/2}}{(\text{head coefficient})^{5/4}} = \frac{P^{1/2}(ND)^{5/2}}{\rho^{1/2}\ N^{3/2}\ D^{5/2}\ (gh)^{5/4}} = \frac{N\sqrt{P}}{\rho^{1/2}\ (gh)^{5/4}}$$

$$\left(\text{Dimensional Specific speed } \frac{N\sqrt{P}}{h^{5/4}} \text{ is commonly used as water is used in most cases}\right)$$

These are the popularly used dimensionally numbers in hydraulic turbo machinery.

Problem 8.17. *In forced convection in pipes heat transfer coefficient h is found to depend on thermal conductivity, viscosity, density, specific heat, flow velocity and the diameter. Obtain dimensionless parameters to correlate experimental results.*

The variables with dimensions are listed below using $MLT\,\theta$ set of dimensions, where θ is temperature.

S.No.	Variable	Unit	Dimension
1	Convection coefficient, h	W/m^2K	$M/T^3\,\theta$
2	Diameter D	m	L
3	Thermal conductivity, k	W/mK	$ML/T^3\,\theta$
4	Density, ρ	kg/m^3	M/L^3
5	Viscosity, μ	kg/ms	M/LT
6	Specific heat, c	Nm/kgK	$L^2/T^2\,\theta$
7	Flow velocity, u	m/s	L/T

Three π terms are possible as there are seven variables and four dimensions. Choosing k, μ, ρ and D as repeating variables.

Let $$\pi_1 = h\,k^a\,\mu^b\,\rho^c\,D^d \quad \text{or} \quad M^0L^0T^0\,\theta^0 = \frac{M}{T^3\theta}\,\frac{M^a}{T^{3a}}\,\frac{L^a}{\theta^a}\,\frac{M^b}{L^bT^b}\,\frac{M^c}{L^{3c}}\,L^d$$

$\therefore$ $$1 + a + b + c = 0,\ a - b - 3c + d = 0,\ -3 - 3a - b = 0,\ -1 - a = 0$$

$\therefore$ $$a = -1,\ b = 0,\ c = 0,\ d = 1$$

$\therefore$ $$\boldsymbol{\pi_1 = hD/k} \quad \textbf{(Nusselt number)}$$

Let $$\pi_2 = u\,k^a\,\mu^b\,\rho^c\,D^d \quad \text{or} \quad M^0L^0T^0\,\theta^0 = \frac{L}{T}\,\frac{M^a}{T^{3a}}\,\frac{L^a}{\theta^a}\,\frac{M^b}{L^bT^b}\,\frac{M^c}{L^{3c}}\,L^d$$

$\therefore$ $$a + b + c = 0,\ 1 + a - b - 3c + d = 0,\ -1 - 3a - b = 0,\ -a = 0,$$

$\therefore$ $$a = 0,\ b = -1,\ c = 1,\ d = 1$$

$\therefore$ $$\boldsymbol{\pi_2 = u\rho D/\mu} \quad \textbf{(Reynolds number)}$$

Let $$\pi_3 = ck^a\,\mu^b\,\rho^c\,D^d \quad \text{or} \quad M^0L^0T^0\,\theta^0 = \frac{L^2}{T^2\theta}\,\frac{M^a}{T^{3a}}\,\frac{L^a}{\theta^a}\,\frac{M^b}{L^bT^b}\,\frac{M^c}{L^{3c}}\,L^d$$

$\therefore$ $$a + b + c = 0,\ 2 + a - b - 3c + d = 0,\ -2 - 3a - b = 0,\ -1 - a = 0,$$

$\therefore$ $$a = -1,\ b = 1,\ c = 0,\ d = 0$$

$\therefore$ $$\boldsymbol{\pi_3 = c\mu/k} \quad \textbf{(Prandtl number)},\quad \frac{hD}{k} = f\left[\frac{uD\rho}{\mu}, \frac{c\mu}{k}\right]$$

These are popular dimensionless numbers in convective heat transfer.

Problem 8.18. *The temperature difference θ at a location x at time τ in a slab of thickness L originally at a temperature difference θ_0 with outside is found to depend on the thermal diffusivity α, thermal conductivity k and convection coefficient h. Using dimensional analysis determine the dimensionless parameters to correlate the situation.*

The influencing parameters with dimensions are listed below, choosing $MLT\,\theta$ set.

S.No.	Parameter	Unit	Dimension
1	Slab thickness, L	m	L
2	Location distance, x	m	L
3	Initial temperature difference, θ_0	deg K	θ_0
4	Temperature difference at time τ, θ	deg K	θ
5	Time, τ	s	T
6	Thermal diffusivity, α	m^2/s	L^2/T
7	Thermal conductivity, k	W/mK	$ML/T^3\theta$
8	Convection coefficient, h	W/m^2 K	$M/T^3\theta$

There are eight variables and four dimensions. Hence four π terms can be identified. Choosing θ_0, L, α and k as repeating variables,

Let $\pi_1 = \theta\,\theta_0^{\,a}\,L^b\,\alpha^c\,k^d$ or $M^0L^0T^0\theta^0 = \theta\,\theta^a\,L^b\,\dfrac{L^{2c}}{T^c}\,\dfrac{M^d}{T^{3d}}\,\dfrac{L^d}{\theta^d}$

$\therefore \quad d = 0,\ b + 2c + d = 0,\ -c - 3d = 0,\quad 1 + a - d = 0,$

$\therefore \quad d = 0,\ c = 0,\ b = 0,\ a = -1$

$\therefore \quad \boldsymbol{\pi_1 = \theta/\theta_0}$

Let $\pi_2 = x\,\theta_0^{\,a}\,L^b\,\alpha^c\,k^d$ or $M^0L^0T^0\theta^0 = L\theta^aL^b\,\dfrac{L^{2c}}{T^c}\,\dfrac{M^d}{T^{3d}}\,\dfrac{L^d}{\theta^d}$

$\therefore \quad a - d = 0,\ d = 0,\ 1 + b + 2c + d = 0,\ -c - 3d = 0,$

$\therefore \quad a = 0,\ b = -1,\ c = 0,\ d = 0$

$\therefore \quad \boldsymbol{\pi_2 = x/L}$

Let $\pi_3 = h\,\theta_0^{\,a}\,L^b\,\alpha^c\,k^d$ or $M^0L^0T^0\theta^0 = \dfrac{M}{T^3\theta}\,\theta^aL^b\,\dfrac{L^{2c}}{T^c}\,\dfrac{M^d}{T^{3d}}\,\dfrac{L^d}{\theta^d}$

$\therefore \quad 1 + d = 0,\ b + 2c + d = 0,\quad -3 - c - 3d = 0,\ -1 + a - d = 0,$

$\therefore \quad a = 0,\ b = 1,\ c = 0,\ d = -1$

$\therefore \quad \boldsymbol{\pi_3 = hL/k}$ **(Biot number)**

Let $\pi_4 = \tau\,\theta_0^{\,a}\,L^b\,\alpha^c\,k^d$ or $M^0L^0T^0\theta^0 = T\,\theta^a\,L^b\,\dfrac{L^{2c}}{T^c}\,\dfrac{M^d}{T^{3d}}\,\dfrac{L^d}{\theta^d}$

$\therefore \quad d = 0,\ b + 2c + d = 0,\ 1 - c - 3d = 0,\ a - d = 0,$

$\therefore \quad a = 0,\ b = -2,\ c = 1,\ d = 0$

$\therefore \quad \boldsymbol{\pi_4 = \alpha\,\tau/L^2}$ **(Fourier number)**

$$\therefore \qquad \frac{\theta}{\theta_0} = f\left[\frac{x}{L}, \frac{hL}{k}, \frac{\alpha\tau}{L^2}\right]$$

There are the popular dimensionless numbers is conduction heat transfer.

This problem shows that the method is not limited to fluid flow or convection only.

Problem 8.19. *Convective heat transfer coefficient in free convection over a surface is found to be influenced by the density, viscosity, thermal conductivity, coefficient of cubical expansion, temperature difference, gravitational acceleration, specific heat, the height of surface and the flow velocity. Using dimensional analysis, determine the dimensionless parameters that will correlate the phenomenon.*

The variables with dimensions in the *MLT* θ set is tabulated below.

S.No.	Variable	Unit	Dimension
1	Height, x	m	L
2	Temperature difference, ΔT	deg K	θ
3	Coefficient of cubical expansion, β	(m^3/m^3)/deg K	1/θ
4	Acceleration due to gravity, g	m/s^2	L/T^2
5	Density, ρ	kg/m^3	M/L^3
6	Viscosity, μ	kg/ms	M/LT
7	Specific heat, c	J/kgK	$L^2/T^2\theta$
8	Thermal conductivity, k	W/mK	$ML/T^3\theta$
9	Convective heat transfer coefficient, h	W/m^2K	$M/T^3\theta$

There are nine variables and four dimensions. Hence five π terms can be identified. ρ, μ, x and k are chosen as repeating variables.

Let $$\pi_1 = \Delta T\, \rho^a\, \mu^b x^c k^d \quad \text{or} \quad M^0L^0T^0\theta^0 = \theta \frac{M^a}{L^{3a}} \frac{M^b}{L^bT^b} L^c \frac{M^dL^d}{T^{3d}\theta^d}$$

$\therefore$ $$a + b + d = 0, -3a - b + c + d = 0, b - 3d = 0, 1 - d = 0,$$

$\therefore$ $$a = 2, b = -3, c = 2, d = 1$$

$\therefore$ $$\boldsymbol{\pi_1 = \Delta T \rho^2 x^2 k/\mu^3}$$

Let $$\pi_2 = \beta\, \rho^a\, \mu^b x^c k^d \quad \text{or} \quad M^0L^0T^0\theta^0 = \frac{1}{\theta} \frac{M^a}{L^{3a}} \frac{M^b}{L^bT^b} L^c \frac{M^dL^d}{T^{3d}\theta^d}$$

$\therefore$ $$a + b + d = 0, -3a - b + c + d = 0, \quad -b - 3d = 0, -1 - d = 0,$$

$\therefore$ $$a = -2, b = 3, c = -2, d = -1$$

$\therefore$ $$\boldsymbol{\pi_2 = \beta \mu^3/\rho^2 x^2 k}$$

Let $$\pi_3 = g\, \rho^a\, \mu^b x^c k^d \quad \text{or} \quad M^0L^0T^0\theta^0 = \frac{L}{T^2} \frac{M^a}{L^{3a}} \frac{M^b}{L^bT^b} L^c \frac{M^dL^d}{T^{3d}\theta^d}$$

$\therefore$ $a + b + d = 0, 1 - 3a - b + c + d = 0, -2 - b - 3d = 0, d = 0,$

$\therefore$ $a = 2, b = -2, c = 3, d = 0$

$\therefore$ $\boldsymbol{\pi_3 = g\,\rho^2\,x^3/\mu^2}$

Let $\pi_4 = c\,\rho^a\,\mu^b x^c k^d$ or $M^0L^0T^0\theta^0 = \dfrac{L^2}{T^2\theta}\dfrac{M^a}{L^{3a}}\dfrac{M^b}{L^bT^b}L^c\dfrac{M^dL^d}{T^{3d}\theta^d}$

$\therefore$ $a + b + d = 0, 2 - 3a - b + c + d = 0, -2 - b - 3d = 0, -1 - d = 0,$

$\therefore$ $a = 0, b = 1, c = 0, d = -1$

$\therefore$ $\boldsymbol{\pi_4 = c\,\mu/k}$ **(Prandtl number)**

Let $\pi_5 = h\,\rho^a\,\mu^b x^c k^d$ or $M^0L^0T^0\theta^0 = \dfrac{M}{T^3\theta}\dfrac{M^a}{L^{3a}}\dfrac{M^b}{L^bT^b}L^c\dfrac{M^dL^d}{T^{3d}\theta^d}$

$\therefore$ $1 + a + b + d = 0, -3a - b + c + d = 0, -3 - b - 3d = 0, -1 - d = 0,$

$\therefore$ $a = 0, b = 0, c = 1, d = -1$

$\therefore$ $\boldsymbol{\pi_5 = hx/k}$ **(Nusselt number)**

As the π terms are too many π_1, π_2 and π_3 are combined as $\pi_1 \times \pi_2 \times \pi_3$ to form the group known as Grashof number.

$$\boldsymbol{\pi_6} = \frac{\Delta T\rho^2x^2k}{\mu^3} \times \frac{\beta\mu^3}{\rho^2x^2k} \times \frac{g\rho^2x^3}{\mu^2} = \frac{\Delta T\beta g x^3\rho^2}{\mu^2} = \frac{\Delta T g\beta x^3}{v^2}$$

$\therefore$ $$\frac{hx}{k} = f\left[\frac{c\mu}{k}, \frac{\Delta T g\beta\rho^2x^3}{\mu^2}\right]$$

Note : When there are more than three π parameters the set should be reduced to three by judicial combination.

OBJECTIVE QUESTIONS

O Q. 8.1. Fill in the blanks:

1. The dimension for force in the *MLT* set is _____.
2. The dimension for mass in the *FLT* set is _____.
3. If there are *n* variables and *m* dimensions, π-theorem states that _____ dimensionless parameters can be obtained.
4. The dimension for thermal conductivity in the *MLT* θ system is _____ and in *FLT* θ system is _____.
5. For an expression to be dimensionally homogeneous, each additive term in the equations should have _____.
6. One of the methods to check the correctness of an equation is to check for _____ for each of the additive terms.
7. The limitation of dimensional analysis is that the _____ has to be determined by experiments.
8. The approximate number of experiments to evaluate the influence of 5 parameters separately is _____ assuming that 10 experiments are needed for each variable.

9. The choice of dimension set has _____ on the final dimensionless numbers determined for a phenomenon.
10. When there are more than 3 Pi terms determined for a phenomenon, for experimentation the set must be reduced to 3 by _____.

Answers

(1) ML/T^2 (2) FT^2/L (3) $n - m$ (4) $ML/T^3\ \theta$, $F/T\theta$ (5) the same dimensions (6) the sameness of dimensions (homogeneous) (7) the exact functional relation (8) 10^5 (9) no effect (10) Combining terms in excess of two into a single π term.

O Q. 8.2. Fill in the blanks:

1. Reynolds number is used in the study of _____ flow.
2. Froude number is used in the study of _____ flow.
3. Euler number is used in the study of _____ flow.
4. Weber number is used in the study of _____ flow.
5. Cauchy number is used when _____ is important.
6. The number used in the study of oscillating flows is _____.
7. Lift coefficient is the ratio of lift force to _____.
8. Drag coefficient is the ratio of drag force to _____.
9. The ratio of inertia to viscous force is called _____ number.
10. The ratio of pressure force to inertia force is called _____ number.

Answers

(1) All types of flow (2) Free surface flow (3) Pressure driven flow (4) Free surface flow, Surface tension effects (5) Compressibility (6) Strouhal number (7) dynamic force (8) dynamic force (9) Reynolds numbers (10) Euler number.

O Q. 8.3. State correct or incorrect:

1. Dimensional analysis can be used to reduce the number of variables for investigation of a phenomenon.
2. Dimensional analysis can provide an exact functional relationship between variables affecting a phenomenon.
3. Dimensional analysis by clubbing variables into groups facilitates presentation of results of experiments in a compact form.
4. It is easier to investigate a problem varying the value of a group as a whole rather than individual variables.
5. Grouping of variables into dimensionless parameters reduces number of experiments.
6. Fluid dynamics problems can be completely solved by dimensional analysis.
7. Reynolds number is the ratio between gravitational force and the viscous force.
8. Froude number is the ratio between inertia force and gravitational force.
9. Euler number is the ratio between Inertia force and viscous force.
10. Weber number is used to study oscillating flow.

Answers

(1) Correct : 1, 3, 4, 5, 8 Incorrect : 2, 6, 7, 9, 10

EXERCISE PROBLEMS

E 8.1. Check whether the equation is dimensionally homogeneous. Find the unit conversion factor to make it dimensionally homogeneous.

$P = P_0 - (1/2)\ \rho u^2 - \rho g z$ (P-pressure, ρ-density, u-velocity, z-height above datum)

($g_0 = kgm/Ns^2$)

E 8.2. The centripetal acceleration of a particle in circular motion is dependent on velocity u and radius r. Using dimensional analysis determine the functional relation. **[$a_r = f(r\omega^2)$]**

E 8.3. The velocity of sound, c in a gaseous medium, depends on the pressure P and density ρ of the gas. Find using dimensional analysis a functional relation. **($c\sqrt{P/\rho}$ = constant)**

E 8.4. In flow over a smooth flat plate, the boundary layer thickness δ is found to depend on the free stream velocity u, fluid density and viscosity and the distance x from the leading edge. Express the correlation in the form of dimensionless groups. **(δ/x) = f($ux\ \rho/\mu$)**

E 8.5. In flow over a smooth flat plate, the wall shear τ_w in the boundary layer depends on the free stream velocity, density and viscosity of the fluid and the distance from the leading edge. Determine the dimensionless parameters to express the relation between the variables.

($\tau_w/\rho u^2$) = f($\mu/\rho ux$)

E 8.6. The volume flow Q over a weir depends on the upstream height h, the width of the weir b, and acceleration due to gravity. Obtain a relationship between the variables in terms of dimensionless parameters. **($Q/h^2\ (gh)^{1/2}$) = f(b/h)**

E 8.7. The speed u of free surface gravity wave in deep water depends on the depth D, wave length λ, density ρ and acceleration due to gravity. Determine, using dimensional analysis, a functional relationship between the varaibles. **($u/(gD)^{1/2}$ = f(λ/D)**

E 8.8. Obtain a relationship for the torque τ to rotate a disk of diameter D in a fluid of viscosity μ at an angular speed ω over a plate, with clearance h. **($\tau/D^3\ \mu\omega$) = f(h/D)**

E 8.9. The power P to drive a fan is found to depend on the diameter D, density of the gas ρ, volume flow rate Q, and the speed N. Using the method of dimensional analysis obtain a correlation in terms of dimensionless numbers. **($P/(\rho\ D^5N^3)$) = f($Q/D^3\ N$)**

E 8.10. Oil is moved up in a lubricating system by a rope dipping in the sump containing oil and moving up. The quantity of oil pumped Q, depends on the speed u of the rope, the layer thickness δ, the density and viscosity of the oil and acceleration due to gravity. Obtain the dimensionless parameters to correlate the flow. **($Q/u\delta^2$) = f** $\left(\dfrac{\mu}{\rho u\delta}, \dfrac{u^2 g}{\delta}\right)$

E 8.11. The time τ to drain a circular tank of diameter D by an orifice of diameter d is found to depend on the initial head h, the density and viscosity of the liquid and gravitational acceleration. Determine dimensionless parameters to correlate experimental results. (**Note** : Used in the determination of viscosity) **($\tau\mu/h^2\ \rho$) = f** $\left(\dfrac{D}{h}, \dfrac{d}{h}, \dfrac{gh^3\rho^2}{\mu^2}\right)$

E 8.12. The instantaneous volume Q drained by an orifice of diameter d from a circular tank of diameter D, when the head is h depends on the density and viscosity of the fluid and acceleration due to gravity. Determine the π terms to correlate the flow. **($Q\rho/d\ \mu$) = f** $\left(\dfrac{D}{d}, \dfrac{h}{d}, \dfrac{gh^3\rho^2}{\mu^2}\right)$

E 8.13. In the atomization of a fluid by passing it through an orifice under pressure, the droplet size d is believed to depend on the jet diameter D, jet velocity u, and the density, viscosity and surface tension of the lqiuid. Determine dimensionless parameters for the phenomenon.

$$\mathbf{(d/D) = f}\left(\frac{\rho uD}{\mu}, \frac{\sigma}{\rho Du^2}\right)$$

E 8.14. The pressure P, developed in a jet pump is found to depend on the jet diameter d, diffuser diameter D, the velocity u of the jet, the volume flow Q and the density and viscosity of the fluid. Determine the dimensionless parameters to organize experimental results.

$$\left[\frac{Q}{ud^2}, \frac{\mu}{\rho uD}, \frac{d}{D}, \frac{P}{\rho u^2}\right]$$

E 8.15. A spherical ball of diameter D and weight w is balanced at the tip of a jet of diameter d at a height h. The velocity of the jet is u. If the other parameters are the liquid density and viscosity, find the π terms that can be used to characterize the phenomenon.

$$\left[\frac{h}{d}, \frac{D}{d}, \frac{\rho uD}{\mu}\right]$$

E 8.16. Players use spin in ball plays like tennis, golf etc. As the ball moves the spin rate will decrease. If the aerodynamic torque τ on the ball in flight depends on the forward speed u, density and viscosity of air, the ball diameter D, angular velocity of spin, ω and the roughness height e on the ball surface, determine the dimensionless parameters to correlate situation.

$$\left[\frac{\tau}{\rho u^2 D^3}, \frac{\mu}{\rho uD}, \frac{\omega D}{u}, \frac{e}{D}\right]$$

E 8.17. The high pressure generated due to sudden closing of a valve in a pipeline (known as water hammer) is found to depend on the velocity of flow, the density of fluid and the bulk modulous E_v. Determine the functional relationship among the variables in terms of dimensionless parameters.

$$\left[\frac{P}{\rho u^2} = f(E_v/\rho u^2)\right]$$

E 8.18. The power required to drive a propeller in a gas medium depends upon the forward speed u, the rotational speed N, diameter D, density and viscosity of the gas and the speed of sound c in the medium. Obtain dimensionless parameters to correlate experimental results.

$$\frac{P}{\rho D^2 u^2} = f\left[\frac{ND}{u}, \frac{\mu}{\rho uD}, \frac{c}{u}\right]$$

E 8.19. In an oven where materials are heated by convection, the heat transfer rate Q (W) is believed to depend on the specific heat of air, temperature difference $\Delta\theta$ (between gas and heated body) a length parameter L, density, viscosity and flow velocity u of the fluid. Determine the π parameters for the situation.

$$\left[\frac{Q}{\rho u^3 L^2}, \frac{c\Delta\theta}{u^2}, \frac{\mu}{\rho uL}\right]$$

E 8.20. The deflection δ at the center due to fluid flow at velocity u, over a thin wire held between rigid supports is found to depend on length of the wire L, diameter of the wire, d elastic modulus of the wire, E and fluid properties density and viscosity and the velocity u.

Determine the π terms for the problem. (use ρ, d and μ as repeating variables)

$$\left[\frac{\delta}{d}, \frac{L}{d}, \frac{\rho u d}{\mu}, \frac{E \rho d^2}{\mu^2}\right] \textbf{ Also } E/\rho u^2$$

E 8.21. The volume of flue gas Q flowing through a chimney of height h and diameter d is influenced by the density of the gas inside ρ_g, the density of air outside ρ_a and acceleration due to gravity.

Determine π terms to correlate the variables.

$$\left[\frac{Q}{d^2\sqrt{gD}}, \frac{h}{d}, \frac{\rho_g}{\rho_a}\right]$$

9 Similitude and Model Testing

9.0 INTRODUCTION

Fluid flow analysis is involved in the design of aircrafts, ships, submarines, turbines, pumps, harbours and tall buildings and structures. Fluid flow is influenced by several factors and because of this the analysis is more complex. For many practical situations exact soluations are not available. The estimates may vary by as much as ± 20%. Because of this it is not possible to rely solely on design calculations and performance predictions. **Experimental validation of the design is thus found necessary.** Consider the case of a hydraulic turbine of 50 MW size. It will be a very costly failure if the design performance and the actual performance differ. If we can predict its performance before manufacturing the unit it will be very useful. Model testing comes to our aid in this situation. **Constructing and testing small versions of the unit is called model testing. Similarity of features enable the prediction of the performance** of the full size unit from the test results of the smaller unit. The application of dimensional analysis is helpful in planning of the experiments as well as prediction of the performance of the larger unit from the test results of the model.

9.1 MODEL AND PROTOTYPE

In the engineering point of view model can be defined as the representation of physical system that may be used to predict the behavior of the system in the desired aspect. **The system whose behavior is to be predicted by the model is called the prototype**. The discussion in this chapter is about physical models that resemble the prototype but are generally smaller in size. These may also operate with different fluids, at different pressures, velocities etc. As models are generally smaller than the prototype, these are cheaper to build and test. Model testing is also used for evaluating proposed modifications to existing systems. The effect of the changes on the performance of the system can be predicted by model testing before attempting the modifications. Models should be carefully designed for reliable prediction of the prototype performance.

9.2 CONDITIONS FOR SIMILARITY BETWEEN MODELS AND PROTOTYPE

Dimensional analysis provides a good basis for laying down the conditions for similarity. The *PI* theorem shows that the performance of any system (prototype) can be described by a functional relationship of the form given in equation 9.2.1.

$$\pi_{1p} = f(\pi_{2p}, \pi_{3p}.....\pi_{np}) \quad ...(9.2.1)$$

The *PI* terms include all the parameters influencing the system and are generally ratios of forces, lengths, energy etc. If a model is to be similar to the prototype and also function similarly as the prototype, then the *PI* terms for the model should also have the same value as that of the prototype or the same functional relationship as the prototype. (eqn. 9.2.1)

$$\pi_{1m} = f(\pi_{2m}, \pi_{3m}.....\pi_{nm}) \quad ...(9.2.2)$$

For such a condition to be satisfied, the model should be constructed and operated such that simultaneously

$$\pi_{1m} = \pi_{1p}, \pi_{2m} = \pi_{2p},.....\pi_{nm} = \pi_{np} \quad ...(9.2.3)$$

Equation 9.2.3 provides the model design conditions. It is also called similarity requirements or modelling laws.

9.2.1 Geometric Similarity

Some of the *PI* terms involve the ratio of length parameters. All the similar linear dimension of the model and prototype should have the same ratio. This is called geometric similarity. The ratio is generally denoted by the scale or scale factor. One tenth scale model means that the similar linear dimensions of the model is 1/10 th of that of the prototype. For complete similarity all the linear dimensions of the model should bear the same ratio to those of the prototype. There are some situations where it is difficult to obtain such similarity. Roughness is one such case. In cases like ship, harbour or dams distorted models only are possible. In these cases the depth scale is different from length scale. Interpretation of the results of the tests on distorted models should be very carefully done. Geometric scale cannot be chosen without reference to other parameters. For example the choice of the scale when applied to the Reynolds number may dictate a very high velocity which may be difficult to achieve at a reasonable cost.

9.2.2 Dynamic Similarity

Similitude requires that π terms like Reynolds number, Froude number, Weber number etc. be equal for the model and prototype. These numbers are ratios of inertia, viscous gravity and surface tension forces. This condition implies that the ratio of forces on fluid elements at corresponding points (homologous) in the model and prototype should be the same. This requirement is called dynamic similarity. This is a basic requirement in model design. If model and prototype are dynamically similar then the performance of the prototype can be predicted from the measurements on the model. In some cases it may be difficult to hold simultaneously equality of two dimensionless numbers. In such situations, the parameter having a larger influence on the performance may have to be chosen. This happens for example in the case of model tasting of ships. Both Reynolds number and Froude number should be simultaneously

held equal between the model and prototype. This is not possible as this would require either fluids with a very large difference in their viscosities or the use of very large velocities with the model. This is illustrated in problem 9.14.

9.2.3 Kinematic Similarity

When both geometric and dynamic similarities exist, then velocity ratios and acceleration ratios will be the same throughout the flow field. This will mean that the streamline patterns will be the same in both cases of model and prototype. This is called kinematic similarly. **To achieve complete similarity between model and prototype all the three similarities - geometric, dynamic and kinematic should be maintained.**

9.3 TYPES OF MODEL STUDIES

Model testing can be broadly classified on the basis of the general nature of flow into four types. These are

(1) Flow through closed conduits

(2) Flow around immersed bodies

(3) Flow with free surface and

(4) Flow through turbomachinery

9.3.1 Flow through Closed Conduits

Flow through pipes, valves, fittings and measuring devices are dealt under this category. The conduits are generally circular, but there may be changes along the flow direction. As the wall shear is an important force, Reynolds number is the most important parameter. The pressure drop along the flow is more often the required parameter to be evaluated. Compressibility effect is negligible at low mach numbers. (M<0.3).

From dimensional analysis the pressure drop can be established as

$$\Delta P/\ \rho u^2 = f\left(\frac{\rho u L}{\mu}, \frac{\varepsilon}{L}, \frac{D}{L}\right) \qquad ...(9.3.1)$$

The geometric scale is given by the ratio, scale = L_m / L_p.

This requires $$\frac{D_m}{D_p} = \frac{\varepsilon_m}{\varepsilon_p} = \frac{L_m}{L_p} = \lambda.$$

Reynolds number similarity leads to the condition for velocity ratio as

$$\frac{u_m \rho_m L_m}{\mu_m} = \frac{u_p \rho_p L_p}{\mu_p} \quad \therefore \quad \frac{u_m}{u_p} = \frac{\mu_m}{\mu_p}\ \frac{\rho_p}{\rho_m}\ \frac{L_p}{L_m} \qquad ...(9.3.2)$$

If the fluid used for the model and prototype are the same, then $\frac{u_m}{u_p} = \frac{L_p}{L_m}$ or $u_m = u_p / \lambda$.

As λ is less than one, the velocity to be used with the model has to be higher compared to the

prototype. Otherwise a different fluid with higher viscosity should be chosen to satisfy the requirements.

The pressure drop in the prototype is calculated as in equation (9.3.3)

From equality of, $\Delta P / \rho u^2$, $\Delta P_P = \frac{\rho_p}{\rho_m}\left(\frac{u_p}{u_m}\right)^2 \Delta P_m$ (9.3.3)

As ΔP_m is measured, using the model, the pressure drop in the prototype can be predicted.

When Reynolds numbers are large the inertia forces are predominant and viscous forces will be small in comparison. In such cases, the Reynolds number similarity becomes unimportant. However, the model should be tested at various Reynolds numbers to determine the range at which its effect on pressure drop becomes negligible. After this is established the model test results can be applied without regard to Reynolds number similarity, in this range.

Another condition is the onset of cavitation at some locations in the flow, particularly in testing components where at some points the local velocity may become high and pressure may drop to a level where cavitation may set in. Unless cavitation effects are the aim of the study, such condition should be avoided. In case cavitation effects are to be studied, then similarity of cavitation number should be established. *i.e.* $(p_r - p_v)/(\rho u^2/2)$. Where p_r is the reference pressure and p_v is the vapour pressure at that temperature.

9.3.2 Flow Around Immersed Bodies

Aircraft, Submarine, cars and trucks and recently buildings are examples for this type of study. In the sports area golf and tennis balls are examples for this type of study. Models are usually tested in wind tunnels. As viscous forces over the surface and inertia forces on fluid elements are involved in this case also, Reynolds number of the model and prototype should be equal. Gravity and surface tension forces are not involved in this case and hence Froude and Weber numbers need not be considered. Drag coefficient, defined by [Drag force /$(1/2)\ \rho u^2 l^2$)] is the desired quantity to be predicted. Generally the following relationship holds in this case.

$$C_D = \frac{D}{(1/2)\rho u^2 l^2} = f\left[\frac{l_1}{l}, \frac{\varepsilon}{l}, \frac{\rho u l}{\mu}\right] \tag{9.3.4}$$

where l is a characteristic length of the system and l_1 represents the other length parameter affecting the flow and ε is the roughness of the surface.

When the flow speed increases beyond Mach number 0.3 compressibility effect on similarity should be considered. Using the similitude, measured values of drag on model is used to estimate the drag on the prototype.

$$D_p = D_m \frac{\rho_p}{\rho_m}\left(\frac{u_p}{u_m} \times \frac{l_p}{l_m}\right)^2 \tag{9.3.5}$$

From Reynolds number similitude

$$u_m = \frac{\mu_m}{\mu_p}\frac{\rho_p}{\rho_m}\frac{l_p}{l_m}u_p = \frac{v_m}{v_p}\frac{l_p}{l_m}u_p \tag{9.3.6}$$

When same fluid is used for both prototype and model

$$u_m = (l_p/l_m)\, u_p \qquad ...(9.3.7)$$

The model velocity should be higher by the geometric scale.

If the prototype is to operate at 100 kmph and if the scale is 1:10, then the model should operate at 1000 kmph, which will mean a high Mach number. The model will be influenced by compressibility effect due to the operation at high Mach numbers. The prototype however will be operating at low Mach numbers where compressibility effect is negligible. Hence the performance prediction will be in error.

This may be overcome by using different fluids say water in place of air. Using equation 9.3.6, as kinematic viscosity of air is about 10 times that of water, the velocity will now be at a reasonable level. Another method is to pressurise the air in the wind tunnel, thus increasing the density, and reducing the required velocity of the model.

Where expense is of no consideration due to the requirement of utmost reliability as in space applications and development of new aircraft, full scale models are also used.

In some cases at higher ranges, the Reynolds number is found to have little influence on drag. Strict Reynolds similarity need not be used in such situations. The variation of drag due to variation in Reynolds number for cylinder and sphere is shown as plotted in Fig. 9.3.1. It may be seen that above Re = 10^4 the curve is flat. If the operation of the prototype will be at such a range, then Reynolds number equality will not be insisted for model testing.

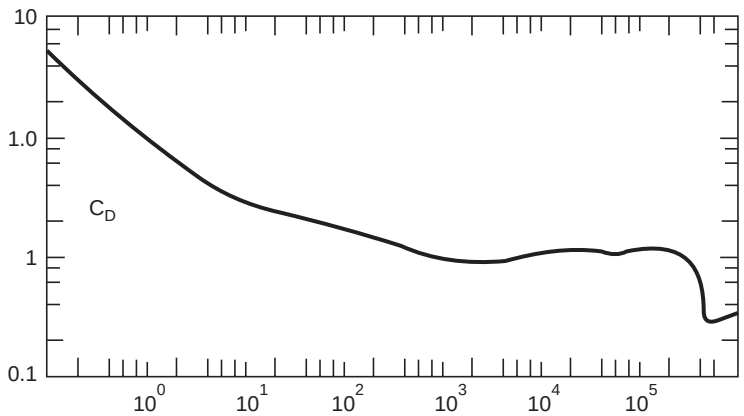


Figure 9.3.1 Variation of drag with Reynolds number for flow over cylinder

Another situation arises in testing of models of high speed aircraft. In this case the use of Mach number similitude requires equal velocities while the Reynolds number similarity requires increased velocity for the model as per geometric scale. In such cases distorted model is used to predict prototype performance.

9.3.3 Flow with Free Surface

Flow in canals, rivers as well as flow around ships come under this category. In these cases gravity and inertia forces are found to be governing the situation and hence Froude number becomes the main similarity parameter.

In some cases Weber number as well as Reynolds number may also influence the design of the model.

Considering Froude number, the velocity of the model is calculated as below.

$$\frac{u_m}{\sqrt{gl_m}} = \frac{u_p}{\sqrt{gl_p}} \tag{9.3.8}$$

$$\therefore \qquad u_m = u_p \sqrt{\frac{l_m}{l_p}} = u_p \sqrt{\text{scale}} \tag{9.3.9}$$

In case Reynolds number similarity has to be also considered, substituting this value of velocity ratio, the ratio of kinematic viscosities is given as

$$\frac{v_m}{v_p} = (\text{scale})^{3/2} \tag{9.3.10}$$

As these situations involve use of water in both model and prototype, it is impossible to satisfy the condition of equations 9.3.9 and 9.3.10 simultaneously. In such a case distorted model may have to be selected.

If surface tension also influences the flow, it is still more difficult to choose a fully similar model.

In many practical applications in this type of situation the influence of Weber and Reynolds number is rather small. Hence generally models are designed on the basis of Froude number similarity.

A special situation arises in the case of ships. The total drag on the ship as it moves is made up of two components: (1) The viscous shearing stress along the hull, (2) Pressure induced drag due to wave motion and influenced by the shape of the hull.

As it is not possible to build and operate a model satisfying simultaneously the Reynolds number similarity and Froude number similarity ingenious methods have to be adopted to calculate the total drag. The total drag on the model is first measured by experiment. The shear drag is analytically determined and the pressure drag on the model is calculated by subtracting this value. The drag on the prototype is determined using Froude number similarity. The calculated value of viscous drag is then added to obtain the total drag.

In case of design of river model, if the same vertical and horizontal scales are used, the depth will be low for the model and surface tension effects should be considered. But the use of distorted model, (vertical scaling smaller than horizontal scaling) overcomes this problem.

9.3.4 Models for Turbomachinery

Pumps as well as turbines are included in the general term turbomachines. Pumps are power absorbing machines which increase the head of the fluid passing though them. Turbines are power generating machines which reduce the head of the fluid passing through them.

The operating variables of the machines are the flow rate Q, the power P and the speed N. The fluid properties are the density and viscosity. The machine parameters are the diameter and a characteristic length and the roughness of the flow surface. Power, head and efficiency can be expressed as functions of π terms as in equation 9.3.11 (refer problem 9.21).

$$\text{Power} = f_1\left(\frac{l}{D}, \frac{\varepsilon}{D}, \frac{Q}{N D^3}, \frac{\rho N D^2}{\mu}\right) \quad (9.3.11)$$

The term ε / D is not important due to the various sharp corners in the machine. The dimensionless term involving power is defined as power coefficient, defined as $C_p = P/\rho N^3 D^3$. The head coefficient is defined as $C_h = gh/N^2 D^2$. The term $Q/N D^3$ is called flow coefficient. If two similar machines are operated with the same flow coefficient, the power and head coefficients will also be equal for the machines. This will then lead to the same efficiency. Combining flow and head coefficients in the case of pumps will give the dimensionless specific speed of the pump.

$$N_{sp} = \frac{N\sqrt{Q}}{(gh)^{3/4}} \quad (9.3.12)$$

Popularly used dimensional specific speed for pumps is defined as

$$N_{sp} = \frac{N\sqrt{Q}}{h^{3/4}} \quad (9.3.12a)$$

In the case of turbines, combining power and flow coefficients, the specific speed is obtained as

$$N_{st} = \frac{N\sqrt{P}}{\rho^{1/2}(gh)^{5/4}} \quad (9.3.13)$$

Popularly used dimensional speed for turbines is

$$N_{st} = \frac{N\sqrt{P}}{h^{5/4}} \quad ...(9.3.13a)$$

In model testing at a particular speed, the flow rate at various delivery heads can be measured. This can be used to predict the performance of the pump at other speeds using the various coefficients defined. The procedure for turbines will also be similar. The model can be run at a constant speed when the head is varied, the power and flow rate can be measured. The performance of the prototype can be predicted from the results of the tests on the geometrically similar model.

9.4 NONDIMENSIONALISING GOVERNING DIFFERENTIAL EQUATIONS

When differential equations describing the phenomenon is not available, the method of dimensional analysis is used to obtain similarity conditions. When differential equations describing the system are available, similarity parameters can be deduced by non dimensionalising the equations.

Consider the continuity and x directional momentum equations for two dimensional flow,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$$

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial P}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

The various quantities can be made dimensionless by dividing by reference quantities, as given below

$$u^*=\frac{u}{U},\ v^*=\frac{v}{V},\ P^*=\frac{p}{P_0},\ x^*=\frac{x}{L},\ y^*=\frac{y}{L},\ t^*=\frac{t}{\tau}$$

Then $$\frac{\partial u}{\partial x}=\frac{U}{L}\frac{\partial u^*}{\partial x^*},\quad \frac{\partial^2 u}{\partial x^2}=\frac{U}{L^2}\frac{\partial^2 u^*}{\partial x^{*2}}$$

Similar method is used in the case of other terms.

Substituting, the momentum equation reduces to the form

$$\left[\frac{L}{\tau U}\right]\frac{\partial u^*}{\partial t^*}+u^*\frac{\partial u^*}{\partial t^*}+v^*\frac{\partial u^*}{\partial t^*}=-\left[\frac{P_0}{\rho U^2}\right]\frac{\partial P^*}{\partial x^*}+\left[\frac{\mu}{\rho UL}\right]\left(\frac{\partial^2 u^*}{\partial x^{*2}}+\frac{\partial^2 u^*}{\partial y^{*2}}\right)$$

It may be noted that the non dimensionalised equation is similar to the general equation except for the terms in square brackets. These are the similarity parameters thus identified.

$$\frac{L}{\tau U},\frac{P}{\rho U^2},\frac{\mu}{\rho UL}$$

In case gravity force is added, gL/U^2 will be identified. These are forms of Strouhal, Euler, Reynolds and Froude numbers. As the equation describes the general unsteady flow all the numbers are involved. If other forms of forces like surface tension is added. Weber number can be identified. If equations for compressible flow is used, Mach number can be obtained by a similar method.

9.5 CONCLUSION

In all the problems in this chapter on model testing the π terms identified in chapter 8 are used. Reference may be made to the problems in chapter 8. The discussions in this chapter is limited to basics. In actual model making and testing as well as interpretation of results many other finer details have to be considered for obtaining accurate predictions about the performance of the prototype.

SOLVED PROBLEMS

Problem 9.1 *To study the pressure drop in flow of water through a pipe, a model of scale 1/10 is used.* ***Determine the ratio of pressure drops*** *between model and prototype if water is used in the model. In case air is used determine the ratio of pressure drops.*

Case (i) Water flow in both model and prototype.

Reynolds number similarity is to be maintained.

$$\frac{u_m d_m \rho_m}{\mu_m} \frac{u_p d_p \rho_p}{\mu_p} \quad \therefore \quad \frac{u_m}{u_p} = \frac{\mu_m}{\mu_p} \times \frac{d_p}{d_m} \times \frac{\rho_p}{\rho_m}$$

As viscosity and density values are the same,

$$\frac{u_m}{u_p} = \frac{d_p}{d_m} = 10,$$

The pressure drop is obtained using pressure coefficient

$$[\Delta P/(1/2)\ \rho u^2]_m = [\Delta P/(1/2)\ \rho u^2]_p$$

$$\therefore \quad \frac{\Delta \mathbf{p_m}}{\Delta \mathbf{p_p}} = \frac{\rho_\mathbf{m} \mathbf{u_m}^\mathbf{2}}{\rho_\mathbf{p} \mathbf{u_p}^\mathbf{2}}, \text{ As } \rho_m = \rho_p \text{ and } u_m/u_p = 10, \Delta P_m/\Delta P_P = 10^2 = \mathbf{100.}$$

Case (ii) If air is used in the model, then

$$\frac{u_m}{u_p} = \frac{\mu_m}{\mu_p} \times \frac{d_p}{d_m} \times \frac{\rho_p}{\rho_m}, \quad \frac{\Delta p_m}{\Delta p_p} = \frac{\rho_m}{\rho_p} \left(\frac{\mu_m}{\mu_p} \times \frac{d_p}{d_m} \times \frac{\rho_p}{\rho_m} \right)^2$$

$$= 100\ \frac{\rho_p}{\rho_m} \left(\frac{\mu_m}{\mu_p} \right)^2$$

From data tables at 20^0 C, $\rho_{air} = 1.205$ kg/m^3, $\mu_{air} = 18.14 \times 10^{-6}$ kg/ms,

$$\rho_w = 1000 \text{ kg/m}^3, \mu_w = 1.006 \times 10^{-3} \text{ kg/ms}$$

$$\therefore \quad \frac{\Delta \mathbf{P_m}}{\Delta \mathbf{P_p}} = 100 \times \frac{1000}{1.205} \times \left(\frac{18.14 \times 10^{-6}}{1.006 \times 10^{-3}} \right)^2 = \mathbf{26.98}$$

This illustrates that it may be necessary to use a different fluid in the model as compared to the prototype.

Problem 9.2 *To determine the pressure drop in a square pipe of 1 m side for air flow, a square pipe of 50 mm side was used with water flowing at 3.6 m/s. The pressure drop over a length of 3 m was measured as 940 mm water column. Determine the corresponding* ***flow velocity*** *of air in the larger duct and also the* ***pressure drop*** *over 90 m length. Kinematic viscosity of air = 14.58 ×10^{-6} m^2/s. Density = 1.23 kg/m^3.* Kinematic viscosity of water = 1.18 × 10^{-6} m^2/s

For pipe flow, Reynolds number analogy should be used. Also the drag coefficients will be equal.

For square section hydraulic mean diameter = $4A/P = 4a^2/4a = a$ (side itself)

$$Re = uD/v = 3.6 \times 0.05/1.18 \times 10^{-6} = 152542$$

For air $$152542 = \frac{1 \times u}{14.58 \times 10^{-6}} \quad \therefore \quad \mathbf{u = 2.224\ m/s}$$

Drag coefficient $F/\rho u^2$ should be the same for both pipes.

$$\frac{F_{air}}{F_w} \frac{\rho_{air} u_{air^2}}{\rho_w u_w^{\ 2}}$$

The pressure drop equals the shear force over the area. For square section, area = a^2, perimeter = $4a$

$$\therefore \qquad \Delta P = \frac{4FL}{a},\ \Delta P_{air} = \frac{4F_{air}L_{air}}{a_{air}},\ \Delta P_{w} = \frac{4F_{w}L_{w}}{a_{w}}$$

Dividing and substituting for F_{air}/F_w

$$\frac{\Delta P_{air}}{\Delta P_w} = \frac{L_{air}}{L_w} \times \frac{a_w}{a_{air}} \times \frac{F_{air}}{F_w} = \frac{L_{air}}{L_w} \times \frac{a_w}{a_{air}} \times \frac{\rho_{air}}{\rho_w} \left(\frac{u_{air}}{u_w}\right)^2$$

$$= \frac{90 \times 0.05^2 \times 1.23}{1 \times 3 \times 1000} \left(\frac{2.224}{3.6}\right)^2 = 3.521 \times 10^{-5}$$

$$\mathbf{\Delta P_{air}} = 940 \times 3.521 \times 10^{-5} = \mathbf{0.033\ mm\ of\ water\ column}$$

Problem 9.3 *Water at 15°C flowing in a 20 mm pipe becomes turbulent at a velocity of 0.114 m/s. What will be the* ***critical velocity of air at 10⁰C*** *in a similar pipe of 40 mm diameter. Density of air = 1.23kg/m³. Dynamic viscosity of air = 17.7 × 10⁻⁶ kg/ms.*

Density of water = 1000 kg/m³. Dynamic viscosity of water = 1.12 × 10⁻³ kg/ms.

As roughness etc are similar, for pipe flow, reynolds number similarity is to be used.

$$\frac{114 \times 0.02 \times 1000}{112 \times 10^{-3}} = \frac{u_{air} \times 0.04 \times 1.23}{127 \times 10^{-6}}, \quad \therefore \quad \mathbf{u_{air} = 0.732\ m/s}$$

Problem 9.4 *A model of 1/8 geometric scale of a valve is to be designed. The diameter of the prototype is 64 cm and it should control flow rates upto 1m³/s.* ***Determine the flow required for model*** *testing. The valve is to be used with brine in a cooling system at –10⁰C. The kinematic viscosity of brine at the saturated condition is 6.956 × 10⁻⁶ m²/s. For model testing water at 30⁰C is used. Kinematic viscosity is 0.8315 × 10⁻⁶ m²/s*

This is a situation of flow through closed conduits. Reynolds number similarity is required.

$$\frac{u_m d_m}{v_m} = \frac{u_p d_p}{v_p}, \qquad \therefore \quad \frac{u_m}{u_p} = \frac{d_p}{d_m} \times \frac{v_m}{v_p}, \qquad Q_p = \frac{\pi D_p^{\ 2}}{4} u_p$$

$$1 = \pi \times \frac{0.64^2}{4} u_p \qquad \therefore \quad \mathbf{u_p = 3.1085\ m/s}$$

$$\therefore \qquad \mathbf{u_m} = 3.1085 \times 8 \times 0.8315 \times 10^{-6}/6.956 \times 10^{-6} = \mathbf{2.9726\ m/s,}$$

$$d_m = d_p/8 = 0.64/8 = 0.08\ \text{m}$$

$$\mathbf{Q_m} = \frac{\pi d_m^{\ 2}}{4} u_m = \frac{\pi}{4} \times 0.08^2 \times 2.9726 = \mathbf{0.0149\ m^3/s}$$

If the valve is to be used with water, then the model velocity has to be 8 × 3.1085 m/s. *i.e.* 24.87 m/s, which is rather high.

The pressure drop can also be predicted from the model measurements using

$$\left(\frac{\Delta p}{\rho u^2}\right)_p = \left(\frac{\Delta p}{\rho u^2}\right)_m$$

Problem 9.5 *To predict the drag on an aircraft at a flight speed of 150 m/s, where the condition of air is such that the local speed of sound is 310 m/s, a pressurised low temperature tunnel is used. Density, viscosity and local sonic velocity at tunnel condition are 7.5 kg/m³, 1.22 × 10⁻⁵ Ns/m² and 290 m/s.* ***Determine the flow velocity and the scale*** *of the model. Assume full dynamic similarity should be maintained. Density and viscosity at the operating conditions are 1.2 kg/m³ and 1.8 × 10⁻⁵ Ns/m².*

In addition to Reynolds number similarity compressibility effect should be considered. For Mach number similarity,

$$M_p = M_m, \frac{u_p}{c_p} = \frac{u_m}{c_m} \quad \therefore \quad u_m = 150 \times 290/310 = \mathbf{140.32\ m/s}$$

For Reynolds number similarity

$$\frac{u_m \rho_m L_m}{\mu_m} = \frac{u_p \rho_p L_p}{\mu_p}$$

$$\frac{\mathbf{L_m}}{\mathbf{L_p}} = \frac{u_p}{u_m} \times \frac{\rho_p}{\rho_m} \times \frac{\mu_m}{\mu_p} = \frac{150}{140.32} \times \frac{1.2}{7.5} \times \frac{1.8 \times 10^{-5}}{1.22 \times 10^{-5}} = \mathbf{0.252}$$

or about 1/4th scale. When both Match number similarity and Reynolds number similarity should be maintained, generally the size of the model has to be on the higher side Drag force similarity is given by $(F/\rho u^2 L^2)_m = (F/\rho u^2 L^2)_p$

$$\frac{F_m}{F_p} = \frac{\rho_m u_m^{\ 2} L_m^{\ 2}}{\rho_p u_p^{\ 2} L_p^{\ 2}} = \frac{7.5}{1.2} \times \frac{140.32^2}{150^2} \times (0.252)^2 = 0.347$$

As the model size is larger, the force ratio is high.

Problem 9.6 *An aircraft fuselage has been designed for speeds of 380 kmph. To estimate power requirements the drag is to be determined. A model of 1/10 size is decided on. In order to reduce the effect of compressibility, the model is proposed to be tested at the same speed in a pressurized tunnel.* ***Estimate the pressure required.*** *If the drag on the model was measured as 100 N, predict the drag on the prototype.*

This is fully immersed flow. Hence Reynolds number similarity is required.

$$\frac{u_m L_m \rho_m}{\mu_m} = \frac{u_p L_p \rho_p}{\mu_p}$$

A viscosity is not affected by pressure and as velocities are equal,

$$L_m \rho_m = L_p \rho_p \quad \therefore \quad \rho_m / \rho_m = L_p / L_m = 10$$

At constant temperature, pressure ratio will be the same as density ratio.

$$\therefore \qquad \mathbf{P_m} = \frac{L_p}{L_m} P_p = \mathbf{10 \times P_p}$$

or 10 times the operating pressure of the aircraft.

The π parameter for drag force, D, gives

$$\frac{D_m}{(1/2)\rho_m u_m{}^2 L_m{}^2} = \frac{D_p}{(1/2)\rho_p u_p{}^2 L_p{}^2} \text{ as } u_m = u_p$$

$\therefore$ $\mathbf{D_p} = D_m\,(\rho_p L_p{}^2/\rho_m L_m{}^2) = 100 \times (1/10)\ 10^2 =$ **1000 or 1 kN**

Problem 9.7 *The performance of an aeroplane to fly at 2400 m height at a speed of 290 kmph is to be evaluated by a 1/8 scale model tested in a pressurised wind tunnel maintaining similarity. The conditions at the flight altitude are temperature = – 1^0C, pressure = 75 kN/m^2.*

μ = 17.1 × 10^{-6} kg/ms. The test conditions are 2150 kN/m^2, and 15^0C.

μ = 18.1 × 10^{-6} kg/ms. The drag resistance on the model measured at 18 m/s and 27 m/s. are 4.7N and 9.6N. ***Determine the drag on the prototype.***

At the given flight conditions, Velocity of sound is

$$C = \sqrt{g_0 kRT} = \sqrt{1 \times 1.4 \times 287 \times 272} = 330\ m/s = 1190 \text{ kmph}$$

Mach number = 290/1190 = 0.24 < 0.3

Hence Reynolds number similarity only need be considered.

Density at test conditions = 2150 × 10^3/(287 × 288) = 26.01 kg/m^3

Density at flight conditions = 75 × 10^3/(287 × 272) = 0.961 kg/m^3

Equating Reynolds numbers, assuming length L,

Velocity at flight condition = 290000/3600 = 80.56 m/s

$$\frac{80.56 \times L \times 0.961}{17.1 \times 10^{-6}} = u \times \frac{L}{8} \times \frac{26.01}{18.1 \times 10^{-6}} \qquad \therefore \quad \mathbf{u = 25.195\ m/s}$$

This is also low subsonic. Drag can be obtained using drag coefficient $F/\rho\, Au^2$

$$\frac{F_m}{\rho_m A_m u_m{}^2} = \frac{F_p}{\rho_p A_p u_p{}^2} \qquad \therefore \quad \frac{F_p}{F_m} = \frac{\rho_p}{\rho_m} \times \left(\frac{u_p}{u_m}\right)^2 \times \frac{A_p}{A_m}$$

$$= \frac{26.01}{0.961} \times \left(\frac{80.56}{25.195}\right)^2 \times 8^2 = 24.165$$

By interpolation using equality of F/u^2, **drag at 25.195 m/s** model speed is obtained as **8.78 N.** $\therefore$ **Drag on prototype =** 8.78 × 24.165 = **212 N**

Problem 9.8 *In a test in a wind tunnel on 1:16 scale model of a bus, at an air speed of 35m/s, the drag on the model was measured as 10.7N, If the width and frontal area of the prototype was 2.44 m and 7.8 m^2,* ***estimate the aerodynamic drag*** *force on the bus at 100 kmph. Conditions of air in the wind tunnel are the same as at the operating conditions of the bus. Assume that coefficient of drag remains constant above Reynolds number 10^5.*

v = 1.006 ×10^{-6} m^2/s. Also ***determine the power required.***

The width of the model = 2.44/16 = 0.1525 m.

$$Re = \frac{0.1525 \times 35}{15.06 \times 10^{-6}} = 3.5 \times 10^5, \quad \text{This condition is above } 10^5.$$

Area of the model $= 7.8/16^2$

$$C_D = \frac{F}{(1/2)\rho u^2 A} = \frac{10.7 \times 2 \times 16^2}{1.205 \times 35^2 \times 7.8} = 0.4758$$

Drag force on the prototype at 100 kmph. (27.78 m/s)

$$0.4758 = \frac{F}{(1/2)1.205 \times 7.8 \times (27.78)^2} \qquad \therefore \quad \mathbf{F = 1725\ N \quad or \quad 1.725\ kN}$$

Power required $= 1725 \times 27.78$ W $= 47927$ W or **47.927 kW.**

Problem 9.9 *A water tunnel operates with a velocity of 3m/s at the test section and power required was 3.75 kW. If the tunnel is to operate with air,* ***determine for similitude the flow velocity and the power required.***

$\rho_a = 1.25$ kg/m^3, $v_a = 14.8 \times 10^{-6}$ m^2/s, $v_w = 1.14 \times 10^{-6}$ m^2/s

In this case Reynolds number similarity is to be maintained. The length dimension is the same.

$$\frac{u_a}{v_a} = \frac{u_w}{v_w}$$

$\therefore$ Velocity of air, $\mathbf{u_a} = \dfrac{u_w}{v_w}\, v_a = \dfrac{3 \times 14.8 \times 10^{-6}}{1.14 \times 10^{-6}} = \mathbf{38.95\ m/s}$

Power can be determined from drag coefficient, by multiplying and dividing by u as $F \times u$ power $\dfrac{F \times u}{\rho A u^2 u} = \dfrac{P}{\rho A u^3}$ As A is the same,

$$P_{air} = P_w \frac{\rho_{air}}{\rho_w} \frac{u_{air}^3}{u_w^3} = 3.75 \times \frac{1.28}{1000} \times \left(\frac{38.95}{3}\right)^3 = \mathbf{10.5\ kW}$$

Problem 9.10 *The performance of a torpedo, 1 m diameter and 4 m long is to be predicted for speeds of 10 m/s. If a scale model of 1/25 size is used to predict the performance using a water tunnel,* ***determine the flow velocity*** *required. The ratio of density between sea water and fresh water is 1.02 and the viscosity ratio is 1.05. Also* ***determine the value of Reynolds number,*** *if the density of water was 1000 kg/m^3 and kinematic viscosity was 0.832 × 10^{-6} m^2/s.*

This is a fully submerged flow. Hence Reynolds number similarity should be maintained in the test. *i.e.* $\dfrac{D_p u_p \rho_p}{\mu_p} = \dfrac{D_m u_m \rho_m}{\mu_m}$,

$$u_m = u_p \times \frac{D_p}{D_m} \times \frac{\rho_p}{\rho_m} \times \frac{\mu_m}{\mu_p} = 10 \times 25 \times 1.02/1.05 = 242.85 \text{ m/s}$$

This is a very high speed generally not achievable in water tunnel.

$$Re = D_m\, u_m\, \rho_m/\mu_m = \frac{1}{25} \times \frac{242.85 \times 1000}{1000 \times 0.832 \times 10^{-6}} = 11.67 \times 10^6$$

For values of $Re > 10^5$ the coefficient of drag remains constant. Hence strict Reynolds number similarity need not be insisted on beyond such value.

In this case for example, velocity around 2.5 m/s may be used for the test which corresponds to $Re = 1.2 \times 10^5$.

Problem 9.11 *A 1/6 scale model of a submarine is tested in a wind tunnel using air of density 28.5 kg/m³ and viscosity 18.39 × 10 ⁻⁶ kg/ms at a speed of 36.6 m/s.* ***Calculate the corresponding speed and drag*** *of the prototype when submerged in sea water with density 1025 kg/m³ and viscosity 1.637 × 10⁻³ kg/ms if the model resistance was 67 N.*

Reynolds number similarity should be considered in this case. Let L be the length of the prototype.

$$\frac{36.6 \times L \times 28.5}{6 \times 18.39 \times 10^{-6}} = \frac{u_p \times L \times 1025}{1.637 \times 10^{-3}} \quad \therefore \mathbf{u_p = 15.2\ m/s}$$

Using drag coefficient

$$\frac{F_m}{\rho_m u_m^{\,2} L_m^{\,2}} = \frac{F_p}{\rho_p u_p^{\,2} L_p^{\,2}}$$

$$\therefore \qquad \mathbf{F_p} = F_m \times \frac{\rho_p}{\rho_m}\left(\frac{u_p}{u_m}\right)^2\left(\frac{L_p}{L_m}\right)^2 = 67 \times \frac{1025}{28.5}\left(\frac{15.2}{36.6}\right)^2 (6)^2 = \mathbf{14961\ N}$$

Problem 9.12 *A sonar transducer in the shape of a sphere of 200 mm diameter is used in a boat to be towed at 2.6 m/s in water at 20°C. To determine the drag on the transducer a model of 100 mm diameter is tested in a wind tunnel, the air being at 20° C. The drag force is measured as 15 N. Determine the* ***speed of air for the test. Estimate the drag*** *on the prototype.*

As it is fully immersed type of flow, Reynolds number similarity should be maintained.

The density and kinematic viscosity values are :

$$\rho_{air} = 1.205\ \text{kg/m}^3, \quad v_{air} = 15.06 \times 10^{-6}\ \text{m}^2/\text{s}$$
$$\rho_w = 1000\ \text{kg/m}^3, \quad v_w = 1.006 \times 10^{-6}\ \text{m}^2/\text{s}$$

$$\frac{u_m D_m}{v_m} = \frac{u_p D_p}{v_p}$$

$$\mathbf{u}_m = u_p \frac{D_p}{D_m}\frac{v_m}{v_p} = 2.6 \times \frac{200}{100} \times \frac{15.06 \times 10^{-6}}{1.006 \times 10^{-6}} = \mathbf{77.85\ m/s}$$

Mach number will be about 0.25. Hence compressibility effect will be negligible.

The coefficient of drag should be same for this condition. As $A \propto D^2$

$$\frac{F_m}{\rho_m u_m^{\,2} D_m^{\,2}} = \frac{F_p}{\rho_p u_p^{\,2} D_p^{\,2}},$$

$$\therefore \qquad \mathbf{F_p} = 15 \times \frac{1000}{1.205} \times \left[\frac{2.6 \times 200}{77.85 \times 100}\right]^2 = \mathbf{55.54\ N}$$

Problem 9.13 *In order to predict the flow conditions after the turbine outlet (tail race) of a hydroelectric plant delivering 2400 m³/s, a model of 1/75 scale is proposed.* ***Determine the flow rate required.***

This is a free surface flow. Hence Froude number similarity is to be maintained.

$$Fr_m = Fr_p \qquad \text{or} \qquad \frac{u_m}{\sqrt{l_m}} = \frac{u_p}{\sqrt{l_p}} \qquad \text{or} \qquad \frac{u_m}{u_p} = \sqrt{\frac{l_m}{l_p}}$$

As flow ($Q = Au$) depends on area which varies as L^2

$$\therefore \quad \frac{Q_m}{Q_p} = \frac{A_m u_m}{A_p u_p} = \frac{{L_m}^2}{{L_p}^2}\sqrt{\frac{L_m}{L_p}} = \left(\frac{L_m}{L_p}\right)^{2.5}$$

$$\therefore \quad \mathbf{Q_m} = 2400\left(\frac{1}{75}\right)^{2.5} = \mathbf{0.04927\ m^3/s}$$

Problem 9.14 *The total drag on a ship having a wetted hull area of 2500 m^2 is to be estimated. The ship is to travel at a speed of 12 m/s. A model 1/40 scale when tested at corresponding speed gave a total resistance of 32 N. From other tests the frictional resistance to the model was found to follow the law $F_{sm} = 3.7\ u^{1.95}\ N/m^2$ of wetted area. For the prototype the law is estimated to follow $F_{sp} = 2.9\ u^{1.8}\ N/m^2$ of wetted area* ***Determine the expected total resistance.***

The total resistance to ships movement is made up of (*i*) wave resistance and (*ii*) frictional drag. For wave resistance study Froude number similarity should be maintained. For frictional resistance Reynolds number similarity should be maintained. But it is not possible to maintain these similarities simultaneously. In the case of ships the wave resistance is more difficult to predict. Hence Froude number similarity is used to estimate wave resistance. Frictional drag is estimated by separate tests. From the Froude number similarity,

$$\mathbf{u_m} = u_p\sqrt{\frac{l_m}{l_p}} = 12/40^{0.5} = \mathbf{1.897\ m/s}$$

The **skin friction drag** for the model is calculated using this velocity.

$$F_{sm} = 3.7 \times 1.897^{1.95} \times A_m \quad \text{as} \quad A_m = 2500/40^2$$
$$= 3.7 \times 1.897^{1.95} \times 2500/40^2 = \mathbf{20.16\ N}$$

Wave drag on the model = 32 – 20.16 = **11.84 N**

The wave drag is calculated using $(F/\rho\ u^2L^2)_m = (F/\rho\ u^2L^2)_p$

Noting that sea water is denser with $\rho = 1025\ kg/m^3$

$$F_{wp} = F_{wm}\frac{\rho_p}{\rho_m}\left(\frac{L_p}{L_m}\right)^2\left(\frac{u_p}{u_m}\right)^2 = 11.84 \times \frac{1025}{1000}(40)^2\left(\frac{12}{1.897}\right)^2 = 774.38 \times 10^3\ \text{N}$$

Skin friction drag for the prototype

$\mathbf{F_{sp}} = 2.9\ {u_p}^{1.8} \times A_p = 2.9 \times 12^{1.8} \times 2500 = 635.13 \times 10^3$ N

$\therefore$ Total resistance $= 1.41 \times 10^6$ N or **1.41 MN**

Problem 9.15 *A scale model of a ship of 1/30 size is to be towed through water. The ship is 135 m long. For similarity* ***determine the speed with which the model should be towed.*** *The ship is to travel at 30 kmph.*

Froude number similarity is to be maintained.

$$\frac{u_m}{\sqrt{gL_m}} = \frac{u_p}{\sqrt{gL_p}} \quad \therefore \quad \boldsymbol{u_m} = u_p\sqrt{\frac{L_m}{L_p}} = \frac{30 \times 1000}{3600} \times \frac{1}{\sqrt{30}} = \mathbf{1.52\ m/s}$$

Problem 9.16 *The wave resistance of a ship when travelling at 12.5 m/s is estimated by test on 1/40 scale model. The resistance measured in fresh water was 16 N.* ***Determine the speed of the model and the wave resistance of the prototype*** *in sea water. The density of sea water = 1025 kg/m³.*

Froude number similarity is to be maintained.

$$\therefore \qquad \mathbf{u_m} = u_p \sqrt{\frac{L_m}{L_p}} = 12.5 \sqrt{\frac{1}{40}} = \mathbf{1.976\ m/s}$$

The wave resistance is found to vary as given below.

$$\frac{F_m}{\rho_m u_m{}^2 L_m{}^2} = \frac{F_p}{\rho_p u_p{}^2 L_p{}^2}$$

$$\therefore \qquad \mathbf{F_p} = F_m \times \frac{\rho_p}{\rho_m}\left(\frac{u_p}{u_m}\right)^2\left(\frac{L_p}{L_m}\right)^2 = 16 \times \frac{1025}{1000}\left(\frac{12.5}{1.976}\right)^2 (40)^2$$

$$= \mathbf{1049.6 \times 10^3\ N} \qquad \text{or} \qquad \mathbf{1050\ kN}$$

Problem 9.17 *Vortex shedding at the rear of a structure of a given section can create harmful periodic vibration. To predict the shedding frequency, a smaller model is to be tested in a water tunnel. The air speed is expected to be about 65 kmph. If the geometric scale is 1:6 and if the water temperature is 20° C determine the speed to be used in the tunnel. Consider air temperature as 40° C. If the shedding frequency of the model was 60 Hz* ***determine the shedding frequency*** *of the prototype. The dimension of the structure are diameter = 0.12 m, height = 0.36 m.*

The frequency of vortex shedding can be related by the equation

$$\omega = F\,(d, h, u, \rho, \mu)$$

Dimensional analysis leads to the π terms relation, (refer Chapter 8)

$$\frac{\omega D}{u} = f\left(\frac{D}{H}, \frac{\rho u D}{\mu}\right)$$

The model dimension can be determined as

$$D_m = 1/6\ D_p = 0.12/0.6 = 0.02\ \text{m},\ H_m = 1/6\ H_p = 0.36/0.6 = 0.06\ m$$

$$\therefore \qquad \frac{D}{H} = \frac{0.02}{0.06} = \frac{1}{3}, \qquad \text{Reynolds similarity requires}$$

$$\frac{\rho_m u_m D_m}{\mu_m} = \frac{\rho_p u_p D_p}{\mu_p} \quad \therefore \quad u_m = u_p \frac{\rho_p}{\rho_m}\frac{\mu_m}{\mu_p}\frac{D_p}{D_m}$$

The property values of air and water at the given temperatures are,

$$\rho_p = 1.128\ \text{kg/m}^3, \quad \mu_p = 19.12 \times 10^{-6}\ \text{kg/ms}$$

$$\rho_m = 1000\ \text{kg/m}^3, \quad \mu_p = 1.006 \times 10^{-6}\ \text{kg/ms}$$

$$u_p = 65 \times 1000/3600 = 18.056\ \text{m/s}$$

$$\therefore \qquad \mathbf{u_m} = 18.056 \times \frac{1.128}{1000} \times \frac{1.006 \times 10^{-3}}{19.12 \times 10^{-6}} \frac{6}{1} = \mathbf{6.43\ m/s}$$

Vortex shedding frequency is determined. Using the third π parameter,

$$\frac{\omega_m D_m}{u_m} = \frac{\omega_p D_p}{u_p}$$

$$\therefore \qquad \boldsymbol{\omega_p} = \frac{u_p}{u_m}\frac{D_p}{D_m}\ \omega_m = \frac{18.06}{6.43} \times \frac{1}{6} \times 60 = \mathbf{28.08\ Hz.}$$

The drag also can be predicted from the model. The drag for unit length can be expressed in the dimensionless from as $D/d\rho u^2$ where D is the drag and d is the diameter. Thus

$$\frac{D_p}{d_p \rho_p {u_p}^2} = \frac{D_m}{d_m \rho_m {u_m}^2} \quad \therefore \quad D_p = D_m \cdot \frac{d_p}{d_m} \cdot \frac{\rho_p}{\rho_m} \cdot \frac{{u_p}^2}{{u_m}^2}$$

Problem 9.18 *In order to determine the drag on supporting columns (of a bridge) of* 0.3 *m diameter, due to water flowing at a speed of 14.5 km/hr, a column of 0.25 m diameter was tested with air flow. The resistance was measured as 227 N/m, under similar conditions of flow.* ***Determine the force on the bridge*** *column per m length.* $v_{air} = 1.48 \times 10^{-5}\ m^2/s$. $v_w = 1.31 \times 10^{-6}\ m^2/s, \rho_a = 1.23\ kg/m^3$

Similarity requires equal Reynolds numbers

Velocity of flow of water = 14.5 × 1000/3600 = 4.028 m/s

$$\frac{4.028 \times 0.3}{1.31 \times 10^{-6}} = \frac{u_a \times 0.25}{1.48 \times 10^{-5}}$$

∴ Velocity of air, u_a = 54.61 m/s

The force can be obtained by the dimensional parameter (drag coefficient)

$F/\rho A u^2$, here $A = 1 \times D$

∴ The parameter in this case for force is

$$\frac{F}{\rho D u^2}\ \frac{F_w}{\rho_w D_w {u_w}^2} = \frac{F_a}{\rho_a D_a {u_a}^2} \quad \text{or}$$

$$\mathbf{F_w} = F_a\left(\frac{\rho_w}{\rho_a}\right)\left(\frac{D_w}{D_a}\right)\left(\frac{u_w}{u_a}\right)^2 = 227 \times \frac{1000}{1.23} \times \frac{0.3}{0.25} \times \left(\frac{4.028}{54.61}\right)^2 = \mathbf{1205\ N}$$

Problem 9.19 *To ascertain the flow characteristics of the spillway of a dam, 1/20 geometric scale model is to be used. The spillway is 40 m long and carries 300* m^3/s *at flood condition.* ***Determine the flow rate required*** *to test the model. Also* ***determine the time scale*** *for the model. Viscous and surface tension effects may be neglected.*

This situation is open surface flow. Froude number similarity is required.

$$\frac{u_m}{(gL_m)^{0.5}} = \frac{u_p}{(gL_p)^{0.5}} \quad \text{or} \quad \therefore \ \frac{u_m}{u_p} = \left(\frac{L_m}{L_p}\right)^{0.5}$$

As $$Q = uA = uL^2,\ Q_m = u_m L_m^2,\ Q_p = u_p L_p^2,\ \frac{Q_m}{Q_p} = \frac{u_m L_m^2}{u_p L_p^2}$$

$$\therefore \qquad \mathbf{Q_m} = Q_p \frac{u_m}{u_p}\frac{{L_m}^2}{{L_p}^2} = Q_p \left(\frac{L_m}{L_p}\right)^{0.5}\left(\frac{L_m}{L_p}\right)^2 = Q_p \left(\frac{L_m}{L_p}\right)^{2.5}$$

$$= 300\left(\frac{1}{20}\right)^{2.5} = \mathbf{0.168\ m^3/s}$$

Time scale can be determined from velocities, as velocity = length/time.

$$\frac{u_m}{u_p} = \frac{L_m}{L_p}\frac{t_p}{t_m} \quad \therefore \quad \frac{\mathbf{t_m}}{\mathbf{t_p}} = \frac{L_m}{L_p}\frac{u_p}{u_m} = \frac{L_m}{L_p}\left(\frac{L_p}{L_m}\right)^{0.5} = \left(\frac{L_m}{L_p}\right)^{0.5} = \left(\frac{1}{20}\right)^{0.5} = \mathbf{0.2236}$$

Problem 9.20 *A fan when tested at ground level with air density of 1.3 kg/m³, running at 990 rpm was found to deliver 1.41 m³/s at a pressure of 141 N/m². This is to work at a place where the air density is 0.92 kg/m³, the speed being 1400 rpm.*

Determine the volume delivered and the pressure rise.

For similarity condition the flow coefficient Q/ND^3 should be equal.

As D is the same,

$$\frac{Q_1}{N_1} = \frac{Q_2}{N_2} \qquad \text{or} \qquad \mathbf{Q_2} = Q_1\frac{N_2}{N_1} = 1.41 \times \frac{1400}{990} = \mathbf{2\ m^3/s}$$

The head coefficient $H/\rho N^2D^2$ is used to determine the pressure rise.

$$\mathbf{\Delta P_2} = \Delta P_1 \frac{\rho_2 {N_2}^2}{\rho_1 {N_1}^2} = 141 \times \frac{0.92}{1.3} \times \left(\frac{1400}{990}\right)^2 = \mathbf{199.55\ N/m^2}$$

Problem 9.21 *A centrifugal pump with dimensional specific speed (SI) of 2300 running at 1170 rpm delivers 70 m³/hr. The impeller diameter is 0.2 m. Determine* ***the flow, head and power if*** *the pump runs at 1750 rpm . Also* ***calculate the specific speed*** *at this condition.*

The head developed and the power at test conditions are determined first. (At 1170 rpm).

$$N_s = N\sqrt{Q}/H^{3/4} = 1170\sqrt{70}/H^{3/4} = 2300 \quad \therefore \mathbf{H = 6.9\ m}$$

Power = mg H = 9.81 × 70000 × 6.9/3600 = **1316 W**

When operating at 1750 rpm, using flow coefficient Q/ND^3, as D is the same

$$\mathbf{Q_2} = 70\left(\frac{1750}{1170}\right) = \mathbf{104.7\ m^3/hr}$$

Using head coefficient, H/N^2D^2, $\mathbf{H_2} = H_1 (N_1/N_2)^2 = 6.9 \times (1750/1170)^2 = \mathbf{15.44\ m}$

Using power coefficient : P/rN^3D^5,

$$\mathbf{P_2} = P_1 \times \left[\frac{N_2}{N_1}\right]^3 = 1316 \times \left[\frac{1750}{1170}\right]^3 = \mathbf{4404\ W}$$

Specific speed for the model

$$N_s = N\sqrt{Q}/H^{3/4} = 1750\sqrt{104.7}/(15.44)^{3/4} = 2300$$

Note: Specfic speeds are the same.

Problem 9.22 *A pump running at 1450 rpm with impeller diameter of 20 cm is geometrically similar to a pump with 30 cm impeller diameter running at 950 rpm. The discharge of the larger pump at the maximum efficiency was 200 litres / s at a total head of 25m.* ***Determine the discharge*** *and head of the smaller pump at the maximum efficiency conditions. Also determine the* ***ratio of power*** *required.*

The *PI* terms of interest are the head coefficient, power coefficient scale and $Q/\omega D^3$ called flow coefficient ($\omega \propto N$) (Refer chapter 8, Problem 8.16).

Considering flow coefficient, denoting the larger machines as 1 and the smaller as 2,

$$\frac{Q_1}{\omega_1 D_1^{\,3}} = \frac{Q_2}{\omega_2 D_2^{\,3}} \quad \therefore \quad \mathbf{Q_2} = Q_1 \frac{\omega_2}{\omega_1}\frac{D_2^{\,3}}{D_1^{\,3}} = 200 \times \frac{1450}{950}\left(\frac{20}{30}\right)^3 = \mathbf{90.45\ l/s}$$

Considering head coefficient, (g being common)

$$\frac{gh_1}{\omega_1^{\,2} D_1^{\,2}} = \frac{gh_2}{\omega_2^{\,2} D_2^{\,2}} \quad \therefore \quad h_2 = h_1 \left(\frac{\omega_2}{\omega_1}\right)^2 \left(\frac{D_2}{D_1}\right)^2$$

$$\therefore \qquad \mathbf{h_2} = 25 \times \left[\frac{1450}{950}\right]^2 \left(\frac{20}{30}\right)^2 = \mathbf{25.885\ m}$$

Consider power coefficient $\dfrac{P_1}{\rho_1 \omega_1^{\,3} D_1^{\,5}} = \dfrac{P_1}{\rho_1 \omega_1^{\,3} D_1^{\,5}}$, as $\rho_1 = \rho_2$,

$$\frac{\mathbf{P_2}}{\mathbf{P_1}} = \left(\frac{\omega_2}{\omega_1}\right)^3 \left(\frac{D_2}{D_1}\right)^5 = \left(\frac{1450}{950}\right)^3 \left(\frac{20}{30}\right)^5 = \mathbf{0.468}$$

As efficiencies should be the same, $Q_1\, \rho_1\, h_1 = Q_2\, \rho_2\, h_2$, with $\rho_1 = \rho_2$

$$0.200 \times 25 = 0.09045 \times 25.885/0.468,\ 5.00 = 5.00 \text{ (checks)}$$

Specific speed $= N\sqrt{Q}/H^{3/4} = 1450\sqrt{0.09045}\,/\,25.885^{3/4} = 38$ (dimensional)

For larger pump, specific speed $= 950\sqrt{0.2}/25^{3/4} = 38$, checks.

Problem 9.23 *A V notch is to be used with utectic calcium chloride solution at 30°C. Density = 1282 kg / m³, $v_e = 2.267 \times 10^{-6}$ m² / s. The flow rate has to be found for various heads. Water was used for the test at 20°C. Density = 1000 kg / m³, $v_w = 1.006 \times 10^{-6}$ m² / s. Neglecting the effect of surface tension,* ***determine the ratio of corresponding heads*** *and* ***mass flow rates of water and*** *the solution at the corresponding heads.*

Dimensional analysis shows (Neglecting surface tension effects), Q being volume flow rate that for similarity the following parameters should be equal. (suffix c refers to the solution properties) (Refer chapter 8, Problem 8.13).

$$\frac{Q}{g^{1/2} h^{5/2}} = f\left[\frac{g^{1/2} h^{3/2}}{v}, \theta\right]$$

$$\therefore \qquad \frac{h_w^{\,3/2}}{v_w} = \frac{h_c^{\,3/2}}{v_c}$$

$$\therefore \quad \frac{\mathbf{h_c}}{\mathbf{h_w}} = \left(\frac{v_c}{v_w}\right)^{2/3} = \left[\frac{2.267\times10^{-6}}{1.006\times10^{-6}}\right]^{2/3} = \mathbf{1.71884}$$

$$\frac{Q_c}{h_c^{5/2}} = \frac{Q_w}{h_w^{5/2}}$$

$$\therefore \quad \frac{\mathbf{Q_c}}{\mathbf{Q_w}} = \left(\frac{h_c}{h_w}\right)^{5/2} = (1.71884)^{5/2} = \mathbf{3.873}$$

Ratio of mass flow rates = 3.873 × (1282/1000) = **4.97**

Problem 9.24 *The discharge Q through an orifice is found to depend on the parameter* $\rho D\sqrt{gH}\,/\mu$, *when surface tension effect is neglected. Determine the ratio of flow rates of water and refrigerant 12 at 20°C under the same head.* ***What should be the ratio of heads for the*** *same flow rate.* $\mu_R = 2.7 \times 10^{-4}$ *kg/ms.,* $\mu_w = 1.006 \times 10^{-3}$ *kg/ms. Density of refrigerent = 923 kg/m*^{3}.

$$Q_w \propto \rho_w D\sqrt{gH}\,/\mu_w \quad \text{and} \quad Q_R \propto \rho_R\, D\sqrt{gH}\,/\mu_r, \text{ Dividing}$$

$$\frac{\mathbf{Q_R}}{\mathbf{Q_w}} = \frac{\rho_R}{\rho_w}\times\frac{\mu_R}{\mu_w} = \frac{923}{1000}\times\frac{1.006\times10^{-3}}{2.7\times10^{-4}} = \mathbf{3.44}$$

For the same flow rate

$$\frac{\rho_w D\sqrt{gH_w}}{\mu_w} = \frac{\rho_R D\sqrt{gH_R}}{\mu_R}$$

$$\therefore \quad \frac{\mathbf{H_w}}{\mathbf{H_R}} = \left(\frac{\mu_w}{\mu_R}\times\frac{\rho_R}{\rho_w}\right)^2 = \left(\frac{1.006\times10^{-3}}{2.7\times10^{-4}}\times\frac{923}{1000}\right)^2 = \mathbf{11.827}$$

OBJECTIVE QUESTIONS

O Q. 9.1 Fill in the blanks.

1. The representation of a physical system used to predict the behaviour of the system is called _____.
2. The system whose behaviour is predicted by the model is called _____.
3. Models are generally _____ in size compared to prototype.
4. When the prototype is very small _____ model is used.
5. Models may also be used to predict the effect of _____ to an existing system.
6. Dimensionless parameters provide _____ conditions for model testing.
7. For geometric similarity ratio of _____ should be equal.
8. For dynamic similarity ratio of _____ should be equal.
9. If stream lines are similar between model and prototype it is called _____ similarity.
10. When geometric and dynamic similarities exist then automatically _____ will exist.

Answers

1. Model 2. Prototype 3. Smaller 4. Enlarged/larger 5. Modifications 6. Similarity 7. Linear dimensions of model and prototype 8. Forces at corresponding locations of model and prototype 9. kinematic 10. kinematic similarity.

O Q. 9.2 Fill in the blanks.

1. For rivers, harbours etc. _____ models are used.
2. Reason for distorted models for rivers etc. is because of _____.
3. For complete similarity in general the _____ number for the model and prototype should be the same.
4. When viscous and inertia forces are important _____ number similarity should be used.
5. When gravity and inertia forces are important _____ number similarity is used.
6. When the ratio of velocities and accelerations are equal at corresponding locations in the model and prototype it is called _____ similarity.
7. When gravity and surface tension are important, then _____ number similarity is used.
8. When wave resistance is important _____ similarity is used.
9. For flow with free surface _____ number similarity is used.
10. To consider compressible flow effects _____ number similarity is used.
11. When periodic motion is to be considered _____ number similarity is used.

Answers

1. Distorted 2. Very small flow height for model 3. Reynolds, Froude, Weber, Mach 4. Reynolds 5. Froude 6. Dynamic 7. Weber 8. Froude 9. Froude 10. Mach 11. Strouhal

O Q. 9.3 Fill in the blanks.

1. For incompressible flow through closed ducts _____ similarity is used.
2. For compressible flow through closed ducts _____ and _____ similarity are used.
3. For flow around immersed bodies _____ similarity is used in case of incompressible flow.
4. For flow around immersed bodies _____ and _____ similarities are used in the case of compressible flow.
5. In case of same fluid properties it is _____ to have Reynolds and Mach analogy simultaneously.
6. _____ Wind tunnels are used to have simultaneously Reynlods and Mach analogy.
7. In flow with free surface _____ similarity is used.
8. For studying wave drag _____ similarity is used.
9. At high Reynolds numbers the coefficient of drag does not significantly _____.
10. Froude and Reynolds similarities _____ be maintained simultaneously.

Answers

1. Reynolds 2. Reynolds and Mach 3. Reynolds 4. Reynolds, Mach 5. Impossible 6. Pressurised 7. Froude 8. Froude 9. vary 10. cannot

O Q. 9.4 State correct or incorrect.

1. Geometric similarity will automatically lead to kinematic similarity.
2. Under kinematic similarity conditions dynamic similarity will result automatically.
3. Geometric similarity will lead to dynamic similarity.
4. Webr number similarity is used to study wave drag.

5. Froude number similarity is used to study wave drag.
6. Mach number similarity need not be considered for low velocities.
7. For obtaining simultaneously Reynolds number and Mach number similarity pressurised tunnels are used.
8. When surface tension forces prevail Froude number similarity should be used.
9. When gravity forces prevail Froude number similarity should be used.
10. For fluctuating flow Strouhal number similarity should be used.
11. Distorted models are used to study river flow.
12. At high Reynolds numbers viscous drag coefficient remains constant.

Answers

Correct : 2, 5, 6, 7, 9, 10, 11, 12
Incorrect : 1, 3, 4, 8

EXERCISE PROBLEMS

Note: Property values are not specified. These should be obtained from tables of properties.

E 9.1. An airship is to operate in air at 20°C and 1 bar at 20 m/s speed. A model of scale 1/20 is used for tests in a wind tunnel, the test speed being 75 m/s. Determine the pressure of the tunnel for dynamic similarity. The air temperatures are equal. If the drag force on the model was 250 N. Determine the drag on the prototype **(539 kPa, 1340 N)**

E 9.2. One fifth scale model of an automobile is tested in a towing water tank. Determine the ratio of speeds of the model and prototype. Assume 20°C in both cases. It is found that the coefficient of drag remains constant for the model after speeds of 4 m/s and the drag at this speed was 182 N. Estimate the drag on the prototype when operating at 90 kmph.

((u_m/u_p) = 0.345, 219 N)

E 9.3. A torpedo 533 mm dia and 6700 mm long is to travel in water at 28 m/s. A model of 1/5 scale is to be tested in a wind tunnel. Air speed in the tunnel should not exceed 110 m/s to avoid compressibility effect. If air can be pressurised with temperature remaining at 20°C determine the minimum pressure required. Dynamic viscosity is not affected by pressure. If the drag force at this condition on the model was 618N, Determine the drag on the prototype.

(1.93 Mpa, 43. 7 kN)

E 9.4. A dynamically similar model of an airfoil of 1/10 scale was tested in a wind tunnel at zero angle of attack at a Reynoles number of 5.5×10^6 (based on chord length). The temperature and pressure in the wind tunnel are 15°C and 10 atm absolute. The prototype has a chord length of 2 m and it is to fly at 15°C and 1 atm. Determine the speed in the wind tunnel and the prototype speed. **(both 39.2 m/s)**

E 9.5. A 3 m dia weather balloon of spherical shape is to travel at 1.5 m/s. To determine the drag a model of dia 50 mm is to be tested in a water tunnel. Under dynamically similar conditions the drag on the model was measured as 3.78 N. Calculate the test speed and estimate the drag on the balloon. **(6.21 m/s, 0.978 N)**

E 9.6. A model of an automobile of scale 1/5 is tested in water tunnel for obtaining the performance of the prototype. If the prototype is to travel at 27.78 m/s in air at 15°C, determine the speed in the tunnel. Also determine the ratio of drag forces. If at a point the pressure coefficient was –1.4, determine the static pressure at the point.

E 9.7. The power requirement of a tractor tailor with a frontal area of 0.625 m^2 when travelling at 22.4 m/s is to be estimated by a model to be tested in a wind tunnel. The scale for the model is 1/4. When tested at 1 atm and 15°*C* at a speed of 89.6 m/s, the drag on the model was measured as 2.46 kN. Determine the drag on the prototype and the power required.

(2.46 kN, 55.1 kW)

E 9.8. Water flows at the rate of 40 m^3/s through a spillway of a dam. The width of the spillway is 65 m. A model of 1.5 m width is proposed for laboratory test. Determine the flow rate of the model.

(3.24×10^{-3} m^3/s)

E 9.9. The pressure drop through an elbow of 150 mm diameter is to be determined by test on a model. The flow through the elbow is water at 20°C. The flow velocity is 5 m/s. The model is to be tested with water at 20°C and the velocity is limited to 10 m/s. Determine the dimeter of the model. If the pressure drop in the model was measured as 15 kpa. Determine the pressure drop in the prototype.

E 9.10. In a flow the geometric scale is 1/4. The density scale is 1. If inertial gravitational, surface tension and viscous effects are important determine the viscosity and surface tension scales.

E 9.11. The drag on a solid body having a characteristic length of 2.5 mm, moving with a velocity of 15 m/s through water is to be estimated by a study on a model having 50 times this characteristic length.

(*i*) In case a water tunnel is used determine the velocity. Also determine the ratio between model and prototype drag.

(*ii*) Repeat the above in case a wind tunnel is used.

Assume a temperature 20°C in all cases.

E 9.12. The drag characteristics of a new design of an automobile at 32 kmph and 144 kmph. speeds are to be estimated. The characteristic length of the unit is 7 m. If a model of 1.2 m is to be used in an atmospheric pressure wind tunnel, determine the air velocity required. In case a pressurised tunnel with a pressure of 8 atm is used, determine the velocity required. In this case the model size is to be 0.6 m. Assume same temperature in all cases.

E 9.13. An open channel of rectangular section of width 7 m carries water to a depth of 1 m and a flow rate of 2 m^3/s. A model to have Froude number similarity is to be designed. The discharge scale is 1/1000. Determine the depth of flow in the model.

E 9.14. A 1/50 scale model of a ship is tested in a towing tank to determine the wave drag on the ships hull. The ship is to designed to cruise at 18 knots (knot = 1852 m). Determine the velocity with which the model is to be towed. Also determine the ratio of drag values on model and prototype. Neglect viscous drag. **(1.31m/s, 1.25×10^5)**

E 9.15. It is proposed to design a centrifugal pump to deliver 4.1 m^3/s of water at 200 *m* head when running at 1200 rpm. The diameter of the impeller is to be 1 m. A laboratory model of 1/5 th scale is proposed for testing. The model is to run at the same speed. Determine the operating head and discharge of the model. Assume both model and prototype operate at the same efficiency. **(8 m, 0.0328 m^3/s)**

E 9.16. Oil flows over a submerged body horizontally at a velocity 15 m/s. The property values for oil are: kinematic viscosity = 3.45×10^{-5} m^2/s, density = 833 kg/m^3. An enlarged model is used with 8:1 scale in a water towing tank. Determine the velocity of the model to achieve dynamic similarity. If drag force on the model is 3.5 N, predict the drag force on the prototype. Kinematic viscosity of water = 1.14×10^{-6} m^2/s. **(0.062 m/s, F = 2666 N)**

E 9.17. A model 1/50 scale of a boat when tested at 1 m/s in water gave a wave resistance of 0.02 N. Determine the velocity of operation of the boat for similarity. Also determine the drag force and the power required for cruising the boat. **(7.1 m/s, 2500 N, 17.75 kW)**

E 9.18. A model of an aeroplane of 1/20 size is to be tested in a pressurised wind tunnel at the same speed as that of the prototype to get over compressibility effects. If the temperatures are the same, determine the pressure in the wind tunnel in atm. The aeroplane is to be operated at 0.8 atm.

E 9.19. The drag force on a sphere submerged in water at 20°C, when moved at 1.5 m/s was measured as 10 N. An enlarged model of 3:1 scale was tested in a pressurised wind tunnel at a pressure of 1.5 MN/m^2 and temperature of 20°C. Determine the velocity for dynamic similarity. Also determine the drag force on the model. Kinematic viscosity of water = 1.006×10^{-6} m^2/s. Viscosity of air = 18.14×10^{-6} kg/ms. Density of air = 17.83 kg/m^3.

E 9.20. Determine the flow rate of air at 80°*C* in a 50 mm diameter pipe that will give dynamic similarity for flow of 50 l/s of water at 60°C in a 400 mm diameter pipe if the pressure of air is 4 bar. Kinematic viscosity of water = 0.478×10^{-6} m^2/s. Dynamic viscosity of air = 20.1×10^{-6} kg/ms. Density = 4.24 kg/m^3.

E 9.21. The flow rate over a spillway of a dam was 150 m^3/s. If the flow rate over the model was 1.35 m^3/s. Determine the linear scale. If the force at a certain point on the model was measured as 5 N, determine the force at the corresponding point on the prototype.

E 9.22. A ship 180 m long is to cruise at a speed of 40 kmph in sea water whose viscosity is 1.2 cp and specfic weight is 10 kN/m^3. If a model 3 m length is to satisfy both Reynolds number and Froude number similarity calculate the kinematic viscosity of the fluid to be used with the model. Comment on the results.

E 9.23. A small insect of about 1 mm dia moves slowly in sea water. To determine the drag an enlarged model of 100:1 scale, tested in glycerin at a velocity of 30 cm/s measured a drag of 1.3 N, Determine the speed of travel of the insect and also the drag force on it. μ_w = 0.001 kg/ms, ρ_w = 998 kg/m^3, μ_w = 1.5 kg/ms, ρ_g =1263 kg/m^3 **(2.53 cm/s, 7.31 × 10^{-7} N)**

E 9.24. The slope of the free surface of a steady wave in one dimensional flow in a shallow liquid layer is described by the equation below. Non dimensionless the equation and obtain dimensionless groups to characterise the flow. Determine the condition for dimensional similarity.

$$\frac{\partial h}{\partial x} = -\frac{u}{g} \cdot \frac{\partial u}{\partial x}.$$

E 9.25. One dimensional unsteady flow in a thin liquid layer is described by the equation below. Non dimensionlise the equation and obtain dimensionless groups to characetrise the flow.

$$\frac{\partial u}{\partial \tau} + u \frac{\partial u}{\partial x} = -g \frac{\partial h}{\partial x}.$$

E 9.26. Steady incompressible two dimensional flow, neglecting gravity is described by the equations below. Non dimensioalise the equation and obtain the diminsionless groups that characterise the flow.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \; u\frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + v\frac{\partial^2 u}{\partial y^2}.$$

E 9.27. A large veturimeter is calibrated using 1/10 scale model. If the same fluid conditions are used for the model and prototype determine the discharge ratio. Assume dynamic similarity conditions. **(1/10)**

E 9.28. The scale ratio between model and prototype of a spillway is 1/25. Determine the ratio of velocities and discharges. If the prototype discharges 3000 m^3/s. Calculate the model discharges. **(1/5, 1/3125, 0.96 m^3/s)**

E 9.29. The performance of a spherical balloon to be used in air at 20°C is to be obtained by a test in a water tank using 1/3 scale model. The diameter of the model is 1 m and when dragged at 1.2 m/s measured a drag of 200 N. Determine the expected drag on the prototype if the water temperature was 15°C.

$v_a = 15.06 \times 10^{-6}$ m²/s, $\rho_a = 1.205$ kg/m³ $v_w = 1.2015 \times 10^{-6}$ m²/s **(42.2 N)**

E 9.30. In model testing if both Reynolds number similarity and Froude number similarity should be simultaneously maintained, determine the ratio of kinematic viscosity of fluids to be used with the model and prototype. **[v_m/v_p = (geometric scale)$^{1.5}$]**

E 9.31. Determine the ratio of drag coefficients of prototype to model when tested at 1/3 rd density of air, but at the same Mach number, the geometric scale being 1/5. **(25:1)**

E 9.32. A ship model of scale 1/50 showed a wave resistance of 30 N at its design speed. Determine the prototype wave resistance.

E 9.33. A turbine model of 1/5 scale uses 2 m³/s of water. The prototype turbine has to work with a flow rate of 15 m³/s. Determine the speed ratio and power ratio. **(16.67, 1.48)**

E 9.34. A centrifugal fan in operation when tested gave the following data. Volume delivered : 2.75 m³/s. Total pressure 63.5 mm of wate column. Power absorbed : 1.75 kW. A geometrically similar fan of 1/4 size is to be used running at twice the speed of the operating fan. Assuming same conditions for the air determine the volume delivered, total pressure and power absorbed.

(2.32 m³/s, 142.8 mm w.c, 3.2 kW)

E 9.35. The drag on a ship 122 m long and with 2135 m² wetted area is to be estimated. A model towed at 1.3 m/s through fresh water had a total drag resistance at 15.3 N. The skin resistance was separately analyzed and found to follow the law $F = c\, u_m^{1.9}$. When tested at 3 m/s, the skin resistance was 14.33 Nm² and the ships skin resistance is estimated to follow the law $F = c\, u_s^{1.85}$ and has a value of 43 N/m². Determine the corresponding speed of the ship and the power needed to propel it. Density of the sea water is 1025 kg/m³ **(7.51 m/s, 6530 kW)**

E 9.36. The prototype weir is to discharge under a head of 1.2 m. A scale model of 1/10 size is proposed and the available flow is 42.5 l/s. Determine the heads necessary and the corresponding discharge of the prototype weir. **(0.12 m, 13.44 m³/s)**

E 9.37. The skin resistance of a ship model is given by $5.22\, u_m^{1.95}$ N/m². For the prototype the skin resistance is given by $4.95\, u_p^{1.9}$ N/m². The model of 1/20 scale with wetted area of 4 m² when towed in fresh water at 1.2 m/s. measured a total resistance of 46.2 N. Determine the total resistance of the ship in sea water at speeds corresponding to that of the model. Density of sea water = 1025 kg/m³.

10 Boundary Layer Theory and Flow Over Surfaces

10.0 INTRODUCTION

Ideal inviscid fluids do not exert any force on the surfaces over which they flow. Real fluids have viscosity. When these fluids flow over surfaces, "no slip condition" prevails. The layer near the surface has to have the same velocity as the surface. If the surface is at rest, then this layer comes to rest. The adjacent layer is retarded to a lesser extent and this proceeds to layers more removed from the surface at rest. A velocity gradient forms leading to shear force being exerted over the layers. The velocity gradient is steepest at the interface and the shear is also highest at the interface. Work is to be done to overcome the force. The equations for the analysis of the complete flow field has been formulated by Navier and Stokes. But solutions for these equations for practical boundary conditions were not available. For a long time empirical equations based on experimental results were used in designs.

The development of boundary layer theory enabled the analysis of such flows to be fairly easy. The theory was proposed by Ludwig Prandtl in 1904. He observed that **in the case of real fluids velocity gradient existed only in a thin layer near the surface. This layer was named as boundary layer**. Beyond this layer the effect of viscosity was found negligible. This was supported by measurement of velocity. The flow field now can be divided into two regions, one in which velocity gradient and shear existed and another where viscous effects are negligible. This region can be dealt with as flow of inviscid fluid or ideal fluid. In the study of flow over immersed bodies like aircraft wings the analysis can be limited to the boundary layer, instead of the field extending to long distances for the determination of forces exerted on the surface by the fluid flowing over it.

10.1 BOUNDARY LAYER THICKNESS

In the solution of the basic equations describing the flow namely continuity and momentum equations of the boundary layer, one boundary is provided by the solid surface. The need for the other boundary is met by edge of the boundary layer determined by the thickness. The determination of the velocity variation along the layer enables the determination

of velocity gradient. This is made possible by these two boundary conditions. Once the velocity gradient at the surface is determined, the shear stress can be determined using the equation

$$\tau = \mu \frac{du}{dy} \tag{10.1.1}$$

This leads to the determination of resistance due to the flow.

10.1.1 Flow Over Flat Plate

The simplest situation that can be analyzed is the flow over a flat plate placed parallel to uniform flow velocity in a large flow field. The layer near the surface is retarded to rest or zero. velocity. The next layer is retarded to a lower extent. This proceeds farther till the velocity equals the free stream velocity. As the distance for this condition is difficult to determine, **the boundary layer thickness is arbitrarily defined as the distance from the surface where the velocity is 0.99 times the free stream velocity.**

There are two approaches for the analysis of the problem.

1. Exact method : Solution of the differential equations describing the flow using the boundary conditions. It is found that this method can be easily applied only to simple geometries.

2. Approximate method : Formulation of integral equations describing the flow and solving them using an assumed velocity variation satisfying the boundary conditions. This method is more versatile and results in easier solution of problems. The difference between the results obtained by the exact method and by the integral method is found to be within acceptable limits.

At present several computer softwares are available to solve almost any type of boundary, and the learner should become familiar with such softwares if he is to be current.

10.1.2 Continuity Equation

The flow of fluid over a flat plate in a large flow field is shown in Fig. 10.1.1. The flow over the top surface alone is shown in the figure.

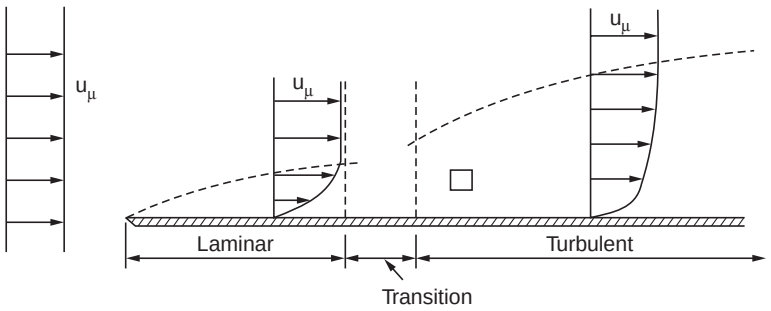


Figure 10.1.1 Formation of boundary layer over flat plate

The velocity is uniform in the flow field having a value of u_∞. Boundary layer begins to form from the leading edge and increases in thickness as the flow proceeds. This is because the viscosity effect is felt at layers more and more removed from the surface. At the earlier stages the flow is regular and layers keep their position and there is no macroscopic mixing between layers. **Momentum transfer resulting in the retarding force is by molecular diffusion**

between layers. This type of flow is called laminar flow and analysis of such flow is somewhat simpler. **Viscous effects prevail over inertial effects in such a layer**. Viscous forces maintain orderly flow. **As flow proceeds farther, inertial effects begin to prevail over viscous forces resulting in macroscopic mixing between layers. This type of flow is called turbulent flow**. Higher rates of momentum transfer takes place in such a flow. For the formulation of the differential equations an element of size $dx \times dy \times 1$ is considered.

An enlarged sectional view of the element is shown in Fig. 10.1.2.

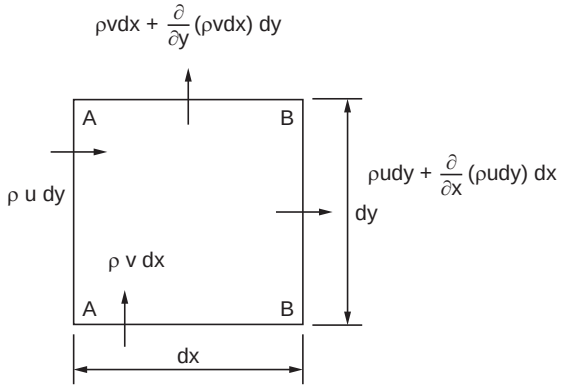


Figure 10.1.2 Enlarged view of element in the boundary layer

The assumptions are (*i*) flow is incompressible or density remains constant, (*ii*) flow is steady, (*iii*) there is no pressure gradient in the boundary layer.

Continuity equation is obtained using the principle of conservation of mass. Under steady flow conditions the net mass flow across the element should be zero. Under unsteady conditions, the net mass flow should equal the change of mass in the elemental volume considered. The values of velocities are indicated in the figure. The density of the fluid is ρ. Unit time and unit *Z* distance are assumed. **Time is not indicated in the equations.**

Flow in across face *AA*, $\rho udy \times 1 = \rho udy$

Flow out across face *BB*, $\rho udy + \dfrac{\partial}{\partial x}(\rho udy)\, dx$

Net flow in the *x* direction = $\dfrac{\partial(\rho u)}{\partial x}\, dxdy$

Similarly the net flow in the *y* direction is given by $\dfrac{\partial(\rho v)}{\partial x}\, dxdy$

Under steady conditions the sum is zero. Also for incompressible flow density is constant. Hence

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (10.1.2)$$

This is known as continuity equation for steady incompressible flow. If *u* decreases, $\dfrac{\partial u}{\partial x}$ is – ve and so $\dfrac{\partial v}{\partial y}$ should be positive. The algebraic sum of *x* and *y* directional flows is zero.

10.1.3 Momentum Equation

The equation is based on Newton's second law of motion. The net force on the surface of the element should equal the rate of change of momentum of the fluid flowing through the element. Here x directional forces are considered with reference to the element shown in Fig. 10.1.3. The flows are indicated on the figure unit time and unit Z distance are assumed. The density of the fluid is ρ

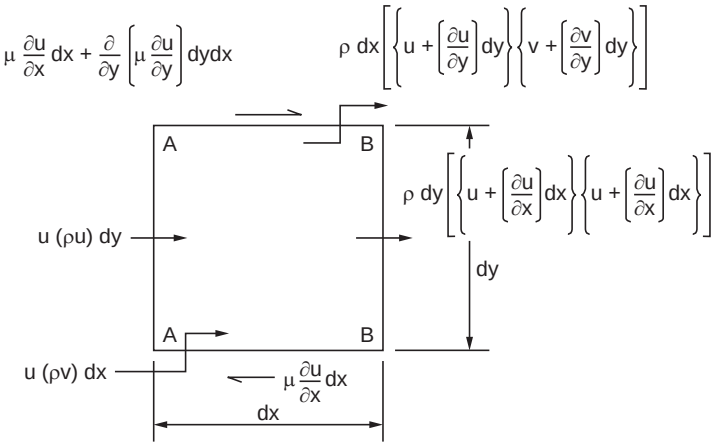


Figure 10.1.3 Momentum analysis

Consider the momentum flow in the x direction :

Across AA momentum flow = $u\ (\rho u)\ dy$

Across BB momentum flow = $u\ (\rho u)\ dy + \dfrac{\partial}{\partial x}\{u(\rho u)\ dy\}\ dx$

Taking the difference, the net flow is (as ρ is constant) (u^2 is written as $u \times u$)

$$\frac{\partial}{\partial x}[u(\rho u)dy]dx = \rho\ dxdy\left[u\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x}\right]$$

Considering the flow in the y direction, the net x directional momentum flow is

$$\frac{\partial}{\partial y}[u(\rho v)dy]dx = \rho\ dxdy\left[u\frac{\partial u}{\partial y} + v\frac{\partial u}{\partial y}\right]$$

Summing up, the net momentum flow is

$$\rho\ dxdy\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \left\{u\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right]\right\}\right]$$

From continuity equation, the second set in the above equation is zero. Hence net x directional momentum flow is

$$\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right]\rho\ dxdy$$

It was assumed that no body forces or pressure forces are present. Only surface forces due to viscosity is considered.

At the bottom surface shear $= dx\ \mu \dfrac{\partial u}{\partial y}$

At the top surface shear $= dx\ \mu \dfrac{\partial u}{\partial y} + \dfrac{\partial}{\partial y}\left[\mu \dfrac{\partial u}{\partial y} dx\right] dy$

The net shear on the element is $\mu \dfrac{\partial^2 u}{\partial y^2} dxdy$, noting $\upsilon = \mu/\rho$

Equating, and simplifying, $u\dfrac{\partial u}{\partial x} + \upsilon \dfrac{\partial u}{\partial y} = \upsilon \dfrac{\partial^2 u}{\partial y^2}$ (10.1.3)

This is known as momentum equation for the boundary layer. υ is also called as momentum diffusivity. In case of pressure gradient along the flow $-\dfrac{1}{\rho}\dfrac{\partial P}{\partial x}$ has to added on the RHS.

10.1.4 Solution for Velocity Profile

The continuity and momentum equations should be simultaneously solved to obtain the velocity profile. The boundary conditions are

(*i*) at $y = 0$, $u = 0$, (*ii*) at $y = \delta$, $u = u_\infty$, $\dfrac{\partial u}{\partial y} = 0$

The solution for these equations was obtained by Blasius in 1908 first by converting the partial differential equation into a third order ordinary differential equation and then using numerical method.

The two new vaiables introduced were

$$\eta = y\sqrt{\frac{u_\infty}{xv}} \text{ and } f(\eta) = \psi/\sqrt{vxu_\infty} \tag{10.1.4}$$

where ψ is the stream function giving

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x} \tag{10.1.5}$$

The resulting ordinary differential equation is

$$2\frac{d^3 f}{d\eta^3} + f\frac{d^2 f}{d\eta^2} = 0 \tag{10.1.6}$$

the boundary conditions with the new variables are

at $y = 0$, $\eta = 0$ and $\dfrac{\partial f}{\partial \eta} = 0$, at $y = \infty$, $\eta = \infty$ and $\dfrac{\partial f}{\partial \eta} = 1$

The results where plotted with u/u_∞ as the dependent vairable and $y\sqrt{\frac{u_\infty}{\nu x}}$ or (η) as the independent variable resulting in a plot as shown in Fig. 10.1.4.

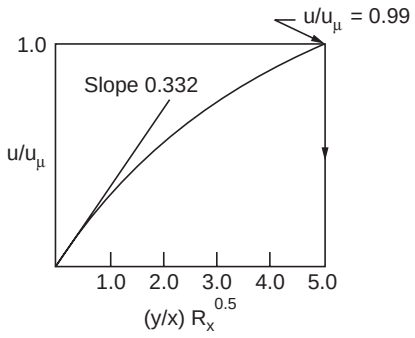


Figure 10.1.4 Velocity distribution in boundary layer

The value of $y\sqrt{\frac{u_\infty}{\nu x}}$ where $u/u_\infty = 0.99$ is found to be 5. This y value is taken as the boundary layer thickness δ as per the definition of thickness of boundary layer.

i.e. $$\delta\sqrt{u_\infty / \nu x} = 5, \quad \text{or} \quad \delta = \frac{5}{\sqrt{u_\infty / \nu x}} = \frac{5x}{\sqrt{u_\infty x / \nu}} \frac{5x}{\sqrt{Re_x}} = 5x\, Re_x^{-0.5} \quad (10.1.7)$$

This equation was more precisely solved in 1983 by Howarth. The significance of Reynolds number has already been explained under dimensional analysis as the ratio of inertia force to viscous force. Velocity gradient at the surface is of greater importance because it decides the shear on the surface at $y = 0$

$$\tau_w = \mu \frac{\partial u}{\partial y} \text{ equals the value of } \mu u_\infty \sqrt{u_\infty / \nu x}\, \frac{d^2 f}{d\eta^2}, \text{ at } \eta = 0$$

From the solution, at η = 0, the value of $\frac{d^2 f}{d\eta^2}$, is obtained as 0.332

Substituting this value and replacing ν by μ/ρ and simplifying

$$\tau_w = 0.332\ \rho u_\infty^2 / \sqrt{Re_x} \quad (10.1.8)$$

Defining skin friction coefficient, C_{fx}, as $\tau_w/(1/2)\rho u_\infty^2$, we obtain

$$C_{fx} = 0.664\ Re_x^{-0.5} \quad (10.1.9)$$

The average value over length L can be obtained by using

$$C_f = \frac{1}{L}\int_0^L C_{fx}\, dx = 1.328\ Re_L^{-0.5} \quad (10.1.10)$$

Not that these results are obtained for laminar flow over flat plate for $Re < 5 \times 10^5$.

Example 10.1. ***Air*** *at 30° C flows over a flat plate at a free stream velocity of 5m/s.* ***Determine the boundary layer thickness*** *at distances 0.2 m, 0.5 m and 0.8 m. Also determine the skin friction coefficients, both local and average, at these locations.*

The property values for air at 30 °C are obtained from tables. $\rho = 1.165$ kg/m^3,

$$v = 16 \times 10^{-6} \text{ m}^2/\text{s}, \ \mu = 18.63 \times 10^{-6} \text{ kg/ms}.$$

$$\delta = 5x\, Re_x^{-0.5}, \ C_{fx} = 0.664\, Re_x^{-0.5}, \ C_{fL} = 1.328\, Re_L^{-0.5}$$

Consider 0.5 m, $$Re_x = \frac{ux}{v} = \frac{5 \times 0.5}{16 \times 10^{-6}} = 1.5625 \times 10^5 < 5 \times 10^5 \quad \therefore \quad \text{Laminar}$$

$\therefore$ $$\delta = 6.325 \text{ mm}, \ C_{fx} = 1.68 \times 10^{-3}, \ C_{fL} = 3.36 \times 10^{-3}$$

The values for other distances are tabulated below.

Distance, m	Re	δ, mm	C_{fx}	C_{fL}
0.2	0.63×10^5	**4.000**	2.66×10^{-3}	5.32×10^{-3}
0.5	1.56×10^5	**6.325**	1.68×10^{-3}	3.36×10^{-3}
0.8	2.5×10^5	**8.000**	1.33×10^{-3}	2.66×10^{-3}

Note that as distance increases the local skin friction factor decreases and the average value is higher than the local value. Also note that the boundary layer thickness increases along the flow direction.

Example 10.2. ***Water*** *at 20° C flows over a flat plate at a free stream velocity of 0.2 m/s.* ***Determine the boundary layer*** *thickness and friction factors at lengths 0.2, 0.5 and 0.8 m from leading edge. The value of kinematic viscosity = 1.006×10^{-6} m^2/s, $\mu = 1.006 \times 10^{-3}$ kg/ms.*

The values calculated using equation 10.1.7, 9 and 10 are tabulated below:

Length, m	Re	δ, mm	C_{fx}	C_{fL}
0.2	0.40×10^5	**5.02**	3.33×10^{-3}	6.66×10^{-3}
0.5	0.99×10^5	**7.93**	2.11×10^{-3}	4.21×10^{-3}
0.8	1.59×10^5	**10.03**	1.67×10^{-3}	3.33×10^{-3}

Note the same trends as in Example 1. Also note that because of higher viscosity the friction values are higher.

10.1.5 Integral Method

In this case flow rate, momentum etc. in the boundary layer are determined using integration over the thickness of the boundary layer. The control volume chosen is shown in Fig. 10.1.5.

There is no flow through the face ad. (consider unit plate width)

Flow through face $$ab = \int_0^H \rho u dy$$

Flow through face $cd = \int_0^H \rho u dy + \frac{d}{dx}\left[\int_0^H \rho u dy\right] dx$

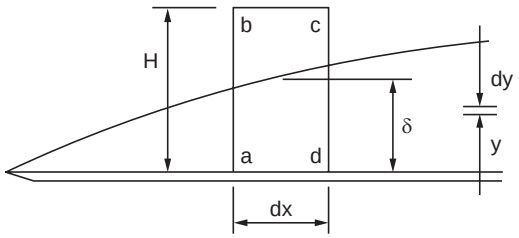


Figure 10.1.5 Boundary layer element for integral analysis

The difference should flow through bc as no flow is possible across ad.

$\therefore$ Flow through face $bc = -\frac{d}{dx}\left[\int_0^H \rho u dy\right] dx$

This is the result of continuity principle. Considering x directional momentum,

Momentum flow through $ab = \int_0^H u\rho u dy$

Momentum flow through $cd = \int_0^H u\rho u dy + \frac{d}{dx}\left[\int_0^H u\rho u dy\right] dx$

The mass crossing the boundary bc has a velocity of u_∞

Momentum flow through $bc = -\frac{d}{dx}\left[\int_0^H u_\infty \rho u dy\right] dx$

Summing up, the net momentum flow through the control volume

$$= \frac{d}{dx}\left[\int_0^H (u - u_\infty)\,\rho u dy\right] dx \qquad \ldots (1)$$

As $(u - u_\infty)$ is zero beyond δ, the integration limit can be taken as δ instead of H. It is assumed that there is no pressure gradient in the boundary layer. The velocity gradient at face bc is zero. So the only force on the control volume surface is

$$-\tau_w\, dx = -\mu \frac{du}{dy}\, dx, \text{ Equating}$$

$$\frac{d}{dx}\left[\int_0^\delta (u_\infty - u)\,\rho u dy\right] = \mu \left.\frac{du}{dy}\right|_{y=0} \qquad (10.1.11)$$

or $$\frac{d}{dx}\left[u_\infty^2 \int_0^\delta \frac{u}{u_\infty}\left(1-\frac{u}{u_\infty}\right)dy\right] = \upsilon \left.\frac{du}{dy}\right|_{y=0} \tag{10.1.12}$$

This is called momentum integral equation. The boundary conditions are

at $y = 0$, $u = 0$; at $y = \delta$, $u = u_\infty$ and $\dfrac{du}{dy} = 0$

Also $\dfrac{d^2u}{dy^2} = 0$ at $y = 0$ (constant pressure gradient)

Equation 10.1.12 can be solved if a velocity profile satisfying the boundary conditions is assumed. Out of the popularly used profiles the results obtained from a cubic profile given below is in closer agreement with the exact solution.

$$\frac{u}{u_\infty} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left[\frac{y}{\delta}\right]^3 \tag{10.1.13}$$

Substituting in equation 10.1.12

$$\frac{d}{dx}\left\{u_\infty{}^2 \int_0^\delta \left[\frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right] \times \left[1 - \frac{3}{2}\frac{y}{\delta} + \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right] dy\right\} = \upsilon \left.\frac{du}{dy}\right|_{y=0}$$

Carrying out the integration, gives

$$\frac{d}{dx}\left[\frac{39}{280}u_\infty^2\,\delta\right] = \frac{3}{2}\upsilon\frac{u_\infty}{\delta} \tag{10.1.14}$$

or $\dfrac{39}{280}u_\infty^2\dfrac{d\delta}{dx} = \dfrac{3}{2}\upsilon\dfrac{u_\infty}{\delta}$, Separating variables and integrating

$$\int_0^x \delta\, d\delta = \int_0^x \frac{140}{13}\frac{\upsilon}{u_\infty}dx \quad \text{at } x = 0,\ \delta = 0. \text{ This leads to}$$

$$\delta = 4.64\, x\sqrt{\frac{\upsilon}{u_\infty x}} = 4.64x/\mathrm{Re}_x^{0.5} \tag{10.1.15}$$

This solution is closer to the exact solution where the constant is 5 instead of 4.64. The value of C_{fx} can be determined using the assumed velocity profile.

$$\frac{u}{u_\infty} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left[\frac{y}{\delta}\right]^3, \quad \therefore \left.\frac{du}{dy}\right|_{y=0} = u_\infty\left[\frac{3}{2\delta}\right] \quad \therefore \quad \tau_w = \mu u_\infty\left[\frac{3}{2\delta}\right]$$

$$C_f = \tau_w/\{(1/2)\,\rho\, u_\infty{}^2\} = \frac{3\mu u_\infty}{2\delta}\frac{2}{\rho u_\infty^2} \text{ As } \delta = 4.64x/\mathrm{Re}^{1/2}$$

$\therefore$ $$C_{fx} = \frac{3\mu u_\infty}{2\times 4.64\times x}\mathrm{Re}_x^{0.5}\frac{2}{\rho u_\infty^2}$$

$$C_{fx} = \frac{3}{4.64}\frac{\mu}{\rho u_\infty x}\mathrm{Re}_x^{0.5} = 0.646/\mathrm{Re}_x^{1/2} \tag{10.1.16}$$

Compared to $0.664/Re_x^{1/2}$ by exact solution.

Due to flexibility this method becomes more versatile as compared to the exact method. Analysis using linear and sine function profiles illustrated under solved problems.

10.1.6 Displacement Thickness

Compared to the thickness δ in free stream, the flow in the boundary layer is reduced due to the reduction in velocity which is the result of viscous forces. In the absence of the boundary layer the flow rate that would pass through the thickness δ will be higher. The idea is illustrated in Fig. 10.1.6.

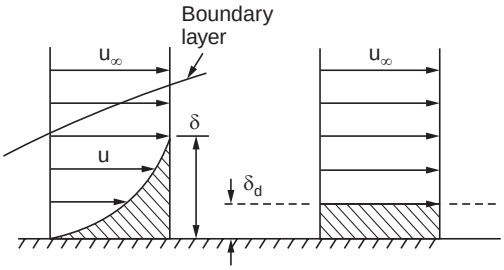


Figure 10.1.6 Displacement thickness

The reduction in volume flow is given by (for unit width)

$$= \int_0^\delta \rho\,(u_\infty - u)\,dy$$

If viscous forces were absent the velocity all through the thickness δ will be equal to u_∞. A thickness δ_d can be defined by equating the reduction in flow to a uniform flow with velocity u_∞ or $\rho u_\infty \delta_d$

$$\delta_d = \int_0^\delta \frac{(u_\infty - u)}{u_\infty}\,dy = \int_0^\delta \left(1 - \frac{u}{u_\infty}\right) dy \qquad (10.1.17)$$

Displacing the boundary by a distance δ_d would pass the flow in the boundary layer at free stream velocity.

Displacement thickness δ_d is the distance by which the solid boundary would have to be displaced in a frictionless flow to give the same mass flow rate as with the boundary layer.

The displacement thickness will equal δ/3. The can be shown by assuming polynomial variation for velocity u in the boundary layer. Assuming (as there are three boundary conditions) the distribution,

$$u = a + by + cy^2, \text{ with boundary conditions,}$$

(*i*) $u = 0$ at $y = 0$, (*ii*) $u = u_\infty$ at $y = \delta$ and $(du/dy) = 0$ at $y = \delta$

The first condition gives $a = 0$ and from the other two conditions

$$c = -u_\infty/\delta^2 \quad \text{and} \quad b = 2u_\infty/\delta$$

Hence the profile is $\frac{u}{u_\infty} = 2\frac{y}{\delta} - \left[\frac{y}{\delta}\right]^2$ (10.1.18)

Note that this is different from the profile previously assumed for the solution of momentum integral equation. Substituting in (10.1.17) and integrating,

$$\delta_d = \int_0^\delta \left(1 - \frac{u}{u_\infty}\right) dy = \int_0^\delta \left\{1 - \frac{2y}{\delta} + \left(\frac{y}{\delta}\right)\right\} dy = \left[y - \frac{y^2}{\delta} - \frac{1}{3}\frac{y^3}{\delta^2}\right]_0^\delta = (1/3)\delta$$

i.e., $\delta_d = \delta/3$ or displacement thickness equals one third of hydrodynamic boundary layer thickness. In case other profiles are adopted, this constant will be different. But this is the value nearer the Blasius solution.

Example 10.3. *Using data of problems* ***Examples 10.1 and 10.2 determine the displacement thickness*** *at the various locations. Also determine the flow out of the boundary layer in the y direction and the average values of velocity v in these sections.*

The deficit flow should go out of the top of the boundary layer. From example 10.1 **air flow at** 30 °C with free stream velocity 5 m/s, (unit width is assumed)

Distance	δ, mm	δ_d, mm	Volume flow, m^3/s	$V_{(0-x)}$, m/s
0.2	4.0	**1.333**	$1.333 \times 5 \times 10^{-3}$	0.0333
0.5	6.325	**2.108**	$2.108 \times 5 \times 10^{-3}$	0.0211
0.8	8.00	**2.666**	$2.666 \times 5 \times 10^{-3}$	0.0167

The volume flow out (deficit flow) equals $\delta_d\, u_\infty \times$ width, assuming 1 m width

$\therefore$ between $x = 0$ and $x = 0.2$ flow is $1.333 \times 5 \times 10^{-3}$ m^3/s.

The average velocity, V = volume/area, Area = 1×0.2 m^2.

$\therefore$ $V = 1.333 \times 5 \times 10^{-3}/0.2 = 0.0333$ m/s. For other lengths values are tabulated above. In the case of **example 10.2, water flow** the values are given below,

Distance, m	δ, mm	δ_d, mm	flow rate, m^3/s	$V_{(0-x)}$, m/s
0.2	5.02	**1.673**	3.35×10^{-4}	1.67×10^{-3}
0.5	7.93	**2.643**	5.29×10^{-4}	1.06×10^{-3}
0.8	10.03	**3.343**	6.69×10^{-4}	0.84×10^{-3}

10.1.7 Momentum Thickness

Similar to the conditions discussed in section (10.1.6) for displacement thickness, there is a reduction in momentum flow through the boundary layer as compared to the momentum flow in a thickness δ at free stream velocity.

The thickness which at free stream velocity will have the same momentum flow as the dificit flow is called momentum thickness. The deficit flow at any thin layer at y of thickness dy is (for unit width) $\rho\,(u_\infty - u)\,dy$

Momentum for this flow is $\rho u\,(u_\infty - u)\,dy$

Hence the deficit momentum = $\int_0^\delta \rho u\,(u_\infty - u)\,dy$

Considering δ_m as momentum thickness,

$$\delta_m\ \rho u_\infty\ u_\infty = \int_0^\delta \rho u\,(u_\infty - u)\,dy$$

$$\delta_m = \int_0^\delta \left[\frac{u}{u_\infty} - \left(\frac{u}{u_\infty}\right)^2\right] dy = \int_0^\delta \frac{u}{u_\infty}\left[1 - \frac{u}{u_\infty}\right] dy \qquad (10.1.19)$$

The concept of reduction in momentum is shown in Fig. 10.1.7.

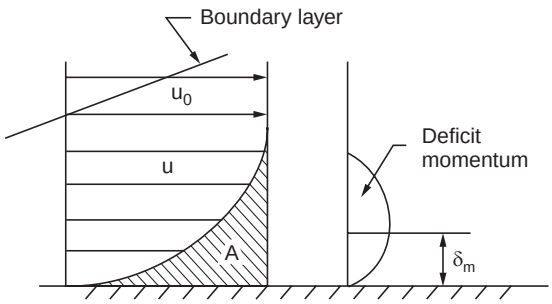


Figure 10.1.7 Momentum thickness

The value of momentum thickness is generally taken as 1/7th of boundary layer thickness in laminar flow. The value will vary with the assumption about velocity distribution. For example if the velocity profiles as in the previous article is used, then

$$\frac{u}{u_\infty} = 2\frac{y}{\delta} - \left[\frac{y}{\delta}\right]^2 \text{ substituting in 10.1.19 and simplyifying}$$

$$\delta_m = \int_0^\delta \left[2\frac{y}{\delta} - 5\left[\frac{y}{\delta}\right]^2 + 4\left[\frac{y}{\delta}\right]^3 - \left[\frac{y}{\delta}\right]^4\right] dy$$

$$= \delta - \frac{5}{3}\delta + \delta - \frac{1}{5}\delta = \frac{2}{15}\delta = \left(\frac{1}{7.5}\right)\delta$$

10.2 TURBULENT FLOW

As flow preceeds farther along the flat plate, inertia forces begin to prevail and viscous forces are unable to keep the flow in an orderly way. **Reynolds number is the ratio of inertia force to viscous force. As inertia force increases Reynolds number increases and the flow becomes turbulent**. Generally the limiting Reynolds number for laminar flow over flat plate is taken as 5×10^5 (for internal flow the critical Reynolds number is 2000).

Turbulent flow is characterized by the variation of velocity with time at any location. The velocity at any location at any time, can be represented by

$$u = \overline{u} + u'$$

where u is the instantaneous velocity, $\overline{u}$ is the average over time and u' is the fluctuating component. The flow is steady as u' is constant at any location. An accurate velocity profile known as universal velocity profile, having different distributions at different heights is available. However it is too cmplex for use with integral method at our level of discussion.

One seventh power law has been adopted as a suitable velocity distribution for turbulent flow.

$$\frac{u}{u_\infty} = \left(\frac{y}{\delta}\right)^{1/7} \tag{10.2.1}$$

Substituting in the integral momentum equation 10.1.2, boundary layer thickness is obtained as

$$\delta = 0.382\, x/\mathrm{Re}_x^{0.2} \tag{10.2.2}$$

For combined laminar and turbulent flow,

$$\delta_L = (0.381x/\mathrm{Re}_L^{0.2}) - (10256/\mathrm{Re}_L) \tag{10.2.2 a}$$

The friction coefficient is obtianed as

$$C_{fx} = 0.0594/\mathrm{Re}_x^{0.2} \tag{10.2.3}$$

for combined laminar turbulent flow

$$C_{fL} = 0.074\mathrm{Re}^{-0.2} - 1742\mathrm{Re}_L^{-1} \tag{10.2.4}$$

Displacement thickness is obtained as $\delta_d = \delta/8$

Example 10.4. ***Water flows*** *at a velocity of 1.2 m/s over a flat plate 1.2 m long. Assume 1/7th power law and* ***determine the boundary layer thickness and displacement thickness.*** *Compare the values with values calculated using laminar flow correlations.*

$v = 1.006 \times 10^{-6}\ m^2/s.$

$$\mathrm{Re} = \frac{ux}{v} = \frac{1.2 \times 1.2}{1.006 \times 10^{-6}} = 1.43 \times 10^6 > 5 \times 10^5 \text{ So the flow is turbulent}$$

$\boldsymbol{\delta_L} = 0.382x/Re_L^{0.2} = 0.0269\ m$ or **26.9 mm**

$$\delta_d = \int_0^\delta \left(1 - \frac{u}{u_\infty}\right) dy = \int_0^\delta \left(1 - \left(\frac{y}{\delta}\right)^{1/7}\right) dy = \left[y - \frac{7}{8}\frac{y^{1+\frac{1}{7}}}{\delta^{1/7}}\right]_0^\partial = \frac{1}{8}\delta$$

$\delta_{dL} = 26.9/8 = 3.37$ mm, $C_f = 0.0594/\mathrm{Re}^{0.2} = 0.003488$

$\tau_w = C_f(1/2)\ \rho\ u_\infty^2 = (0.003488/2) \times 1000 \times 1.2^2 = 2.51\ \mathrm{N/m^2}$

In case laminar flow correlations were used:

$\delta = 5x/\mathrm{Re}^{0.5} = 0.005$ m or 5.0 mm (about 1/5th)

$\delta_d = \delta/7 = 0.72$ mm, $C_f = 0.664/\mathrm{Re}^{0.5} = 5.55 \times 10^{-4}$

$\tau = 5.55 \times 10^{-4} \times 0.5 \times 1000 \times 1.2^2 = 0.40\ \mathrm{N/m^2}$

The boundary layer is thicker and shear stress is higher in turbulent flow.

10.3 FLOW SEPARATION IN BOUNDARY LAYERS

Boundary layer is formed in the case of flow of real fluids. Viscous forces exist in such flows. The shear stress at the wall is given by

$$\tau_w = \mu \left.\frac{du}{dy}\right|_{y=0}$$, The wall shear cannot be zero. Hence at $y = 0$, $\frac{du}{dy}$

cannot be zero. This means that the velocity gradient at the wall cannot be zero.

Separation of flow is said to occur when the direction of the flow velocity near the surface is opposed to the direction of the free stream velocity, which means $(du/dy) \leq 0$. Such a situation does not arise when there is no pressure gradient opposed to the flow direction, *ie.,* the pressure downstream of flow is higher compared to the pressure upstream. An example is subsonic diffuser. In the direction of flow the pressure increases. The increase in area along the flow causes a pressure rise.

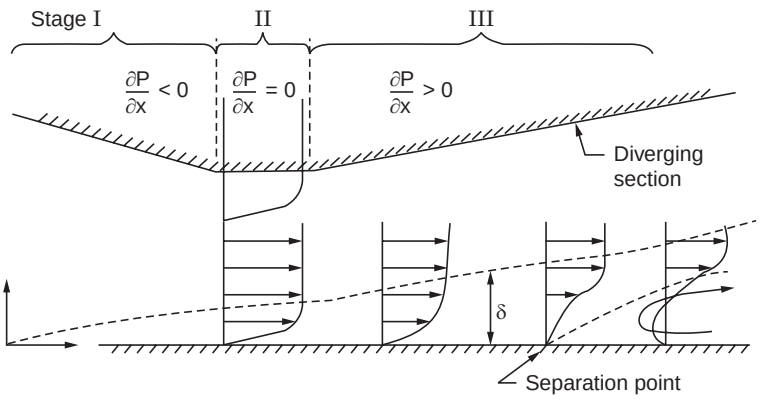


Figure 10.3.1 Flow separation

If (dp/dx) increases to the extent that it can overcome the shear near the surface, then separation will occur. Such a pressure gradient is called adverse pressure gradient. In the case of incompressible flow in a nozzle a favorable pressure gradient exists. Separation will not occur in such flows. In the case of diverging section of a diffuser, separation can occur if the rate of area increase is large. This is shown in Fig. 10.3.1. In turbulent flow, the momentum near the surface is high compared to laminar flow. Hence turbulent layer is able to resist separation better than laminar layer.

In the case of flow over spheres, cylinders, blunt bodies, airfoils etc., there is a change in flow area due to the obstruction and hence an adverse pressure gradient may be produced. Simple analytical solutions are not available to determine exactly at what conditions separation will occur. Experimental results are used to predict such conditions.

10.3.1 Flow Around Immersed Bodies – Drag and Lift

When fluid flows around a body or the body moves in a fluid there is a relative motion between the fluid and the body. The body will experience a force in such a situation. In the case of a flat plate positioned parallel to the direction of the flow, the force is parallel to the surface.

But generally in the case of blunt bodies, the force will neither be paraller nor perpendicular to the surface. The force can be resolved into two components one parallel to the flow and the other perpendicular to the flow. The former may be called shear force and the other, the pressure force.

The component parallel to the direction of motion is called drag force F_D and the component perpendicular to the direction of motion is called lift force, F_L. Determination of these forces is very important in many applications, an obvious example being aircraft wings. Simple analytical methods are found to be insufficient for the determination of such forces. So experimentally measured coefficients are used to compute drag and lift.

10.3.2 Drag Force and Coefficient of Drag

Drag is the component of force acting parallel to the direction of motion. Using the method of dimensional analysis the drage force can be related to flow Reynolds number by

$$\frac{F_D}{\rho A V^2} = f(\text{Re}) \tag{10.3.1}$$

For generality velocity is indicated as V

Defining coefficient of drag as the ratio of drag to dynamic pressure, it is seen that

$$C_D = f(\text{Re}),$$

$$C_D = \frac{F_D}{(1/2)\rho A V^2} \tag{10.3.2}$$

This applies to viscous drag only. In case wave drag is encountered, then

$$C_D = f(Re, Fr) \tag{10.3.3}$$

If compressibility effect is to be considered

$$C_D = f(Re, M) \tag{10.3.4}$$

Friction coefficient over flat plate in laminar flow, at a location was defined by $C_{fx} = \tau_w /(1/2)\rho A V^2 = 0.664/Re_x^{0.5}$. Over a given length the average value is obtained as twice this value. For a flat plate of length L, in laminar flow

$$C_D = 1.328/Re_L^{0.5} \tag{10.3.5}$$

In turbulent flow in the range $5 \times 10^5 > Re < 10^7$

$$C_D = 0.074/ Re_L^{0.2} \tag{10.3.6}$$

For Re_L up to 10^9, an empirical correlation due to Schlichting is

$$C_D = 0.455/(\log Re_L)^{2.58} \tag{10.3.7}$$

For combined laminar and turbulent flow in the range $5 \times 10^5 > Re < 10^7$

$$C_D = \frac{0.074}{\text{Re}_L^{0.2}} - \frac{1740}{\text{Re}_L} \tag{10.3.8}$$

For the range $5 \times 10^5 > Re < 10^9$

$$C_D = \frac{0.455}{(\log \text{Re}_L)^{2.58}} - \frac{1610}{\text{Re}_L} \tag{10.3.9}$$

The values of C_D for laminar flow is in the range 0.002 to 0.004.

Example 10.5. *A ship having a wetted perimeter of 50 m and length of 140 m is to travel at 5 m/s.* ***Determine the power required to overcome the skin friction****. Assume kinematic viscosity* $v = 1.4 \times 10^{-6}$ *m²/s. Density 1025 kg/m³*

$$Re = 5 \times 140/1.4 \times 10^{-6} = 0.5 \times 10^9,$$

So the equation applicable is 10.3.9

$$C_D = \frac{0.455}{(\log 0.5 \times 10^9)^{2.58}} - \frac{1610}{0.5 \times 10^9} = 1.719 \times 10^{-3}$$

$$F_D = C_D A(1/2)\, \rho u^2 = (1.7179 \times 10^{-3})\,(1/2) \times 140 \times 50 \times 1025 \times 5^2 \text{ N}$$

$$= \mathbf{0.154 \times 10^6\ N}$$

$\therefore$ **Power =** $F_D\, u = 0.154 \times 10^6 \times 5 = 0.77 \times 10^6$ W = **0.77 MW**

10.3.3 Pressure Drag

When flow is perpendicular to blunt objects, like a plate or a disk, shear does not contribute to drag force. The drag is then mainly due to pressure difference between the faces. So it is called pressure drag. The drag coefficient is based on the frontal area (or projected area) of the object. In the case of airfoils the plan area is the basis for drag coefficient. The drag coefficient for same geometries are shown in Table 10.3.1 below. These are applicable for $Re > 10^3$.

Table 10.3.1 Drag coefficients for various shapes

Shape	C_D
Square plate	1.18
Rectangle 1:5	1.20
Cube	1.05
Disk	1.17
Hemisphere facing flow	1.42
Parachute	1.20
Hemisphere facing downstream	0.38

It may be seen that the coefficient of pressure drag is independent of Reynolds number.

Example 10.6. *A drag chute is used to slowdown a car with a mass 1800 kg travelling at 60 m/s. The value of coefficient of drag for the car is 0.32 and frontal area is 1.1 m². The chute is of 1.8 m* ***diameter and drag coefficient is*** *1.2 Density of air = 1.2 kg/m³. Determine the speed after 50 secs. Also determine the time for the speed to reach 20 m/s.*

The total drag force at any instant for the car and the chute is given by (subscript C refers to car and P refers to parachute)

$$F_D = \frac{1}{2}\, \rho u^2\, [C_{DC} A_C + C_{DP} A_P]$$ and this force acts to decelerate the car.

Force = mass × Acceleration = $m(du/dt)$ ∴ (du/dt) = force/mass

$$\therefore \quad \frac{du}{dt} = \frac{F_D}{m} = -\frac{k}{m}u^2 \text{ where } k = \frac{\rho}{2}\,[C_{DC}A_C + C_{DP}A_P]$$

Separating variables and integrating,

$$\int_{u_0}^{u} \frac{du}{u^2} = -\frac{k}{m}\int_0^t dt$$

$$\therefore \quad \frac{1}{u_0} - \frac{1}{u} = -\frac{k}{m}t$$

$$\therefore \quad u = \frac{u_0}{1+\left(\frac{k}{m}\right)u_0 t} \qquad \ldots (A)$$

$$k = \frac{1.2}{2}\,[(0.32 \times 1.1) + (1.2 \times \pi \times 1.8^2/4)] = 2.0433$$

$$\therefore \quad (k/m)u_0 = (2.0433 \times 60)/1800 = 0.06811$$

∴ (*i*) After 50 seconds, **u** = 60/(1 + 0.06811 × 50) = **13.62 m/s**

(*ii*) For u = 20, 20 = 60/(1 + 0.06811 × t) ∴ **t = 29.36 seconds**

The distance travelled can be obtained by integrating $u\,dt$.

$$\therefore \quad s = \int_0^t u dt = \int_0^t \frac{u_0 dt}{1+(k/m)u_0 t} = \frac{u_0}{(k/m)u_0}\ln\,[1 + (k/\text{m})u_0 t] = (m/k)\ln\,[1 + (\text{k/m})u_0 t]$$

At t = 50 sec $s = \dfrac{1800}{2.0433}\ln(1 + 0.06811 \times 50) = 1306$ m

At t = 29.36 sec $s = \dfrac{1800}{2.0433}\,\text{In}\,(1 + 0.06811 \times 29.36) = 968$ m

10.3.4 Flow Over Spheres and Cylinders

In these cases both pressure and friction drag contribute to the total drag. The flow separation at the rear and formation of wake contributes to the pressure drag. The flow pattern and the variation of drag coefficient is shown in Fig. 10.3.2. It may be noted that the coefficient of drag is nearly constant from $Re = 10^3$ to 5×10^5. From experiments the boundary layer in the forward portion is found to be laminar in this range. Separation is found to occur at about mid section and a wide wake is found to exist with pressure in the wake below that at the front.

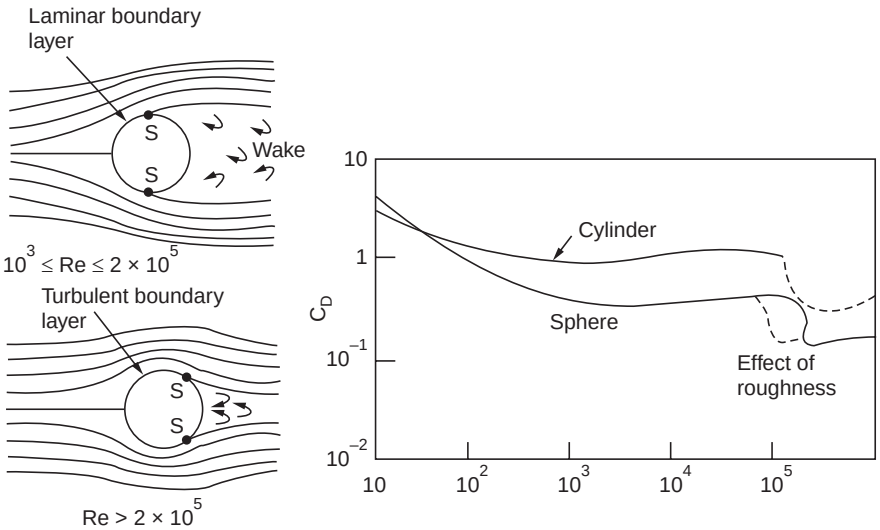


Figure 10.3.2 Flow separation in flow over cylinder/sphere

There is a sharp drop in the value of C_D after the critical Reynolds number. The flow in the forward side is found to turn turbulent and separation moves downstream and wake is now narrow, reducing the net pressure drag leading to the abrupt decrease in the drag coefficient. Turbulent layer has a higher momentum near the surface resisting separation.

Separation can be reduced by streamlining the body shape, reducing the pressure drag. This generally increases the area thus increasing friction drag. An optimum streamlined shape is the one which gives minimum total drag. Stream lining is now adopted not only for aircrafts but almost for all transport vehicles.

Example 10.7. *A model of a bathysphere 50 mm diameter is towed under water at a speed of 1 m/s.* ***Determine the tension in the towline****. Density of water = 1020 kg/m³. Kinematic viscosity = 1.006 × 10⁻⁶ m²/s*

$$Re = uD/v = 1 \times 0.05/1.006 \times 10^{-6} = 4.97 \times 10^4$$

From graph (Fig. 10.3.2) C_D is read as 0.45

$$\therefore \quad \mathbf{F_D} = C_D\,(1/2)\,\rho\,Au^2 = \frac{0.45}{2} \times 1020 \times \frac{\pi \times 0.05^2}{4} \times 1^2 = \mathbf{0.45\ N}$$

10.3.5 Lift and Coefficient of Lift

The force on an immersed body moving in a fluid can be resolved into two components. The component along the flow direction is called drag. The component perpendicular to the flow direction is called lift. The lift on airfoil is an example. The coefficient of lift is defined by

$$C_L = \frac{F_L}{(1/2)\,\rho\,Au^2} \tag{10.3.10}$$

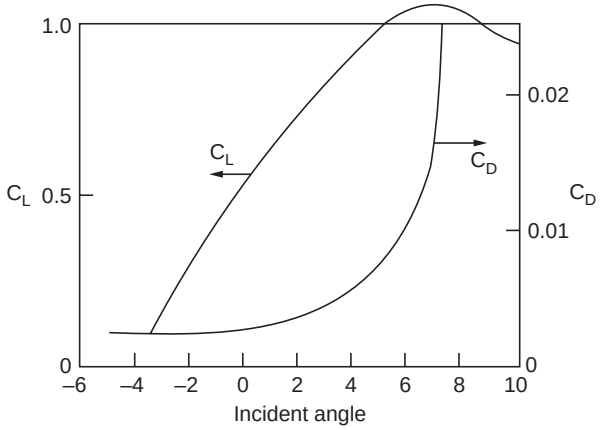


Figure 10.3.3 Variation of Lift and Drag on an airfoil

Lift is of interest mainly in the design of airfoil sections. Airfoil blade shapes are also used in turbomachines. The lift and drag coefficients depend on the Reynolds number and angle of attack. The angle between the airfoil chord and the flow direction is called angle of attack. The chord of an airfoil is the line joining the leading edge and the trailing edge. The planform area (the maximum projected area) is used in the definition of lift and drag coefficients. A typical plot of the variation of lift and drag coefficients with angle of attack for a specified Reynolds number is shown in Fig. 10.3.3.

For each airfoil section such plots are available. Flow separation will result in sudden drop in the lift, known as stall. Presently computer softwares are available for the design of airfoil sections with a very high ratio of lift to drag. These data are for long spans and corrections should be made as per the aspect ratio defined by b^2/A_p. where b is the span length and A_p is the planform area. This will equal the ratio (span/chord) as, $A_p = bc$. The lift to drag ratio varies from 20 to 40 with the lower value applicable for small planes.

10.3.6 Rotating Sphere and Cylinder

In order to reduce skin friction in flow over surfaces, particularly curved surfaces boundary layer control is used. One method of boundary layer control is by the use of moving surfaces at locations where separation may start. This is difficult to apply due to mechanical restrictions. However this principle is used in sports like baseball, golf, cricket and tennis where spin is applied to control the trajectory of the ball. Spin also provides significant aerodynamic lift to increase the distance travelled by the ball. Spin can also be used to obtain a curved path of travel for the ball.

Spin alters the pressure distribution and also the location of boundary layer separation. For spin along the flow direction, separation is delayed on the upper surface and it occurs earlier in the lower surface. Pressure is reduced on the upper surface and is increased on the lower surface and the wake is deflected downwards.

The coefficients of lift and drag are found to be a function of ω D/2u called spin ratio.

Example 10.8. ***Show*** *using dimensional analysis that the* ***lift and drag coefficients are functions of spin ratio and Reynolds number***.

The variables affecting the phenomenon are listed below. As C_L and C_D are dimensionless, these are not listed.

No	Variable	Unit	Dimension
1	Linear Velocity, u	m/s	L/T
2	Radius, R	m	L
3	Angular velocity, ω	Radians/s	$1/T$
4	Kinematic viscosity, v	m^2/s	L^2/T

There are four variables and two dimensions, namely L and T. Hence two π terms can be identified. Choosing linear velocity and radius as repeating variables

Let $$\pi_1 = \omega\, u^a R^b \text{ or } L^0T^0 = \frac{1}{T}\frac{L^a}{T^a} L^b$$

$\therefore$ $$a + b = 0, -1 - a = 0 \quad \therefore \quad a = -1 \quad \therefore \quad b = 1$$

$$\pi_1 = \omega R/u = \omega R/2u, \text{ called } \textbf{spin ratio.}$$

Let $$\pi_2 = vu^a R^b \quad \text{or} \quad L^0T^0 = \frac{L^2}{T}\frac{L^a}{T^a} L^b, \quad \therefore \quad 2 + a + b = 0, -1 - a = 0$$

$\therefore$ $$a = -1 \quad b = -1 \quad \pi_2 = \frac{v}{uR} \quad \text{or} \quad \frac{uD}{v}, \text{ Reynolds number.}$$

Hence $$\mathbf{C_L = f\left[\frac{\omega D}{2u}, \frac{uD}{v}\right]} \text{ and } \mathbf{C_D = f\left[\frac{\omega D}{2u}, \frac{uD}{v}\right]}$$

The variation of C_L and C_D are found to be influenced more by spin ratio than Reynolds number. The trend is shown in Fig. Ex. 10.8. In the case of cylinders the area for definition of C_L and C_D is $L \times D$

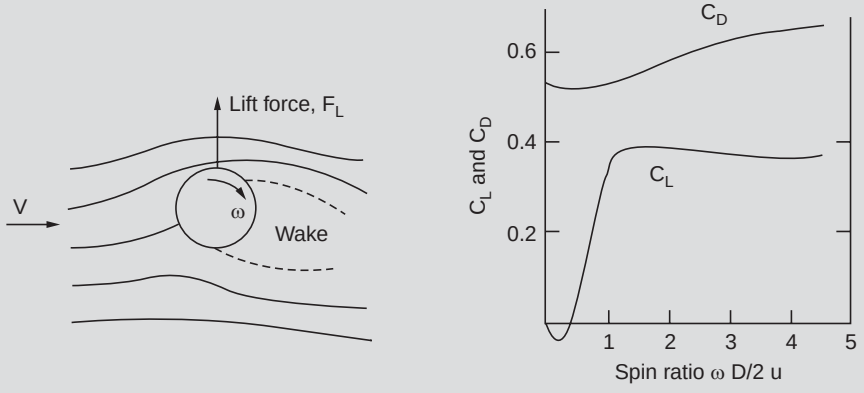


Figure Ex. 10.8 Variation of Lift and Drag with spin ratio

A force perpendicular to both direction of motion and the spin axis is created during the flight. This is known as MAGNUS effect. This can cause drift in the flight path.

SOLVED PROBLEMS

Problem 10.1 *Assuming **linear velocity variation in the boundary layer** and using linear momentum integral equation, determine the thickness of the boundary layer. Also determine the friction coefficient and the displacement and momentum thicknesses.*

Momentum integral equation is

$$\frac{d}{dx}\left[\int_0^{\delta} u(u_\infty - u)dy\right] = \upsilon \left.\frac{du}{dy}\right|_{y=0} . \text{ As } \frac{u}{u_\infty} = \frac{y}{\delta} \quad \therefore \quad u = \frac{u_\infty y}{\delta},$$

$$\frac{du}{dy} = \frac{u_\infty}{\delta} \quad \therefore \tau = \frac{\mu u_\infty}{\delta}, \text{ Considering the integral part}$$

$$\int_0^{\delta}\left[\frac{u_\infty^2}{\delta} y - \frac{u_\infty^2}{\delta^2} y^2\right] dy = \left[\frac{u_\infty^2 \delta}{2} - \frac{u_\infty^2 \delta}{3}\right] = \frac{1}{6} u_\infty^2 \delta$$

$$\therefore \qquad \frac{d}{dx}\left[\frac{u_\infty^2}{6}\delta\right] = \upsilon \frac{u_\infty}{\delta} \text{ or } \frac{u_\infty^2}{6}\frac{d\delta}{dx} = \upsilon \frac{u_\infty}{\delta}$$

Separating variables and integrating, $\delta\, d\delta = (6\upsilon/u_\infty)\, dx$

$$\delta^2 = (12\upsilon x)/u_\infty = 12x^2/(\upsilon/u_\infty x) = 12x^2/\mathrm{Re}_x$$

$$\therefore \qquad \boldsymbol{\delta = 3.464x/Re_x^{0.5}},$$

The constant is 3.464 instead of 5 in the exact solution

$$\mathbf{C_{fx}} = \frac{\tau}{(1/2)\rho u_\infty^2} = \frac{2\mu u_\infty}{\rho u_\infty^2 \delta} = \frac{2\upsilon \mathrm{Re}_x^{0.5}}{u_\infty 3.464x} = \mathbf{0.577/Re_x^{0.5}}$$

The displacement thickness

$$\delta_d = \int_0^{\delta}\left[1 - \frac{u}{u_\infty}\right] dy = \int_0^{\delta}\left[1 - \frac{y}{\delta}\right] dy = (1/2)\delta \text{ or } \delta/2. \text{ As against } \delta/3$$

Momentum thickness is given by

$$\delta_m = \int_0^{\delta} \frac{u}{u_\infty}\left[1 - \frac{u}{u_\infty}\right] dy = \int_0^{\delta}\left[\frac{y}{\delta} - \frac{y^2}{\delta^2}\right] dy = \frac{1}{2}\delta - \frac{1}{3}\delta = \frac{1}{6}\delta$$

By the exact solution, $\delta_m = (1/7)\delta$

Problem 10.2 *Assuming **second degree velocity distribution in** the boundary layer determine using the integral momentum equation, the thickness of boundary layer friction coefficient, displacement and momentum thicknesses.*

Let $u = a + by + cy^2$. The boundary conditions are $u = 0$ at $y = 0$,

At $y = \delta$, $\dfrac{du}{dy} = 0$, and $u = u_\infty$. This gives $\dfrac{u}{u_\infty} = 2\dfrac{y}{\delta} - \left(\dfrac{y}{\delta}\right)^2$ (Refer 10.1.18)

Substituting in the integral momentum equation,

$$\frac{d}{dx}\left[\int_0^{\delta} u(u_\infty - u)dy\right] = v\left.\frac{du}{dy}\right|_{y=0},$$

$$u = u_\infty\left[2\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2\right], \left.\frac{du}{dy}\right|_{y=0} = 2u_\infty/\delta,\ \tau = 2\mu u_\infty/\delta$$

Considering the integral part,

$$u_\infty^2\int_0^{\delta}\left[\left(\frac{u}{u_\infty}\right) - \left(\frac{u}{u_\infty}\right)^2\right]dy = u_\infty^2\int_0^{\delta}\left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 - 4\left(\frac{y}{\delta}\right)^2 + 4\left(\frac{y}{\delta}\right)^3 - \left(\frac{y}{\delta}\right)^4\right]dy$$

$$= u_\infty^2\int_0^{\delta}\left[2\left(\frac{y}{\delta}\right) - 5\left(\frac{y}{\delta}\right)^2 + 4\left(\frac{y}{\delta}\right)^3 - \left(\frac{y}{\delta}\right)^4\right]dy$$

$$= u_\infty^2\left[\delta - \frac{5}{3}\delta + \delta - \frac{1}{5}\delta\right] = \frac{2}{15}u_\infty^2\delta \qquad \ldots (A)$$

Substituting

$$\frac{d}{dx}\left[\frac{2}{15}u_\infty^2\delta\right] = 2vu_\infty/\delta \quad \text{or} \quad \frac{2}{15}u_\infty^2 = \frac{d\delta}{dx} = 2\,vu_\infty/\delta$$

$\therefore$ $\delta d\delta = 15(v/u_\infty)dx$ Integrating

$$\delta^2 = 30vx/u_\infty,\ 30\left(\frac{v}{u_\infty x}\right)x^2 = 30x^2/\mathrm{Re}_x,\ \mathbf{\delta = 5.477x/Re_x^{0.5}}$$

Note that the constant is 5.477 as against 5 by exact solution.

As $\tau = 2\mu u_\infty/\delta$ and $\delta = 5.477\ x/\mathrm{Re}_x^{0.5}$

$$\mathbf{C_{fx}} = \frac{\tau}{(1/2)\rho u_\infty^2} = \frac{4\mu\,u_\infty}{\rho u_\infty^2\,\delta} = \frac{4v}{u_\infty\,\delta} = \frac{4v\,\mathrm{Re}_x^{0.5}}{5.477\,xu_\infty} = \mathbf{0.73/Re_x^{0.5}}$$

instead of $0.644/\mathrm{Re}_x^{0.5}$

$$\delta_d = \int_0^{\delta}\left[1 - \frac{u}{u_\infty}\right]dy = \int_0^{\delta}\left[1 - 2\left(\frac{y}{\delta}\right) + \left(\frac{y}{\delta}\right)^2\right]dy = \frac{\delta}{3}$$

$$\delta_m = \int_0^{\delta}\left[\frac{u}{u_\infty} - \left(\frac{u}{u_\infty}\right)^2\right]dy = \frac{2}{15}\delta \quad \text{(see equation } A\text{)}$$

Problem 10.3 *Assuming the velocity distribution in the boundary layer as* $\frac{u}{u_\infty} = \sin\left(\frac{\pi y}{2\delta}\right)$ *(in the range* $0 \le y \le \delta$*, and* $u/u_\infty = 1$ *beyond* δ*) determine the thickness of the boundary layer, using integral momentum method (Refer equation 10.1.12).*

$$\frac{u}{u_\infty} = \sin\left(\frac{\pi y}{2\delta}\right), \frac{du}{dy} = \frac{\pi}{2\delta}\cos\frac{\pi y}{2\delta} \text{ at } y = 0,\ \frac{du}{dy} = u_\infty\ (\pi/2\delta)$$

$$\frac{d}{dx}\left[\int_0^{\delta} u(u_\infty - u)dy\right] = v\frac{du}{dy}_{y=0}$$

Considering the integral part and substituting the velocity distribution,

$$u_\delta^2 \int_0^{\delta}\left[\left(\frac{u}{u_\infty}\right) - \left(\frac{u}{u_\infty}\right)^2\right]dy = u_\delta^2 \int_0^{\delta}\left[\sin\frac{\pi y}{2\delta} - \sin^2\frac{\pi y}{2\delta}\right]dy$$

Noting $\int \sin^2 ax = \frac{x}{2} - \frac{\sin 2ax}{4a}$

$$= u_\infty^2\left[-\frac{2\delta}{\pi}\cos\frac{\pi y}{2\delta} - \frac{y}{2} + \frac{\delta}{2\pi}\sin\frac{\pi y}{2\pi}\right]_0^{\delta}$$

$$= u_\infty^2\left[0 - \frac{\delta}{2} + 0\right] - \left[-\frac{2\delta}{\pi} - 0 + 0\right] = 0.1366 \times u_\infty{}^2 \times \delta \qquad \ldots (A)$$

$$\frac{d}{dx}[u_\infty^2 \times 0.1366\,\delta] = \frac{\pi}{2\delta}\,u_\infty v,\ [u_\infty^2 \times 0.1366]\frac{d\delta}{dx} = \frac{\pi}{2\delta}\,u_\infty v$$

$\therefore$ $$\delta\, d\delta = \frac{\pi}{2\times 0.1366}\frac{v}{u_\infty}dx$$

Integrating $$\frac{\delta^2}{2} = \frac{\pi}{2\times 0.1366}\frac{vx}{u_\infty} \text{ or } \mathbf{\delta = 4.8x/Re_x^{0.5}}$$

$\mathbf{C_{fx}} = \tau_w/(1/2)\rho\, u_\infty{}^2$, $\tau_w = \mu\,(du/dy)$, at $y = 0$, $\tau_w = \mu\, u_\infty \pi/2\delta$

$\therefore$ $$\mathbf{C_{fx}} = \frac{2\mu u_\infty \pi}{2\delta\rho u_\infty^2} = \frac{\pi}{\delta}\frac{v}{u_\infty} = \frac{\pi}{4.8}\frac{v}{u_\infty x}\mathrm{Re}_x^{0.5} = \mathbf{0.655/Re_x^{0.5}}$$

$$\delta_d = \int_0^{\delta}\left[1 - \frac{u}{u_\infty}\right]dy = \int_0^{\delta}\left[1 - \sin\left(\frac{\pi y}{2\delta}\right)\right]dy = \left[\delta + \frac{2\delta}{\pi}\cos\frac{\pi y}{\delta}\right]_0^{\delta}$$

$= [\delta + 0] - [0 + (2\delta/\pi)] = 0.3625\ \delta = \delta/2.76$, instead of $\delta/3$

$\delta_m = 0.1366\ \delta$, or $\delta/7.32$ (refer result A)

Problem 10.4 *Using the* ***cubic velocity profile*** *determine upto a length* ***L the flow out of the boundary*** *layer in terms of the boundary layer thicknes.*

The free stream flow for thickness of δ is $\rho\, u_\infty\, \delta$.

Assuming cubic velocity profile,

$$u = u_\infty\left[\frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right]$$

Mass flow through the boundary layer

$$= \int_0^{\delta}\rho u dy = \int_0^{\delta}\rho u_\infty\left[\frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right]dy = \rho u_\infty\left[\frac{3}{2}\frac{y^2}{\delta} - \frac{1}{8}\frac{y^4}{\delta^3}\right]_0^{\delta} = \frac{5}{8}\rho u_\infty \delta$$

$\therefore$ Mass flow out of the boundary layer = (1–(5/8)) $\rho\, u_\infty\, \delta$ = 3/8 $\rho\, u_\infty\, \delta$ or displacement thickness times the free stream flow. (Nota : δ_d = (3/8) δ for cubic profile)

The average velocity in the y direction can be obtained by dividing the volume flow by area *i.e.,* 1 × x for unit width. **Volume flow out of the boundary**

$$\mathbf{v} = (3/8)\, u_\infty\, \delta, \text{ velocity} = \frac{3}{8} u_\infty \frac{4.64x}{\mathrm{Re}_L^{0.5}} \frac{1}{x} = \frac{1.74 u_\infty}{\mathrm{Re}_L^{0.5}}$$

This will be low as Reynolds number will be high.

Consider the data from example (10.1) Air flow, u_∞ = 5 m/s, at a distance 0.5 m,

$$\mathrm{Re} = 1.56 \times 10^5 \quad \therefore \mathbf{v} = (1.74 \times 5)/(1.56 \times 10^5)^{0.5} = \mathbf{0.022\ m/s}$$

This can also be calculated in a round about way using the continuity equation $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$. The value of $\frac{\partial u}{\partial x}$ can be obtained from the assumed profile and then equated to $-\frac{\partial v}{\partial y}$. Integrating the same between 0 and δ the same result will be obtained. [Refer Problem 10.6].

Problem 10.5 *Using the continuity and momentum equations show that at y = 0,* $-\frac{\partial^3 u}{\partial y^3} = 0$. *Deduce from the above that the cubic profile is approximate.*

Consider the x directional momentum equation, $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = v\frac{\partial^2 u}{\partial y^2}$. Differentiating with resect to y,

$$\frac{\partial u}{\partial y}\frac{\partial u}{\partial x} + u\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} + v\frac{\partial^2 u}{\partial y^2} = v\frac{\partial^3 u}{\partial y^3}, \text{ Simplifying.}$$

$$\frac{\partial u}{\partial y}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] + u\frac{\partial^2 u}{\partial x \partial y} + v\frac{\partial^2 u}{\partial y^2} = v\frac{\partial^3 u}{\partial y^3}$$

The first term is zero due to continuity equation. At y = 0, u = 0 and v = 0. Hence the second and third terms are also zero. So $\frac{\partial^3 u}{\partial y^3}$ should be zero.

Consider the cubic profile:

$$\frac{u}{u_\infty} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$$

$$\therefore \quad \frac{\partial u}{\partial y} = u_\infty\left[\frac{3}{2}\frac{1}{\delta} - \frac{3}{2}\frac{y^2}{\delta^3}\right] \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = u_\infty\left[-\frac{6}{2}\frac{y}{\delta^3}\right] \frac{\partial^3 u}{\partial y^2} = -3u_\infty/\delta^3$$

This is not zero. Hence profile assume is approximate.

Problem 10.6 *Derive a general expression* ***for the y directional velocity at a location*** *x in the boundary layer in flow over a flat plate. Indicate at what y location this will be maximum. Assume cubic velocity variation.*

Consider continuity equation.

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad \frac{\partial v}{\partial y}=-\frac{\partial u}{\partial x}$$

$$u = u_\infty\left[\frac{3}{2}\frac{y}{\delta}-\frac{1}{2}\left(\frac{y}{\delta}\right)^3\right],\ \delta = 5x/\mathrm{Re}_x^{1/2} = \frac{5xv^{1/2}}{u_\infty^{1/2}x^{1/2}} = cx^{1/2}$$

where $c = [5v^{1/2}/u_\infty^{1/2}]$ substituting and putting $c_1 = 3u_\infty/2\ \delta$ and $c_2 = u_\infty/2\ \delta^3$, velocity expression reduces to $u = c_1 y\, x^{-1/2} - c_2 y^3 x^{-3/2}$, $\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -[c_1 \times (-1/2) \times y\, x^{-3/2} + (3/2)\, c_2\, y^3\, x^{-5/2}]$

Integrating w.r.t. y, and substituting for c_1 and c_2

$$v = \frac{3}{8c}\frac{u_\infty}{x^{3/2}}y^2 - \frac{3u_\infty}{16c^3}\frac{y^4}{x^{5/2}},\ \text{Substituting for } c$$

$$v = \frac{3}{8}\frac{u_\infty u_\infty^{1/2}y^2}{5v^{1/2}x^{3/2}} - \frac{3\times u_\infty^{3/2}}{16\times 125v^{3/2}}\frac{u_\infty y^4}{x^{5/2}}$$

Substituting $\frac{5xv^{1/2}}{u_\infty^{1/2}x^{1/2}} = \delta$

$$v = \frac{3}{8}u_\infty\frac{y^2}{\delta x} - \frac{3}{16}u_\infty\frac{y^4}{\delta^3 x} = \frac{3}{8}\frac{u_\infty}{\delta x}\left[y^2 - \frac{1}{2}\frac{y^4}{\delta^2}\right]$$

(Check for dimensional consistency : dimensions of $y^2/\delta x$ and $y^4/\delta^3 x$ cancel and v has the same unit as u_∞)

Maximum value occurs when $\frac{\partial v}{\partial y} = 0$.

$\frac{\partial}{\partial y}\left[y^2 - \frac{1}{2\delta^2}y^4\right] = 2y - \frac{1}{2\delta^2}4y^3$. Equating to zero and solving $y = \delta$

This is physically explainable as the total flow in y direction should occur at

$y = \delta$. Velocity at $y = \delta$ is $\mathbf{v_{\delta x}} = \frac{3}{16}\frac{u_\infty \delta}{x} = \mathbf{0.87}\ \frac{\mathbf{u_\infty}}{\mathbf{Re_x^{0.5}}}$

Total mass flow when integrated over the length will equal (3/8) $\rho\ u_\infty\ \delta_L$ (Refer Problem 10.4).

Problem 10.7 *The shear at a location 2 m from the leading edge of a flat plate was measured as 2.1 N/m^2. Assuming the flow to be turbulent from the start determine if air at 20°C was flowing over the plate (i) the velocity of air (ii) the boundary layer thickness and (iii) the velocity at 15 mm above the plate. $\rho = 1.205$ kg/m^3, $v = 15.06 \times 10^{-6}$ m^2/s*

Using equation (10.2.3) $C_{fx} = 0.0594/\mathrm{Re}_x^{0.2}$, $\tau_w = C_{fx}\,(1/2)\,\rho\, u^2$,

$$\text{Equating } 2.1 = \frac{0.0594 \times (15.06 \times 10^{-6})^{0.2}}{u_\infty^{0.2}\, 2^{0.2}} \times \frac{1}{2} \times 1.205\, u_\infty{}^2, \text{ Solving,}$$

$$u_\infty{}^{1.8} = 621.04 \text{ or } u_\infty = 35.623 \text{ m/s}$$

$$\text{Re} = 35.623 \times 2/15.06 \times 10^{-6} = 4.808 \times 10^6,$$

Turbulent hence the use of equation (10.2.3) is justified. Using equation (10.2.1),

$$\delta = 0.382x/\text{Re}_x{}^{0.2} = 0.382 \times 2/(4.808 \times 10^6)^{0.2}$$

$$= 0.0352 \text{ m or } \delta = 35.2 \text{ mm}$$

If the velocity profile is assumed as

$$\frac{u}{u_\infty} = \left(\frac{y}{\delta}\right)^{1/7}$$

$$\therefore \qquad \mathbf{u} = 35.623\ (15/35.2)^{1/7} = \mathbf{32.05\ m/s}$$

Problem 10.8 ***Determine the length at which the flow over a flat plate will turn turbulent for air****, water and engine oil if the flow velocity is 3 m / s. Also determine the boundary layer thickness at the location. Temperature of the fluid = 20°C. The kinematic viscosity and density of the fluids are :*

S. No	Density, kg/m^3	Kinematic viscosity	L_{cv}, m	δ, mm
Air	1.205	15.06×10^{-6}	**2.51**	**17.7**
Water	1000	1.006×10^{-6}	**0.17**	**1.2**
Engine oil	888	901×10^{-6}	**150**	**1061**

The flow turns turbulent at $\text{Re} = 5 \times 10^5$

(1) Air : $5 \times 10^5 = 3 \times L_a/15.06 \times 10^{-6} \quad \therefore \quad L_a = 2.51$ m

$$\delta = 5x/\text{Re}_x{}^{0.5} = 5 \times 2.51/(5 \times 10^5)^{0.5} = 0.0177 \text{ m}$$

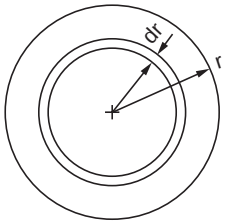
(2) Water : $5 \times 10^5 = 3 \times L_w/1.006 \times 10^{-6} \quad \therefore \quad L_w = 0.1677$ m

$$\delta = 5x/\text{Re}_x{}^{0.5} = 5 \times 0.1677/(5 \times 10^5)^{0.5} = 0.0012 \text{ m}$$

(3) Engine oil : $5 \times 10^5 = 3 \times L_o/901 \times 10^{-6} \quad \therefore \quad L_o = 150.17$ m

$$\delta = 5x/\text{Re}_x{}^{0.5} = 5 \times 150.17/(5 \times 10^5)^{0.5} = 1.061 \text{ m}$$

Problem 10.9 *The pressure distribution on the front and back surfaces of a thin disk of radius, R oriented perpendicular to a fluid stream was measured and the pressure coefficient has been correlated as below.*



Front side : $C_P = 1 - (r/R)^6$. *Rear surface :* $C_P = -0.42$

Determine the drag coefficient for the disk.

$$C_P = \Delta P/(1/2)\ \rho\ AV^2$$

$$\therefore \qquad \Delta P = C_P(1/2)\ \rho\ AV^2 = C_P\, 2\pi\, rdr\ (\rho\ V^2/2)$$

Consider a small strip of width dr at a radius r. The force on the area = $\Delta P \times 2\pi\, rdr$

$$\text{F}_\text{D} = \int_0^R \Delta P \times 2\pi\, rdr = \frac{1}{2}\rho V^2 \int_0^\text{R} C_p\ 2\pi\ rdr = \frac{1}{2}\rho V^2 \int_0^R \left[1 - \left(\frac{r}{R}\right)^6\right] 2\pi\, rdr$$

$$= \frac{1}{2}2\pi \times \rho V^2 \left[\frac{r^2}{2} - \frac{r^8}{8R^6}\right]_0^R = \frac{1}{2}2\pi \times \rho V^2 \left[\frac{R^2}{2} - \frac{R^2}{8}\right]$$

$$= \frac{1}{2}\rho V^2 \pi R^2 \left[3/4\right] = (1/8)\ \rho\ AV^2$$

$$\mathbf{C_D} = F_D/(1/2)\ \rho\ AV^2 \quad \therefore \mathbf{C_D} = 3/4 = \mathbf{0.75}$$

On the otherside, the pressure is independent of radius $\therefore\ C_D = C_p$ and it is in the opposite direction

$$\therefore \qquad C_p = 0.75 + 0.42 = \mathbf{1.17}$$

Problem 10.10 *Air flows along a triagular plate as shown in Fig. P. 10.10. Determine the shear force on both sides of the plate. Assume air temperature. as 20°C. $\rho = 1.205\ kg/m^3$, kinematic viscosity is $15.06 \times 10^{-6}\ m^2/s$.*

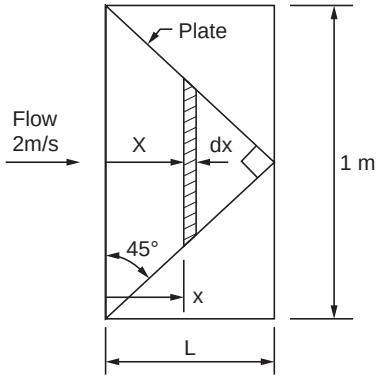


Figure P.10.10 Problem model

Considering the maximum length of 0.5 m

$$Re = 2 \times 0.5/15.06 \times 10^{-6} = 0.66 \times 10^5 \quad \therefore \quad \text{flow is laminar}$$

$$\tau_x = 0.332\ \rho\ u^2/\mathrm{Re}^{0.5} = 0.332\ \rho\ u^2 v^{1/2}/u^{1/2}x^{1/2}$$

$$= 0.332\ \rho\ u^{1.5} v^{1/2} x^{-1/2}$$

$$= 0.332 \times 1.205 \times 2^{1.5} \times (15.06 \times 10^{-6})^{0.5} x^{-1/2}$$

$$= 4.97 \times 10^{-3} x^{1/2}$$

Considering a strip of width dx at a distance x from base, and assuming the length of base as $2L$, height will be L.

$$dA = \frac{L - x}{L} \times 2L \times dx = 2(L - x)dx, \text{ Force on the strip}$$

$$dF = \tau_x\, dA = 4.97 \times 10^{-3} \times 2(L - x)x^{-1/2}dx$$

Integrating between $x = 0$ to $x = L$

$$F = 9.94 \times 10^{-3}\left[\frac{Lx^{1/2}}{0.5} - \frac{x^{1.5}}{1.5}\right]_0^L = 9.94 \times 10^{-3} \times \frac{4}{3} \times L^{1.5}$$

Here **L = 0.5 m.** $\therefore$ **F = 4.68 × 10⁻³ N**

Check for dimensional homogeneity.

$$F = \text{const}\ \rho\, u^{1.5} v^{1/2} L^{1.5},$$

$$N = \text{const}\ \frac{kg}{m^3}\frac{m^{1.5}}{s^{1.5}}\frac{m}{s^{0.5}}\ m^{1.5} = \text{kgm/s}^2 = \text{N},$$

Hence checks.

Problem 10.11 *A water ski is 1.2 m long and 0.2 m wide and moves in water at 10 m/s. the water temperature is 20°C.* ***Determine the viscous drag*** *approximating it as a flat plate.*

$$v = 1.006 \times 10^{-6}\ m^2/s.\ \rho = 1000\ kg/m^3,$$

$$Re = 1.2 \times 10/1.006 \times 10^{-6} = 11.93 \times 10^6$$

$\therefore$ The flow is turbulent considering combined laminar and turbulent flows.

$$C_{fL} = 0.074\text{Re}_L^{-0.2} - 1742\text{Re}_L^{-1} = 2.99 \times 10^{-3},\ \text{Drag} = C_{fL}\,(1/2)\rho\, u^2 \Delta$$

$$\textbf{Drag} = (1/2)\ 1000 \times 10^2 \times 1.2 \times 0.2 \times 2.99 \times 10^{-3} = \textbf{35.88 N}$$

Power required considering 2 skis, $P = 2 \times 35.88 \times 10 = 717.6$ W

Problem 10.12 *In a power plant located near the sea* ***a chimney*** *of 1.2 m diameter and 35 m height has been installed. During a cyclone the wind reaches velocity in the range of 60 kmph.* ***Determine the moment at the base of the chimney.***

$$\rho = 1.2\ \text{kg/m}^3,\ v = 17.6 \times 10^{-6}\ \text{m}^2/\text{s},\ u = 600000/3600 = 16.67\ \text{m/s}$$

$$Re = 16.67 \times 1.2/17.6 \times 10^{-6} = 1.14 \times 10^6$$

From graph for circular cylinder C_D is read as 0.35

$\therefore$ $$F_D = C_D\, \rho\, Au^2/2 = 0.35 \times 1.23 \times 35 \times 1.2 \times 16.66^2/2 = 5022.5\ \text{N}$$

As this is a uniform force, it can be taken to act at the mid point.

$\therefore$ **Moment** = 5022.5 × 35/2 = 87893 Nm or **87.893 kNm.**

Problem 10.13 *A overhead water tank is in the shape of a sphere of 12 m diameter and is supported by a 30 m tall tower of circular section of diameter 2 m.* ***Determine the moment at the base*** *caused by the aerodynamic force due to cyclonic wind of speed 100 kmph. Assume density of air as 1.205 kg/m³ and kinematic viscosity as 15.06 × 10⁻⁶ m²/s.*

For the spherical portion : $$Re = 12 \times \frac{100 \times 1000}{3600} \times \frac{1}{1506 \times 10^{-6}} = 2.21 \times 10^7$$

The value of C_D is read as 0.19 from graph by extrapolation.

For the cylindrical portion : $$Re = 2 \times \frac{100 \times 1000}{3600} \times \frac{1}{1506 \times 10^{-6}} = 3.689 \times 10^6$$

The value of C_D is read as 0.40 from graph by extrapolation.

$$F_D = C_D\,(1/2)\ \rho\, AV^2,\ M = F_D \times \text{distance},$$

$$V = 100 \times 1000/3600 = 27.78\ \text{m/s}.$$

For the spherical portion

$$M = (30 + 6) \times 0.19 \times (1/2) \times 1.205 \times (\pi \times 12^2/4) \times 27.78^2$$

$$= 359.6 \times 10^3\ \text{Nm}$$

For the cylindrical portion

$$M = 15 \times 0.4 \times (1/2) \times 1.205 \times 2 \times 30 \times 27.78^2 = 167.4 \times 10^3 \text{ Nm}$$

Total moment $= (359.6 + 167.4) \times 10^3 =$ **527.0×10^3 Nm**

Problem 10.14 *A parachute moves down at a speed of 6 m/s. The mass of the chute and the jumper is 120 kg.* ***Determine the minimum diameter of the chute****. Density of air = 1.23 kg/m^3.*

For parachute $C_D = 1.2$, (Refer table 10.3.1)

Net force = 120×9.81 N

$120 \times 9.81 = 1.2 \times (1/2) \times 1.23 \times (\pi D^2/4) \times 6^2$ Solving **D = 7.51 m**

Problem 10.15 *Hail stones that are formed in thunder clouds are sopported by the drag due to the air draft upwards and will begin to fall when the size reaches a critical value.* ***Estimate the velocity upwards so that hailstones begin to fall*** *when the diameter reaches a value of 40 mm. The density and dynamic viscosity of air at the altitude of 5000 m where the stones are formed are 0.7364 kg/m^3 and 1.628×10^{-5} kg/ms. Hailstone is assumed to be in the shape of a sphere with a density of 940 kg/m^3.*

The drag force should be just less than the gravity force when the hailstone begins to fall. At the limiting condition thses can be taken as equal. Other body forces like buoyancy forces are negligible.

Drag force = C_D (1/2) ρ Au^2, Gravity force = ρ Vg, V being the volume.

Equating and substituting the values,

$$C_D = (1/2) \times 0.7364 \times \pi \times 0.02^2 \, u^2 = (4/3) \times \pi \times 0.02^3 \times 9.81 \times 940$$

$$\therefore \quad C_D \, u^2 = 667.85$$

C_D depends on Reynolds number which cannot be calculated without the value of velocity. Looking at the graph for C_D for spheres, the value is about 0.45 for $Re = 10^3$ to 5×10^5. Substituting this value, **u = 38.52 m/s or 138.7 kmph.**

$$\text{Re} = 38.52 \times 0.04/1.628 \times 10^{-5} = 0.94 \times 10^5.$$

Hence the assumed value of C_D is acceptable.

Problem 10.16 *A stirrer is constructed as shown in Fig. P. 10.16. The dimensions are indicated in the figure. The stirrer speed is 90 rpm.* ***Determine the torque*** *on the shaft and also* ***the power required****. Assume the vessel is large. Neglect the drag on the rod and the shaft. Density of the fluid is 1025 kg/m^3.*

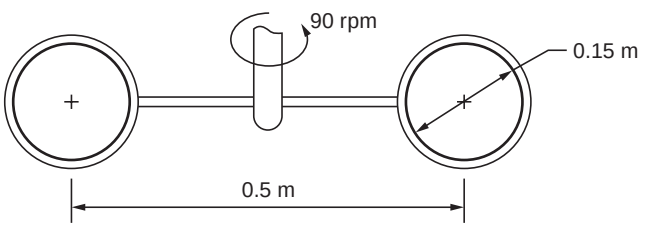


Figure P. 10.16 Stirrer details

For circular plate $C_D = 1.17$

Linear speed of the disk = $\frac{\pi DN}{60} = \frac{\pi \times 0.5 \times 90}{60}$ = 4.7124 m/s

$C_D = F_D/(1/2)\ \rho\ AV^2$, $A = \pi \times 0.15^2/4$

$\therefore$ $F_D = 1.17 \times (1/2) \times 1025 \times (\pi \times 0.15^2/4) \times 4.7124^2 = 235.31$ N

Torque = Force × torque arm = 235.30 × 0.5 = **117.65 Nm.**

Power = 2π NT/60 = $(2\pi \times 90 \times 117.65)/60$ = **1109 W**

Problem 10.17 *An anemometer has hemispherical cups of 80 mm dia with an arm distance from the post to center of 130 mm. If due ot fiction, the cups starts rotating at a wind speed of 3 m/s.* ***Determine the starting torque****. Consider density of air as 1.23 kg/m³.*

The coefficient of drag when the cup faces the wind is 1.42. The coefficient of drag on the back = 0.38.

$\therefore$ Net coefficient = 1.42 – 0.38 = 1.04

$\therefore$ Force = $C_D A\rho\ V^2/2$, Torque = Force × torque arm, Substituting

Starting torque $= 1.04 \times \pi \times \frac{0.08^2}{4} \times \frac{1.23}{2} \times 3^2 \times 0.13 =$ **3.76×10^{-3} Nm**

Problem 10.18 ***Determine the wind force*** *on the antenna shown in Figure P. 10.18. All the components face the wind blowing at 100 kmph. $\rho = 1.205$ kg/m³, kinematic viscosity is 15.06×10^{-6} m²/s.*

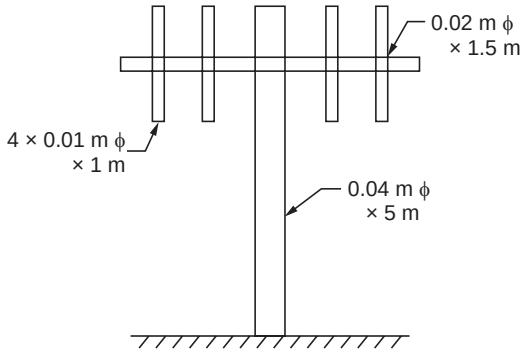


Figure P. 10.18 Antenna details

Velocity of wind = 100000/3600 = 27.78 m/s

The value of Reynolds number is given by

Re = $27.78 \times 0.04/15.06 \times 10^{-6} = 7.38 \times 10^4$ for 40 mm rod and 1.84×10^4 for 10 mm rod. At this value C_D is about 1.4 for both cases.

$$\mathbf{F_D} = 1.4 \times \frac{1.205}{2} \times 27.78^2\ [(5 \times 0.04) + (1.5 \times 0.02) + (4 \times 0.01)]$$

$$= \mathbf{175.7\ N}$$

Problem 10.19 *The total mass of an aircraft is 70000 kg. The wing area is 160 m^2. If the craft travels at 600 kmph,* ***determine the lift coefficient****. Neglect the compressibility effect. Air density at the flight conditions is 0.85 kg/m^3.*

The lift force should be equal the weight at steady flight. Lift force F_L is given by

$$F_L = C_L A(1/2)\,\rho\, V^2, \text{ flight speed, } V = 600000/3600 = 166.7 \text{ m/s}$$

$$70000 \times 9.81 = C_L \times 160 \times (1/2) \times 0.85 \times 166.7^2 \quad \therefore \boldsymbol{C_L = 0.3635}.$$

Problem 10.20 *In championship tennis, balls are hit at speeds exceeding 100 kmph and good amount of spin.* ***Calculate the aerodynamic lift on ball and radius of curvature of path*** *in the vertical plane, when the ball is hit at a speed of 108 kmph and a top spin of 8000 rpm. The ball diameter is 0.064 m and mass is 0.057 kg. For air density = 1.165 kg/m^3 and kinematic viscosity is 16 × 10^{-6} m^2/s.*

The lift force depends on the spin ratio and Reynolds number. Top spin causes downward force. Spin ratio = $\omega D/2u$.

$$u = 108000/3600 = 30 \text{ m/s}, \ \omega = 8000 \times 2\pi/60 = 837.76 \text{ radians/s}$$

$$\therefore \text{ Spin ratio } = (837.76 \times 0.064/2 \times 30) = 0.8936$$

$$Re = 30 \times 0.064/16 \times 10^{-6} = 1.2 \times 10^5$$

By interpolation in Fig. 10.3.4, page 341 C_L is read as 0.25

$$\therefore \quad \textbf{Lift force} = 0.25 \times \pi \times \frac{0.064^2}{4} \times \frac{1}{2} \times 1.165 \times 30^2 = \mathbf{0.4216\ N},$$

This force acts downwards due to top spin.

gravity force = 0.057 × 9.81 = 0.5592 N $\quad \therefore$ Total force = 0.9808 N

Equating it to the z directional acceleration

$F = mu^2/R$ where R is the radius of the path in the vertical plane.

$$\mathbf{R} = 0.057 \times 30^2/0.9808 = \mathbf{52.3\ m}$$

In case only gravity force acts, then $R = 0.057 \times 30^2/0.5592 = 91.7$ m

The ball comes down sharply due to the top spin.

Problem 10.21 *A table tennis ball of mass 2.5 grams and a diameter of 38 mm is hit with a velocity of 12 m/s, with a back spin ω.* ***Determine the value of back spin for the ball to travel in a horizontal path****, not dropping due to gravity. Density and kinematic viscosity of air are 1.165 kg/m^3 and 16 × 10^{-6} m^2/s.*

For this situation, the force due to gravity should equal the lift force. *i.e.,*

$$mg = C_L(1/2)\,\rho\, u^2 A$$

$$\frac{2.5}{1000} \times 9.81 = C_L \times \frac{1}{2} \times 1.165 \times 12^2 \times \pi \times \frac{0.038^2}{4} \quad \therefore C_L = 0.2578$$

From the graph for C_L vs spin ratio, (Fig. 10.3.4) the value of spin ratio is read as 0.91

$$\frac{\omega D}{2u} = 0.91, \quad \therefore \quad \boldsymbol{\omega = 574.7 \text{ rad/s or } 5488 \text{ rpm}}$$

If the velocity is more, for this spin the ball will rise. If the velocity is less then the ball will travel in an arc.

Problem 10.22 *A cork ball 0.3 m diameter with specific gravity 0.21 is tied on the bed of a river. At a certain time it rests at 30° to the horizontal due to the flow.* ***Determine the velocity of flow.***

The forces on the cork ball are shown in Fig. P. 10.22.

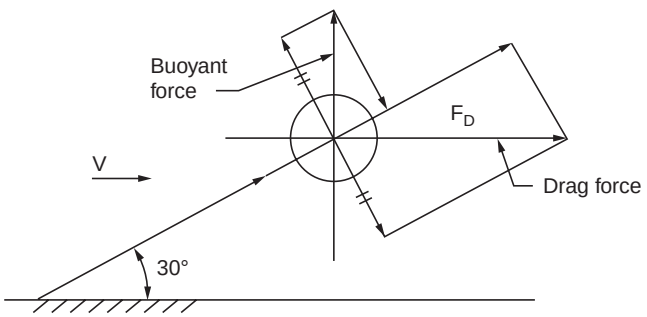


Figure P. 10.22 Force diagram

At equilibrium the components along the rope (at 30° to the horizontal) is taken up by the rope. The components perpendicular to this line should balance.

$\therefore \quad F_D \cos 60 = F_b \cos 30$

F_b = Buoyant force = difference in density × volume × g

$$= 790 \times \frac{4}{3}\pi \times 0.15^3 \times 9.81 \text{ N} = 109.56 \text{ N}$$

$\therefore \quad F_b \cos 30 = 94.88$ N. $\quad \therefore F_D = 94.88/\cos 60 = 189.66$ N

$F_D = C_D\,(1/2)\,\rho\, AV^2$, $\boldsymbol{C_D = 0.45}$ **for sphere**

$$189.66 = \frac{0.45}{2} \times 1000 \times \pi \times \frac{0.3^2}{4} V^2, \text{ Solving } \mathbf{V = 3.45 \text{ m/s}}$$

$$\text{Reynolds number} = \frac{3.45 \times 0.3}{1.06 \times 10^{-6}} = 0.98 \times 10^6$$

For this value $\mathbf{C_D = 0.2}$ Corresponding $\mathbf{V = 5.18}$ **m/s**

Further iteration is necessary as the new value of Re = 1.47×10^6 and $C_D = 0.35$.

Problem 10.23 *Air flows in a square duct of side 0.6 m with a velocity of 3 m/s. The displacement thickness in meter is given by* $\delta_d = 0.0039\, x^{0.5}$ *where x is the distance along the flow.* ***Determine the velocity outside the boundary layer at a distance of 30 m. Density of air*** *= 1.2 kg/m³.*

The flow with boundary layer can be taken as flow at the free stream velocity with the boundary moved by a distance equal to the displacement thickness.

At 30 m, displacement thickness is $\delta_d = 0.0039 \times 30^{0.5} = 0.02136$ m

The side of the square is reduced by twice this thickness.

$\therefore$ Length of side considering displacement thickness is

$$L_d = (0.6 - 2 \times 0.02136) = 0.5573 \text{ m}$$

Equating the volume flow rate $0.6 \times 0.6 \times 3 = 0.5573^2 \times V_2$

$\therefore$ $\mathbf{V_2 = 3.48 \text{ m/s}}$

The pressure drop can be calculated for the flow outside the boundary layer as

$$\Delta P = (1/2)\ \rho\ (V_2^2 - V_1^2) = (1/2) \times 1.2\ [3.48^2 - 3^2] = 1.87 \text{ N/m}^2$$

OBJECTIVE QUESTIONS

O Q. 10.1 Fill in the blanks:

1. In flow over surfaces, fluid at the surface takes on the velocity of the body as a result of _____ condition.
2. The study of non viscous fluid flow is called _____.
3. Equations describing the complete flow field are know as _____ equations.
4. The effect of viscosity is important only in a thin layer adjacant to the surface called _____.
5. The flow outside the boundary layer can be treated as _____ flow.
6. Velocity gradient exists only in the _____.
7. The forces which are important in the boundary layer are _____.
8. In ideal flwo the forces that are important are _____.
9. The pressure gradient at the surface causes _____ on the surface.
10. Initially _____ flow prevails in the boundary layer.

Answers

(1) no slip (2) Theoretical hydrodynamics. (3) Navier-Stokes (4) boundary layer (5) Ideal fluid (6) boundary layer (7) Inertia and viscous forces (8) pressure and inertia (9) shear stress (10) Laminar

O Q. 10.2 Fill in the blanks:

1. Mass and momentum flow in laminar boundary layer is only at the_____ level.
2. The two methods of analysis of boundary layer flow are _____.
3. Macroscopic mixing between layers occurs in _____.
4. The ratio of inertia force to viscous force is called _____ number.
5. Turbulent flow over a flat plate is generally taken to start at a Reynolds number of_____.
6. In laminar flow viscous forces are _____ compared to inertia forces.
7. In turbulent flow viscous forces are _____ compared to inertia force.
8. Boundary layer separation occurs when there is an _____ pressure gradient.
9. Lift is the component of the total force on a body immersed in a flow in the _____ direction.
10. Drag is the component of the total force on a body immersed in a flow in the _____ direction.

Answers

1. microscopic 2. exact differential, approximate integral. 3. Turbulent flow. 4. Reynolds 5. 5×10^5 6. Larger 7. Smaller 8. Adverse 9. Perpendicular 10. Flow

O Q. 10.3 Fill in the blanks:

1. The force perpendicular to both the flow direction and the axis of rotation of an object in flow is known as _____ effect.
2. The drift of a shell fired is due to _____.
3. Rotating cylinders were proposed to propel ships by the use of _____.
4. The coefficient of lift is the ratio between _____ and _____.
5. The coefficcient of drag is the ratio between _____ and _____.
6. The coefficient of lift on an airfoil _____ with angle of incidence upto a limit and then _____.
7. Coefficient of lift on an airfoil decreases when _____ occurs.
8. The coefficient of lift has values about _____.
9. Coefficient of drag has values in the range _____.
10. Top spin _____ the length of travel of a ball.

Answers

1, 2, 3 Magnus effect 4. Lift force, dynamic force 5. Drag force, dynamic force 6. Increases, decreases 7. Separation 8. 1.0, 9. 0.1 10. Shortens

O Q. 10.4 Fill in the blanks:

1. The layer thickness which will have the same flow rate as the boundary layer with free stream velocity is called _____.
2. The layer thickness which will have the same flow momentum as the boundary layer with free stream velocity is called _____.
3. The drag due to boundary layer sparation is called _____ drag.
4. The pressure gradient which will delay boundary layer seperation is called _____ gradient.
5. The pressure gradient which will induce boundary layer seperation is called _____ gradient.
6. The angle between the flow direction and the chord of an airfoil is called _____.
7. The lift and drag coefficient on a spinning sphere is dependent on _____ defined as $\omega D/2V$, V being the forward velocity.
8. In the range of Reynolds numbers 10^3 to 10^5, the drag coefficient on a cylinder or sphere is _____.
9. The velocity profile in laminar flow follows nearly a _____ polynomial.
10. The velocity profile in turbulent flow can be represented by _____ power law.

Answers

1. Displacement thickness 2. Momentum thickness 3. Pressure 4. Favourable 5. Adverse 6. Angle of incidence 7. Spin ratio 8. Nearly constant 9. Cubic 10. One seventh

O Q. 10.5 Fill in the blanks:

1. The distance from the wall where the velocity is 99% of its asymptotic limit is known as _____ of a boundary layer.

2. In laminar boundary layer, the momentum transfer is by _____.
3. A boundary layer in which there is macroscopic mixing is called_____.
4. In flow of real fluids the viscous effects can be considered to be confined to the _____.
5. In flow over a flat plate the boundary layer undergoes transition when the value of Reynolds number is about _____.
6. The velocity gradient in turbulent boundary layer will be _____ than in the laminar boundary layer.
7. In a turbulent boundary layer over a smooth plate, there exits a thin layer in which velocity variation is linear is called as _____.
8. The phenomenon of boundary layer separation takes place at adverse _____ gradient.
9. The disturbed region downstream of boundary layer separation is known as _____.
10. At the separation point of boundary layer, the velocity gradient will be _____.

Answers

1. Thickness 2. Molecular diffusion 3. Turbulent boundary layer 4. Boundary layer 5. 5×10^5 6. greater 7. Laminar sub layer 8. Pressure 9. Wake 10. Zero

O Q. 10.6 Fill in the blanks with increases or decreases:

1. Boundary layer thickness _____ along the flow direction.
2. Local velocity within a boundary layer _____ towards the boundary surface.
3. Shear stress within the boundary layer _____ towards the boundary surface.
4. Reynolds number _____ in the direction of flow over flat plate.
5. Turbulence _____ in the direction of flow.
6. Drag coefficient on a plate _____ with increase in Reynolds number.
7. Local shear stress coefficient _____ with increase in Reynolds number.
8. Reynolds number _____ with increase in kinematic viscosity of a fluid.
9. Drag force _____ with increase in free stream velocity.
10. Boundary layer thickness _____ with increase in Reynolds number.

Answers

Increases : 1, 3, 4, 5, 9 Decreases : 2, 6, 7, 8, 10

O Q. 10.7 State True to False:

1. The velocity gradient in the boundary layer is maximum at the top edge of the layer.
2. The flow above the boundary layer can be treated as invicid flow.
3. The velocity gradient will be zero at the top plane of the boundary layer.
4. Displacement thickness is the thickness by which the plane is to be moved up, so that the flow equals the flow at free stream velocity.
5. Rate of momentum transfers will be higher in turbulent flow.
6. Momentum thickness will be larger compared to displacement thickness.
7. Boundary layer thickness will be smaller compared to displacement thickness.
8. Displacement thickness is defined by $\frac{1}{u_\infty^2}\int_0^\delta (uu_\infty - u^2)dy$.
9. Momentum thickness is given by $\int_0^\delta \left(1 - \frac{u}{u_\infty}\right) dy$.

10. Friction coefficient $C_f = \tau_w/\rho\, u_\infty^2$.
11. In turbulent flow $C_f \propto \text{Re}^{0.2}$.
12. In laminar flow $C_f \propto \text{Re}^{-0.5}$.
13. In turbulent flow $C_f \propto \text{Re}^{-0.2}$.
14. Boundary layer thickness in laminar flow is proportional to x.
15. Boundary layer thickness in laminar flow is proportional to $x^{1/2}$.
16. In turbulent flow boundary layer thickness is proportional to $x^{0.8}$.

Answers

True : 2, 3, 4, 5, 12, 13, 15, 16

False : 1, 6, 7, 8, 9, 10, 11, 14

O Q.10.8 Define

1. Boundary layer thickness 2. Momentum thickness 3. Displacement thickness 4. Coefficient of friction 5. Laminar flow 6. Turbulent flow 7. Drag coefficient 8. Pressure drag 9. Flow separation 10. Chord, wing span of an airfoil. 11. Lift coefficient 12. Drag coefficient 13. Angle of incidence 14. Spin ratio 15. Magnus effect.

EXERCISE PROBLEMS

E 10.1 Assuming that air at 20 °C flows over a flat with a free stream velocity of 6 m/s, determine the velocity at 0.5 m and 0.8 m at a distance of 6 mm from plate surface (*i*) Assuming cubic profile and (*ii*) linear profile.

$\upsilon = 15.06 \times 10^{-6}$ m^2/s $\quad \rho = 1.205$ kg/m^3, In case flow is turbulent use 1/7th power law.

E 10.2 Derive an expression for the displacement thickness and Momentum thickness in flow over a flat plate assuming $\dfrac{u}{u_\infty} = \sin\left(\dfrac{\pi}{2}\dfrac{y}{\delta}\right)$.

E 10.3 Compare the velocities for flow over a flat plate to turn turbulent at a distance of 0.6 m for (*i*) air (*ii*) water and (*iii*) engine oil, all at 20 °C.

E 10.4 Assuming momentum thickness to be constant at the transition point whether laminar or turbulent flow correlation is used, find the ratio of laminar boundary layer thickness to turbulent boundary layer thickness. Assume parabolic profile for laminar correlation $\dfrac{u}{u_\infty} = 2\dfrac{y}{\delta} - \left(\dfrac{y}{\delta}\right)^2$ and 1/7th power law variation for turbulent correlation.

(δ_t/δ_L = 144/105)

E 10.5 Determine the ratio of δ_m/δ_d in the turbulent region of flow over a flat plate, assuming 1/7th power law. **(δ_m/δ_d = 0.0972)**

E 10.6 In a flow of air over a flat plate at a distance 20 cm from leading edge, the boundary layer thickness was measured as 5.7 mm the free stream velocity being 25 m/s. If the velocity profile is given as $\dfrac{u}{u_\infty} = 2\dfrac{y}{\delta} - \left(\dfrac{y}{\delta}\right)^2 + C$, calculate the value of C. Also determine the displacement and momentum thicknesses at this section. **(δ_d = 1.9 mm, δ_m = 0.76 mm)**

E 10.7 In a wind tunnel of square section of side 80 mm, the inlet velocity is 30 m/s. At 0.3 m from inlet the displacement thickness is 1 mm. Determine the change in pressure between the sections. **(59 N/m^2)**

E 10.8 In a wind tunnel square section of side 0.305 m, the boundary layer thickness increases from 9.5 mm to 12.7 mm. At the first location, the velocity of air is 18.3 m/s. The static pressure is – 22.9 mm of water column (gauge). Assuming turbulent flow conditions and 1/7th power law velocity profile, determine the velocity at the second section. Also determine the change in static pressure. **(18.4 m/s, ΔP = 2.19 N/m^2)**

E 10.9 In a water tunnel the freestream velocity is 1.6 m/s. A plate 0.3 m long and 1 m wide is placed parallel to the flow. (0.3 m along flow). Determine the total viscous force on the plate. Use the results of cubic velocity profile assumption. **(1.62 N)**

E 10.10 Show that for flow over a flat plate, drag force upto a length L on one side is given by $F_D = \rho u_\infty^2 \delta_{mL} b$. where b is the width, u_∞ free stream velocity and δ_{mL} momentum thickness, For u_∞ = 2 m/s, and L = 0.3 m, b = 1 m, determine the total drag. **(0.81 N)**

E 10.11 Determine the velocity of water if the drag on a sphere of 12 cm dia is 5 N. The water is at 10 °C. **(V = 1.29 m/s)**

E 10.12 Asuming $(u/u_\infty) = \left(\frac{y}{\delta}\right)^{1/9}$, derive an expression for boundary layer thickness at a distance x from leading edge. **(δ = 0.287 x/R$^{1/6}$)**

E 10.13 Determine the friction drag on an airship 100 m long and 20 m diameter when it travels at 130 kmph. The condition of air is 25 °C and 0.9 bar. **(8370 N)**

E 10.14 A large truck weighing 45000 N is be air dropped using a large parachute, the value of C_D being 1.2. If the terminal speed is 10 m/s in air at 1 bar and 20 °C, determine the diameter of the chute. **(30 m)**

E 10.15 Describe the types of drag when a disc is held parallel to flow direction and perpendicular to flow direction.

E 10.16 An open C section of 30 cm dia and 8 m length is held with concave side facing the flow at 1 m/s of water. Determine the drag on the section. **(2.76 kN)**

E 10.17 A spherical balloon of helium of diameter 3 m is held tied to a rope. The wind flows at 20 kmph. Determine the angle of inclination of the cable. The pressures and temperatures on the inside and outside are 1 bar and 20 °C.

E 10.18 An advertisement board 3 m dia is exposed to a 100 kmph normal wind. Determine the total force on the board. **(3.6 kN)**

E 10.19 A high speed car with a frontal area of 1 m^2 and drag coefficient C_D = 0.3 travelling at 100 m/s is to be decelerated using a drag chute of 2 m dia with C_D = 1.2. Determine the speed 10 seconds after the chute is deployed. Density of air = 1.2 kg/m^3. **(45 m/s)**

E 10.20 A wind tunnel of 1 m square section is 6 m long. Air at 20 °C flows at 30 m/s. In order to maintain the velocity constant, the walls are to be slightly slanted outward. This is to compensate for the growth of the displacement thickness. Determine the angle of slant between 2 m and 4 m distances from entry.

E 10.21 A thin circular plate is held parallel to a flow with velocity u_∞. Derive an expression for the drag assuming the flow to be similar to that on a flat plate.

E 10.22 A flat plate of lenght L and width equal to the boundary layer thickness is held parallel to the flow direction. Derive an expression for the drag on the plate. (*i*) Assume linear velocity profile. (*ii*) Assume cubic profile and (*iii*) Assume 1/7th power law.

E 10.23 A ship 150 m long has a rough wetted area of 5000 m^2. To overcome friction the power required was 5152 kW at a speed of 15 kmph. Determine the reduction in power required if the surface was smooth. Take properties of water at 20 °C.

E 10.24 Determine the frequency of vortex shedding by a wire 5 mm dia, at a wind speed of 30 m/s. The density of air 1.2 kg/m^3 and $\mu = 18 \times 10^{-6}$ kg/ms. **(1.2 kHz)**

E 10.25 The tension of the rope holding a kite is 20 N. When the rope is at 45° to the horizontal. The kite is a square of side 6 m , and weights 5 N. The angle made by the kite is 10° with horizontal. The wind speed is 4 kmph. Air density is 1.2 kg/m^3. Calculate the lift and drag coeficients.

(C_L = 0.72, C_D = 0.53)

E 10.26 A spherical ballon of 1.5 m dia filled with helium when let go ascended at 6.27 m/s in air. The weight of the empty ballon was 50 N. Determine the density of helium in the balloon. Density of air = 1.2 kg/m^3. **(0.2 kg/m^3)**

E 10.27 A chute carrying a bomb totally weighs 1 kN. If C_D = 1.2 and rate of descent is 6 m/s determine the diameter of chute. Density of air = 1.22 kg/m^3 **(7 m)**

E 10.28 Determine the terminal velocity and the maximum diameter of a spherical particle with a relative density of 1.8 that will settle in water with a density of 1000 kg/m^3. Dynamic viscosity = 1.2×10^{-3} Ns/m^2. Use stokes law. **($V = 8.1 \times 10^{-3}$ m/s, d = 0.149 mm)**

E 10.29 Derive the expression for terminal velocity of a particle $V = D^2(f_g - f)/18\mu$, when f_g = gravity force and f is bouyant force N/m^3.

E 10.30 A small stone of dia 5 mm and relative density 2.8 falls in oil of relative density 0.9. Estimate the terminal velocity.μ = 0.3 Pas. **(8.63×10^{-2}m/s)**

11 Flow Measurements

11.1 INTRODUCTION

The performance of engineering equipments and systems should be validated by tests and experiments before these could be commissioned. Tests and experiments involve various measuring instruments. Performance tsting of pumps, teurbines, fans and blowers are typical cases.

Other areas requiring measurement of flow parameters are irrigation systems, chemical process control and research work in fluid mechanics.

Out of the many parameters to be measured, flow velocity and flow rate are involved in almost all cases. Measurement of pressure and pressure difference between locations are discussed in chapter 2.

In this chapter the methods and instruments for the measurement of flow velocity and flow rate are described.

11.2 VELOCITY MEASUREMENTS

The measurement of velocity at a point or a number of points throughout a section in a flow stream is often needed to establish the velocity profile. Measurement of velocity at a point is almost impossible, since any sensing device has a finite dimension. However, if the area of flow occupied by the sensing device is relatively small compared to the total area of flow stream, then it may be considered that the velocity measured is the velocity at a point. It is essential that the presence of the sensing device in the flow stream does not afffect the flow being measured. **Velocity is usually measured indirectly by measuring the difference between the stagnation and static pressures** (pitot tube) or by the rotational speed of wheels (vane anemometer) or by the temperature drop on a thin cylindrical wire in cross flow (hot wire anemometer) and also by optical systems. Velocity is also measured directly, in some instances, by determining the distance travelled by a group of fluid particles during a measured time interval.

11.2.1 Pitot Tube

If a small bore hollow tube bent at 90° is placed in a flow stream with its end facing upstream, fluid will rise in the vertical side of the tube as shown in Fig 11.2.1 (*a*). This method is used as pick-up in velocity measurment.

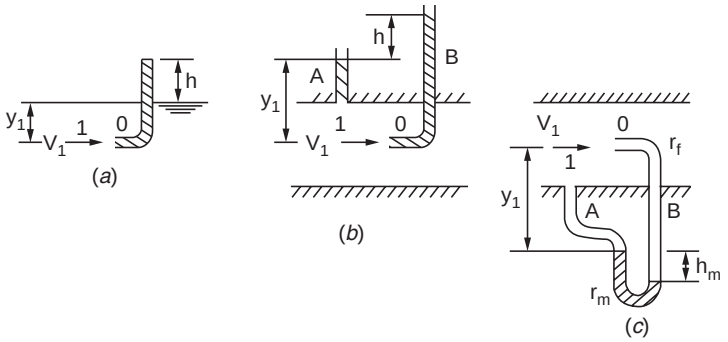


Figure 11.2.1 Pitot tube arrangements

If Bernoulli equation is applied between a point, 1 upstream at the submerged end of the tube and a point, 0 at the other end of the tube, then leaving out P_{atm} on both sides

$$\frac{P_1}{\rho g} + \frac{V_1^2}{2g} = \frac{P_0}{\rho g} \tag{11.2.1}$$

Since stagnation condition exists within the tube

$P_0 = \rho g(y + h)$, at point 1, the static pressure is $P_1 = \rho g y_1$

Substituting and rearranging equation (11.1)

$$\frac{V_1^2}{2g} = \frac{(P_0 - P_1)}{\rho g} = \frac{\rho g[(y_1 + h) - y_1]}{\rho g} = h\ ,$$

$$\therefore \qquad V_1 = \sqrt{2gh} \tag{11.2.2}$$

Note that **h is the head expressed as the column of flowing fluid.**

For velocity measurement in ducts a different arrangement of pick ups is necessary. A typical method is illustrated in Fig. 11.2.1 (*b*). A tapping perpendicular to the flow gives the static pressure. The tube connection at this point is called static tube/probe. The pitot probe held facing upstream measures the total pressure.

The static tube *A* and pitot tube *B* are connected to a *U* tube manometer as shown in Fig. 11.2.1 (*c*) for measurement of velocity in a pipe. Equating the pressure at the left and right side limbs of the manometer,

$$P_1 + \rho g y_1 + \rho_m g\, h_m = P_0 + \rho g y_1$$

$$h_m g(\rho_m - \rho) = P_0 - P_1 \tag{11.2.3}$$

where ρ and ρ_m are the densities of flowing and manometric fluids. Substituting for $(P_0 - P_1)$ from equation 11.2.1,

$$h_m g\,(\rho_m - \rho) = \frac{\rho V_1^2}{2}$$

∴ The velocity of fluid near the tip of the pitot at section 1 is

$$V_1 = \sqrt{2gh_m(\rho_m - \rho)/\rho} = \sqrt{2gh_m((\rho_m/\rho) - 1)} \tag{11.2.4}$$

ρ_m/ρ is to be replaced by s_m/s in terms of specific gravities.

Though the set up illustrates the basic principle involved in the measurement, it is not a practical arrangement. In practice the two tubes are combined together to be used as a single instrument called pitot static tube as shown in Fig. 11.2.2 (*a*) and (*b*). This set up (Fig. 11.2.2(*a*)) is due to Prandtl and it is more accurate over the design of Brabbee as shown in Fig. 11.2.2 (*b*).

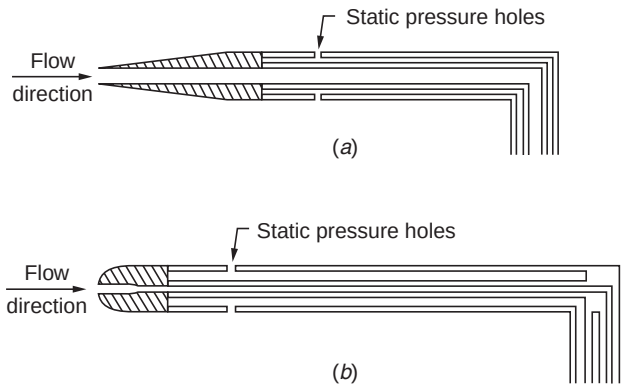


Figure 11.2.2 Prandtl and Brabbee pitot tubes

In both these cases, the inner tube measures the stagnation pessure and the tube with opening on the surface measures the static pressure. The accuracy of the measurement is found to depend on the shape of the tip of the pitot tube. Turbulent flows with more fluctuations at the tip may show a higher reading compared to the time averaged velocity at that location. Hence equation 11.2.4 is modified as

$$V_1 = C_v \sqrt{2gh_m((\rho_m/\rho) - 1)} \tag{11.2.5}$$

where C_v is a coefficient and its value has to be determined by calibrating the device.

In the case of subsonic gas flow, the equation for the velocity should be modified, considering compressibility as

$$V_1 = C_v \sqrt{\frac{2kRT_1}{k-1}\left[\left(\frac{P_0}{P_1}\right) - 1\right]} \tag{11.2.6}$$

where $k = C_P/C_v$ the ratio of specific heats.

This instrument is extensively used for velocity measurements in gas flow.

Standards are available for these instruments, indicating the details of construction.

11.2.2 Vane Anemometer and Currentmeter

Vane anemometers and current meters are used to measure the velocity of air and water in larger flow fields. Hemispherical vanes are fitted on the radial arms of vane anemometer as shown in Fig. 11.2.3 (*b*) and cones are fitted on the current meter as shown in the Fig 11.2.3(*a*). Drag force on the vanes and cones when fluid moves over them causes the rotary movement of the rotor. Impeller type wheels are also used for the measurement of gas flow velocity. The speed of rotation is generally indicated by means of electrical contacts made once in each revolution and the number of contacts made per unit time interval is a direct measure of the average speed of the fluid in the region traversed by the meters. Calibration is done by towing the meters through stagnant water or air at known speeds. Atmospheric wind speed measurement is generally done using such devices.

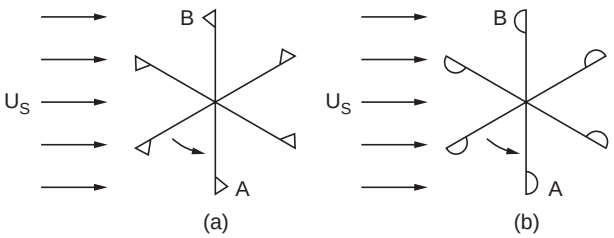


Figure 11.2.3 Vane anemometer

11.2.3 Hot Wire Anemometer

In hot-wire anemometers, an electrically heated thin wire is placed across a flowing stream. The heat transfer depends on the flow velocity. Two methods of measuring flow rate are:

1. Constant resistance method. The wire resistance is kept constant by adjusting the current flow through it and the velocity is determined by measuring the current and calibrating the instrument accordingly.

2. Constant current method. The current flow though the wire is kept constant and the change in wire resistance from convection cooling is measured in terms of the voltage drop across it. Fluctuations in velocity may be detected and recorded by suitable circuitry.

When the hot wire is placed in a flowing stream, heat will be transferred from the wire mainly by convection, radiation and conduction being negligible. The following relationship is used to determine the velocity.

$$\frac{\text{Power/unit length}}{\text{Temperature difference}} = \frac{I^2R}{T_w - T_a} = A + B\sqrt{\rho V}$$

where I is the instantaneous current, R is the resistance of wire per unit length, T_w is the temperature of the wire, T_a is the ambient temperature, V is the free stream velocity and ρ is the density of the fluid with constants A and B to be determined by calibration. The pick ups are shown in Fig. 11.2.4.

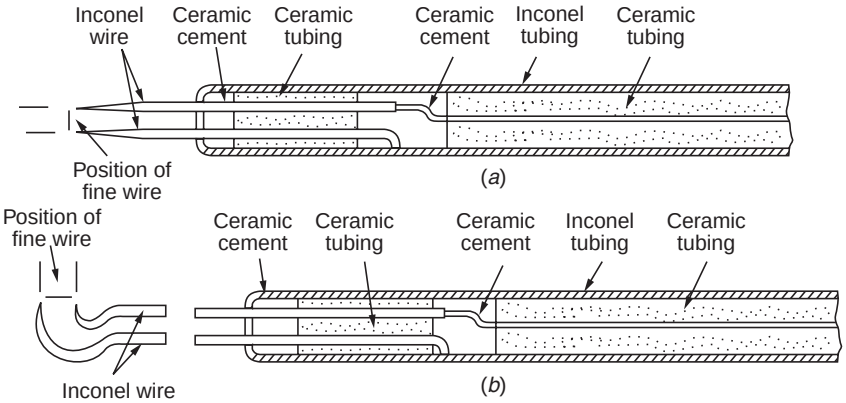


Figure 11.2.4 Hot wire anemometer

11.2.4 Laser Doppler Anemometer

Laser Doppler anemometer is used to measure the velocity of a flow without disturbing the flow. It is often used to measure turbulent velocity and also low volume flow rates. It measures the velocity of small particles that are either naturally present in most liquid flows or are seeded with 1 μm size particles in gas flows. This size of particle can generally follow all motions in the fluid in which it is carried. The reference beam mode with frequency tracking shown in Fig. 11.2.5 is one of the technique used for velocity measurement.

The light scattered off a moving particle has its frequency shifted by an amount that is proportional to the particle speed. A laser of fixed wavelength serves as a source of light and optical components split the laser beam into a reference beam and a secondary beam which are made to intersect at the measurement volume in the flow field. Frequency shifted light that is scattered off particles passing through the measurement volume and the unshifted beam is collected at a photodetector. There is an amplitude variation due to the frequency modification between the two beams. A frequency tracking filter locks onto the modulation frequency in the photodetector output to obtain the Doppler frequency which is linearly related to the velocity component through the optical system geometry. The doppler frequency will vary with time in unsteady laminar or turbulent flow. A continuous velocity signal is possible with this type of measurement system from which turbulence characteristics can be analysed.

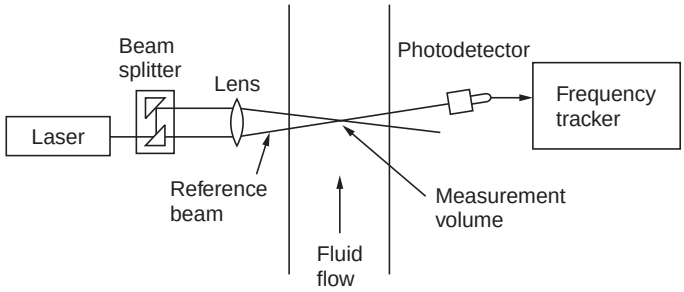


Figure 11.2.5 Laser Doppler anemometer

Multicomponent measuring systems can be developed by using laser with different wave lengths or other frequency shifting techniques. The major disadvantage of this system is its high cost and the requirement of optical access to the flow field.

11.3 VOLUME FLOW RATE MEASUREMENT

Volume flow rate in pipes can be measured either using direct measuring devices such as watermeter or rotameters (float meters) or using a constriction or elbow meters which produce a measurable pressure difference that can be used to determine the flow rate.

Flow meters (watermeter or rotameter) may be calibrated either by the manufacturer or by the user before installation. The same fluid and same range of flows as in the actual installation should be used for the calibration.

In the case of constriction meters Bernoulli equation and continuity equation are applied between the upstream and downstream sections of the constriction to obtain an expression for the flow rate.

11.3.1 Rotameter (Float Meter)

The rotameter is a device whose indication is essentially linear with flow rate. This device is also called as variable area meter or float meter. In this device a flot moves freely inside a tapered tube as shown in Fig. 11.3.1

The flow takes place upward through the tube. The following forces act on the float (*i*) downward gravity force (*ii*) upward buoyant force (*iii*) pressure and (*iv*) viscous drag force.

For a given flow rate, the float assumes a position inside the tube where the forces acting on it are in equilibrium. Through careful design, the effects of changes in viscosity or density may be minimized, leaving only the pressure forces as the main variable. Pressure force depends on flow rate and area available for flow. Hence the position of the float indicates the flow rate.

A major limitation in using rotameters is that these have to be installed in vertical position only. Also it cannot be used with liquids containing large number of solid particles and at high pressure conditions. It is also expensive. The advantage is that its capacity to measure the flow rate can be easily changed by changing the float or the tube.

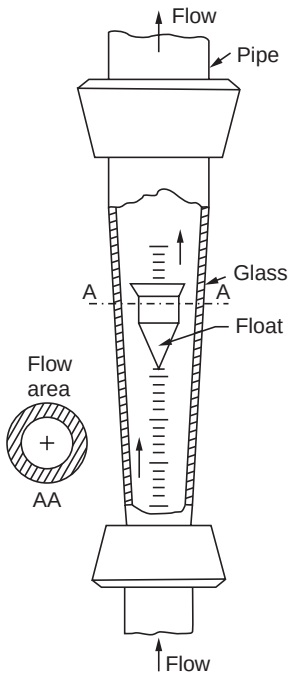


Figure 11.3.1 Rotameter

11.3.2 Turbine Type Flowmeter

Turbine type flow meter is used to measure flow in closed conduits. Propeller rotors are used in this meter. The number of turns of the rotor per unit time is counted and used as a

measure of the flow rate. The rotor movement is sensed by a reluctance pickup coil. A permanent magnet in the rotor body produces a voltage pulse everytime the rotor blade passes the pole of the coil. The pulse rate may be indicated by a frequency meter or displayed on a *CRO* screen or counted by some type of meter which converts the pulses to a proportional *DC* output. The major problem inherent in this type of meter is the reduced accuracy at low flow rates. As there is no intrusion in to the flow this type can be used to measure flow of chemicals also. The arrangement is shown in Fig. 11.3.2.

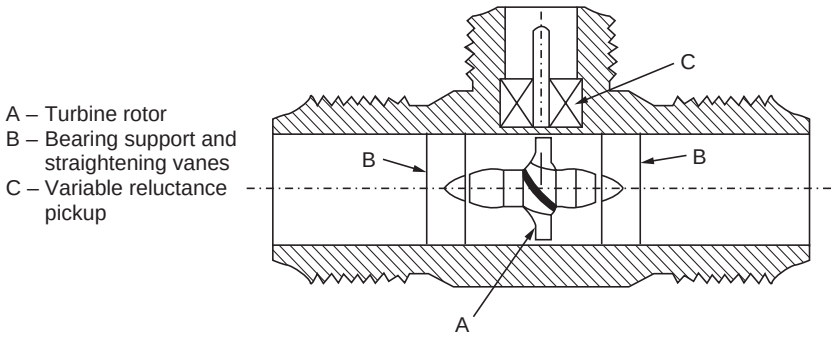


Figure 11.3.2 Turbine flowmeter

11.3.3 Venturi, Nozzle and Orifice Meters

Venturi, Nozzle and Orifice meters are the three obstruction type meters commonly used for the measurement of flow through pipes. In each case the meter acts as an obstacle placed in the path of the flowing fluid causing local changes in pressure and velocity as shown in Fig. 11.3.3.

Applying Bernoulli and continuity equations between sections 11.1 and 11.2

$$\frac{P_1}{\rho g} + \frac{V_1^2}{2g} + Z_1 = \frac{P_2}{\rho g} + \frac{V_2^2}{2g} + Z_2 \text{ and } V_1A_1 = V_2A_2$$

Solving these equations,

$$V_2 = \frac{1}{\sqrt{1-(A_2/A_1)^2}} \sqrt{2g[(p_1/\rho g + Z_1) - (P_2/\rho g + Z_2)]}$$

By connecting a manometer to the tappings at sections 11.1 and 11.2 the difference in pressure levels $\left[\left(\frac{P_1}{\rho g} + Z_1\right) - \left(\frac{P_2}{\rho g} + Z_2\right)\right]$ can be measured by the manometer reading, Δh.

$$\therefore \qquad V_2 = \frac{1}{\sqrt{1-(A_2/A_1)^2}} \sqrt{2g\Delta h}$$

$$\therefore \text{ Flow rate} \qquad Q = \frac{A_2}{\sqrt{1-(A_2/A_1)^2}} \sqrt{2g\Delta h}$$

Refer equations 11.2.3 and 11.2.2.

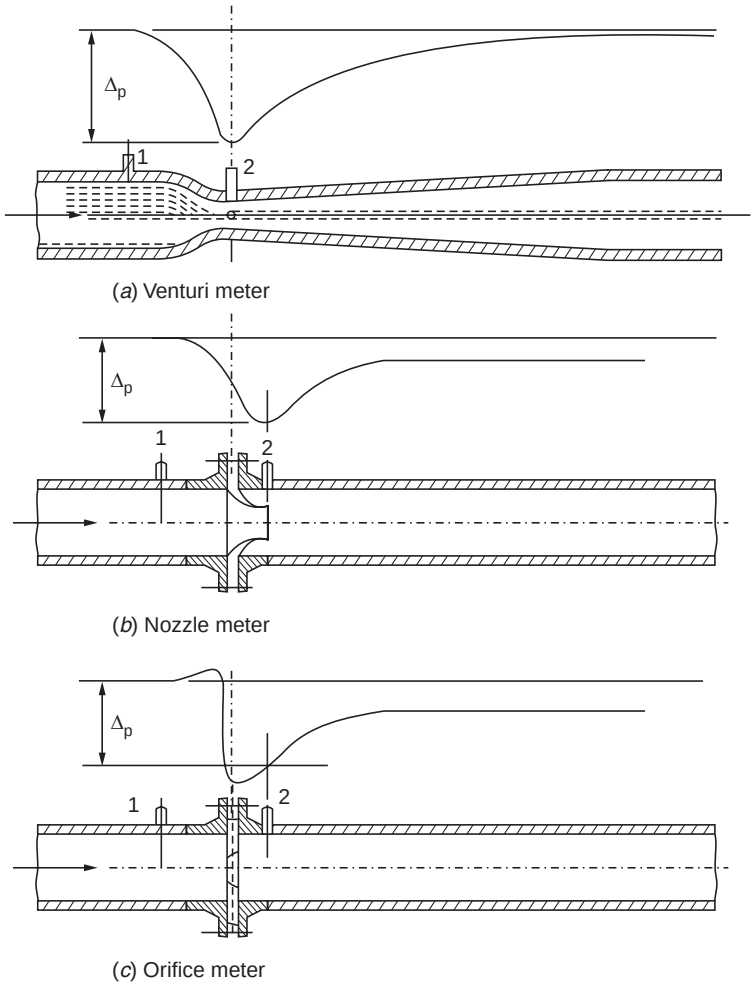


Figure 11.3.3 Pressure variation in obstruction type meters

This equation needs a modifying coefficient as viscous effects and boundary roughness as well as the velocity of approach factor that depend on the diameter ratio have been neglected. The coefficient is defined by,

$$\therefore \qquad Q_{\text{actual}} = Q_{\text{theoretical}} \times C_d$$

where C_d is the coefficient of discharge. Venturimeter is a highly accurate device with discharge coefficient falling within a narrow range depending on the finish of the entrance cone. C_d for venturi meters is in the range 0.95 to 0.98.

The approach curve in the nozzle flow meter must be proportioned to prevent separation between the flow and the well. A parallel section is used to ensure that flow fills the throat. C_d for flow nozzle is in the range 0.7 to 0.9 depending on diameter ratio and Reynolds number to some extent.

Orificemeter is the simple and cheap device compared to the other two. But sudden area of contraction in this device leads to higher pressure loss. The range for coefficient of discharge is 0.6 to 0.65. The value depends on the diameter ratio. Higher the value D_2/D_1 lower the value of the coefficient. In both the above cases for $Re > 10^5$ the effect of Re on C_d is marginal.

11.3.4 Elbow Meter

Elbow meter is used to measure the flow through a pipe. When the fluid flows through the elbow fitted in a pipe line, higher pressure results at the outer wall surface than at the innerwall surface. The difference in pressure at the outer and inner wall is a function of the flow rate. The pressure difference is measured using a manometer as shown in Fig. 11.3.4. The elbow meter is inexpensive and accurate if it is calibrated carefully.

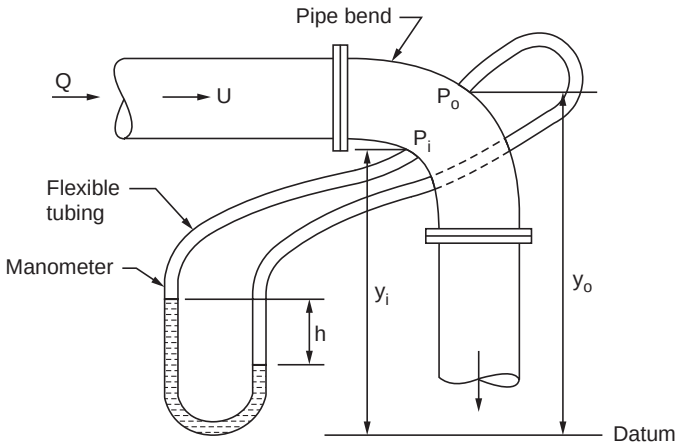


Figure 11.3.4 Elbow meter

11.4 FLOW MEASUREMENT USING ORIFICES, NOTCHES AND WEIRS

Flow out of open tanks are measured using orifices. Flow out of open channels is measured using weirs. Flow from open channels and tanks is due to gravity and the change in velocity produced is due to the change in head.

11.4.1 Discharge Measurement Using Orifices

Fig. 11.4.1 shows an orifice in an open tank through which the flow takes place. Applying Bernoulli equation between points 1 and 2

$$\frac{P_1}{\rho g} + \frac{V_1^2}{2g} + Z_1 = \frac{P_2}{\rho g} + \frac{V_2^2}{2g} + Z_2$$

The velocity a point 1 is zero and the pressures at 1 and 2 are both atmospheric.

$$\therefore \qquad Z_1 - Z_2 = V_2^2/2g$$

$$V_2 = \sqrt{2g(Z_1 - Z_2)} = \sqrt{2gh}$$

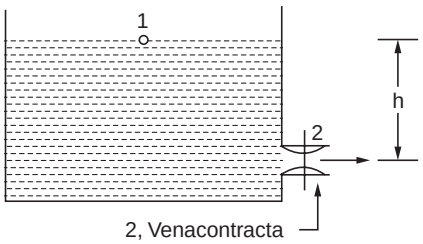


Figure 11.4.1 Orifice meter

The theoretical flow rate is given by $Q_t = A_2 \sqrt{2gh}$

where A_2 is the area of cross-section at section 11.2. The actual flow rate is given by

$$Q_{\text{actual}} = C_d A_0 \sqrt{2gh}$$

where A_0 is the area of orifice and C_d is the coefficient of discharge.

$$C_d = Q_{\text{actual}}/Q_{\text{theoretical}}$$

The values of C_d depends upon the contraction of the jet fromt the orifice to section 2 and on nonideal flow effects such as head losses which depend upon the roughness of the inside surface of the tank near the orifice and the flow rate. Typical value for C_d is 0.62.

Cofficient of velocity (C_v). There is alsways some loss of energy due to viscous effects in real fluid flows. Due to these effects, the actual flow velocity through the orifice will always be less than the theoretical possible velocity. The velocity coefficients C_v is defined as follows.

$$C_v = \frac{\text{Actual velocity of jet at venacontracta}}{\text{Theoritical velocity}} = \frac{V}{\sqrt{2gh}}$$

The value of C_v varies from 0.95 to 0.99 for different orifices depending on their shape and size.

Coefficient of contraction C_c. As water leaves an open tank through an orifice, the stream lines converge and the area just outside the orifice is lower compared to the area of the orifice. This section is called as vena contracta. Area of jet at the vena contracta is less than the area of the orifice itself due to convergence of stream lines. The coefficient of contraction C_c is defined as follows

$$C_c = \frac{\text{Area of the jet at vena contracta}}{\text{Area of orifice}} = \frac{a_c}{a}$$

The value of coefficient of contraction varies from 0.61 to 0.69 depending on the shape and size of the orifice.

Coefficient of discharge (C_d) Coefficient of discharge is defined as

$$C_d = \frac{\text{Actual discharge}}{\text{Theoretical discharge}} = \frac{\text{Actual area}}{\text{Theoretical area}} \times \frac{\text{Actual velocity}}{\text{Theoretical velocity}} = C_c \times C_v$$

Average value of C_d for orifices is 0.62.

11.4.2 Flow Measurements in Open Channels

Rectangular and triangular weirs are used to measure the flow in an open channel. A rectangular notch is shown in Fig. 11.4.2. A weir extends to the full width of the channel while a notch occupies a smaller width.

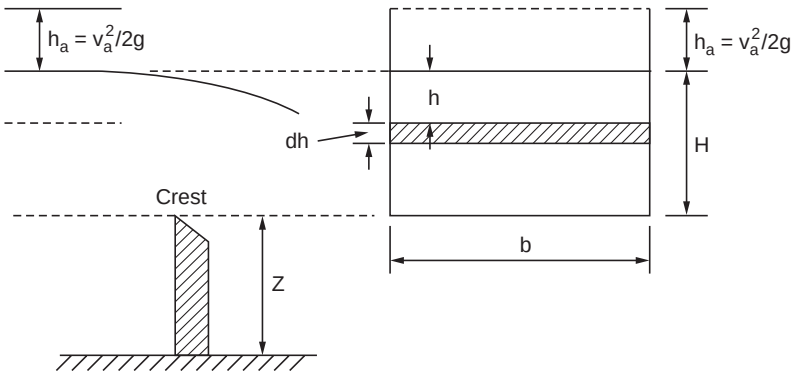


Figure 11.4.2 Rectangular weir

Rectangular weir. Bernoulli equation is applied between upstream and drown stream of the weir. Consider a rectangular strip as shown in figure, with height dh and width B at a height h above the strip.

Flow rate through the elemental strip = $dq = C_d\,(B\,dh)\,\sqrt{2gh}$

Integrating between the weir tip and the water level

Total discharge $$Q = \int_0^H C_d B dh\ \sqrt{2gh} = C_d B\ \sqrt{2g}\ \int_0^H h^{1/2}\,dh$$

$$Q = C_d B\sqrt{2g}\left[\frac{h^{3/2}}{3/2}\right]_0^H \quad Q = \frac{2}{3}C_d B\sqrt{2g}H^{3/2} \tag{11.4.1}$$

The value of C_d depends the approach velocity which in turn depends on the ratio of head H and crest height z. The value of C_d is given by

$$C_d = 0.611 + 0.075\frac{H}{z} \tag{11.4.2}$$

A trapezoidal weir with side slope of 1 horizontal to 4 vertical is used to compensate for flow reduction due to end contraction at the corners. It is called **Cipolletti weir.** The flow equation is the same with B as bottom width. The value of C_d will however be different.

Discharge over a triangular notch. A triangular notch is called V notch as shown in Fig. 11.4.3.

Consider an elemental strip dh, the discharge through the elemental strip dh is

$$dq = C_d\left(2\,(H-h)\tan\frac{\theta}{2}dh\right)\sqrt{2gh}$$

Total discharge $$Q = \int_0^H C_d\left(2(H-h)\tan\frac{\theta}{2}dh\right)\sqrt{2gh}$$

$$= 2\,C_d\,\sqrt{2g}\,\tan\frac{\theta}{2}\int_0^H (H-h)\,h^{1/2}dh$$

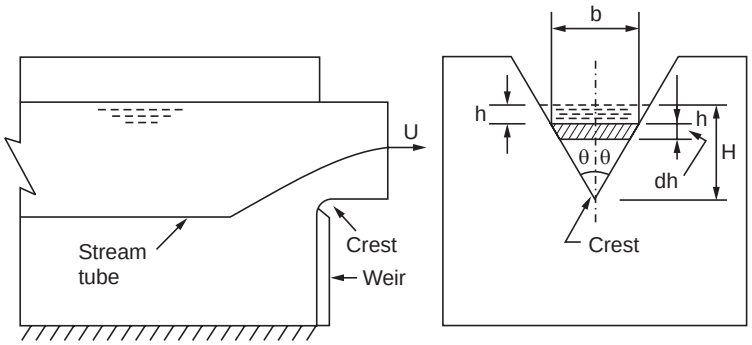


Figure 11.4.3 Triangular notch

$$Q = 2C_d\sqrt{2g}\,\tan\frac{\theta}{2}\left[\frac{Hh^{3/2}}{3/2} - \frac{h^{5/2}}{5/2}\right] = \frac{8}{15}C_d\sqrt{2g}\,\tan\frac{\theta}{2}H^{5/2} \qquad (11.4.3)$$

Table 11.4.1 A comparison of various flow measuring devices

Flow meter	Application	Accuracy	Pressure loss	Relative cost
Orifice	Clean, dirty liquids and some slurry	± 2 to 4% of the meter range	Medium	Low
Venturi meter	Clean, dirty and viscous liquids and some slurries	± 1% of meter range	Low	Medium
Flow nozzle	Clean and dirty liquids	± 1 to 2% of meter range	Medium	Medium
Pitot tube	Clean liquids	± 3 to 5% of meter range	Very low	Low
Elbow meter	Clean and dirty liquids and some slurries	± 5 to 10% of meter range	Very low	Low
Turbine meter	Clean and viscous liquids	± 0.25% of flow rate	High	High
Laser Doppler	Dirty, viscous liquids and slurries	± 5% of meter range	Nil	High
Rectangle and *V* notch	Clean and dirty liquids	± 2 to 5% of meter range	Very low	Medium

SOLVED PROBLEMS

Problem 11.1 *A pitot static tube is used to measure the velocity of air flowing through a duct. The manometer shows a difference in head of 5 cm of water. If the density of air and water are 1.13 kg/m³ and 1000 kg/m³* ***determine the velocity of air.*** *Assume the coefficient of the pitot tube as 0.98.*

Flow velocity, $\mathbf{V} = C_v\sqrt{2gh_m\left[(\rho_m/\rho)-1\right]}$

$$= 0.98\sqrt{2\times 9.81\times\left(\frac{1000}{1.13}-1\right)\frac{5}{100}} = \mathbf{28.86\ m/s}$$

Problem 11.2 *The difference in mercury level of a pitot static tube connected to a submarine is 20 cm.* ***Determine the speed of the submarine.*** *The density of sea water is 1019 kg/m³. Assume $C_v = 0.98$*

Speed of the submarine,

$$\mathbf{V} = C_v\sqrt{2gh_m\left((\rho_m/\rho)-1\right)}$$

$$= 0.98\sqrt{2\times 9.81\times\left(\frac{13600}{1019}-1\right)\frac{20}{100}} = \mathbf{6.821\ m/s\ or\ 25.6\ kmph.}$$

Problem 11.3 *A pitot static tube is mounted on an aircraft travelling at a speed 300 kmph against a wind velocity of 20 kmph. If the specific weight of air is 12 N/m³* ***determine the pressure difference the instrument will register.***

Aircraft velocity $= \dfrac{300\times 1000}{3600} = 83.33$ m/s,

Wind velocity $= \dfrac{20\times 1000}{3600} = 5.56$ m/s

Relative velocity of plane

$V = 83.33 + 5.56 = 88.89$ m/s

Velocity recorded by pitot tube,

$$V = C_v\sqrt{2gh}$$

$88.89 = 0.98\ \sqrt{2\times 9.81\times h}$, Solving for, h which is head of air,

$\mathbf{h_{air} = 419.32\ m,}$ $\Delta P = 419.32 \times 12 =$ **5032 N/m².**

Problem 11.4 *The flow velocity of water in a pipe is measured by a pitot static tube. The tube is placed at the centre of a 30 cm diameter pipe. The difference between the stagnation and static pressures measured as head of mercury and converted to head of water is 10 cm. If the coefficient of velocity $C_v = 0.98$,* ***determine the velocity of water in the pipe.*** *If the mean velocity is 0.7 times the centre line velocity, compute the discharge of water through the pipe. The head causing the flow is given as head of the flowing fluid. Hence the equation used is*

Centre line velocity $= C_v\sqrt{2gh} = 0.98\sqrt{2\times 9.81\times(10/100)}$ = **1.373 m/s**

Mean velocity in pipe = 0.7 × 1.373 = **0.961 m/s**

Discharge through the pipe, Q = Area of cross section × Mean velocity

$$\mathbf{Q} = \frac{\pi}{4} \times 0.3^2 \times 0.961 = \mathbf{0.068\ m^3/s\ or\ 68\ \textit{l}/s}$$

Problem 11.5 *A pitot static tube is used to measure the velocity of air in a duct. The water manometer shows a reading of 8 cm. The static pressure in the duct is 9 kN/m² and the air temperature is 320 K. The local barometer reads 740 mm of mercury.* ***Calculate the air velocity if*** C_v *= 0.98. Assume the gas constant for air as 287 J/kg K.*

Atmospheric pressure $= \rho gh = (13.6 \times 10^3)(9.81)\left(\frac{740}{1000}\right) = 98.73$ kN/m²

∴ Static pressure in the duct = 98.73 + 9 = 107.73 kN/m² (absolute pressure)

Density of air inside the duct,

$$\rho = \frac{P}{RT} = \frac{107.73\times 10^3}{287\times 320} = 1.173\ \text{kg/m}^3$$

Differential pressure head = 8 cm of water = $\frac{8}{100}\times\frac{1000}{1.173}$ = 68.2 m of air

Air velocity $\mathbf{V} = C_v\sqrt{2gh_{air}} = 0.98\sqrt{2\times 9.81\times 68.2}$ = **35.85 m/s**

Problem 11.6 *A venturimeter of 150 mm × 75 mm size is used to measure the flow rate of oil having specific gravity of 0.9. The reading shown by the U tube manometer connected to the venturimeter is 150 mm of mercury column.* ***Calculate the coefficient of discharge*** *for the venturimeter if the flow rate is 1.7 m³/min. (Note : The size of venturimeter generally specified in terms of inlet and throat diameters)*

Refer equation (6.6.2)

Velocity $$V_2 = \frac{C_d A_1}{\sqrt{A_1^2 - A_2^2}}\sqrt{2gh_m\left(\frac{\rho_m}{\rho} - 1\right)} \text{ and } Q = V_2 \times A_2$$

Flow rate $$Q = \frac{C_d A_1 A_2}{\sqrt{A_1^2 - A_2^2}}\sqrt{2gh_m\left(\frac{\rho_m}{\rho} - 1\right)}$$

Inlet area $$A_1 = \frac{\pi}{4} \times 0.15^2 = 0.0177\ \text{m}^2,$$

Throat area $$A_2 = \frac{\pi}{4} \times 0.075^2 = 0.00442\ \text{m}^2$$

Flow rate = (1.7/60) = 0.0283 m³/s, Substituting

$$0.0283 = \frac{C_d \times 0.0177 \times 0.00442}{\sqrt{0.0177^2 - 0.00442^2}} \sqrt{2 \times 9.81 \times 0.15 \left(\frac{13.6}{0.9} - 1 \right)}$$

$\therefore$ $\mathbf{C_d = 0.963}$

Problem 11.7 *A venturimeter is used to measure liquid flow rate of 7500 litres per minute. The difference in pressure across the venturimeter is equivalent to 8 m of the flowing liquid. The pipe diameter is 19 cm.* ***Calculate the throat diameter*** *of the venturimeter. Assume the coefficient of discharge for the venturimeter as 0.96.*

$$Q = C_d \frac{A_1 A_2 \sqrt{2gh}}{\sqrt{A_1^2 - A_2^2}},$$

$$A_1 = \frac{\pi}{4} \times 0.19^2 = 0.0284 \text{ m}^2$$

$$\frac{7500 \times 10^{-3}}{60} = \frac{0.96 \times 0.0284 \, A_2}{\sqrt{0.0284^2 - A_2^2}} \sqrt{2 \times 9.81 \times 8}, \text{ Solving } A_2 = 0.0098 \text{ m}^2$$

Let the diameter be d, $\frac{\pi}{4} \times d^2 = 0.0098$

$\therefore$ $\mathbf{d = \sqrt{\frac{4 \times 0.0098}{\pi}} = 9.9 \text{ cm}}$

Problem 11.8 *A venturimeter is fitted in a pipe of 30 cm diameter inclined at 40° to the horizontal to measure the flow rate of petrol having a specific gravity of 0.8. The ratio of areas of main pipe and throat is 5 and the throat is at 1 m from the inlet along its length. The difference in manometer head is 40 mm of mercury. Assuming the coefficient of discharge as 0.96.* ***Calculate the discharge through the venturimeter*** *and the pressure difference between the throat and the entry point of the venturimeter.*

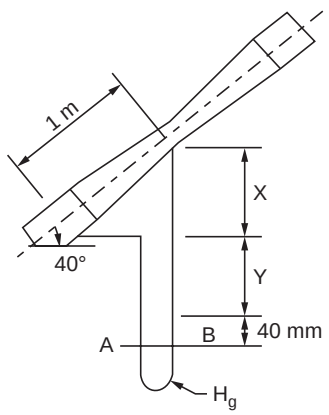


Figure P. 11.8

Refer equation 6.6.2 applicable for all orientations

$$Q = C_d \frac{A_1 A_2}{\sqrt{A_1^2 - A_2^2}} \sqrt{2gh_m \left(\frac{\rho_m}{\rho} - 1\right)}, A_1 = \frac{\pi}{4} \times 0.3^2 = 0.0707 \text{ m}^2$$

$A_1/A_2 = 5$

$\therefore$ $A_2 = 0.0707/5 = 0.0141$ m^2, Substituting,

$$\mathbf{Q} = \frac{0.96 \times 0.0141 \times 0.0707}{\sqrt{0.0707^2 - 0.0141^2}} \sqrt{2 \times 9.81 \times 0.04 \times \left(\frac{13.6}{0.8} - 1\right)}$$

$$= \mathbf{0.0486 \text{ m}^3/\text{s}}$$

Considering points A and B and level at A as datum

$$P_A + \rho gy + \rho g(0.04) = P_B + \rho gx + \rho gy + \rho_m g\,(0.04)$$

$$P_A - P_B = \rho gx + 0.04 \times g \times (\rho_m - \rho)$$

$$= \rho g\,(1 \times \sin 40) + 0.04 \times 9.81 \times (13600 - 800)$$

$$= 800 \times 9.81\,(1 \times \sin 40) + 0.04 \times 9.81 \times (13600 - 800)$$

$$= 10067.32 \text{ N/m}^2 \text{ or } 10.07 \text{ kN/m}^2$$

Problem 11.9 *A venturimeter of 20 cm × 10 cm size is calibrated in a laboratory using a right angled V notch. When a steady head of 0.187 m is maintained over the notch with a coefficient of discharge 0.6, the difference of head between he entrance and throat section of the Venturimeter is found to be 39 cm head of the fluid measured using notch as actual flow,* ***determine the discharge coefficient of venturimeter.***

Discharge over a triangular notch $Q = \frac{8}{15} C_D \sqrt{2g} \tan\frac{\theta}{2} h^{5/2}$

For right angled triangular notch, $\tan(\theta/2) = \tan 45° = 1$

$\therefore$ $$Q = \frac{8}{15} \times 0.6 \times \sqrt{2 \times 9.81} \times 0.187^{5/2} = 0.02143 \text{ m}^3/\text{s}$$

For the venturimeter

$$Q = C_d \frac{A_1}{\sqrt{(A_1^2 / A_2^2) - 1}} \sqrt{2gh}\,, A_1 = \frac{\pi}{4} \times 0.2^2 = 0.0314 \text{ m}^2$$

$$A_2 = \frac{\pi}{4} \times 0.1^2 = 0.0785 \text{ m}^2,$$

$$0.0214 = C_d \frac{0.0314}{\sqrt{(0.0314^2 / 0.00785^2) - 1}} \sqrt{2 \times 9.81 \times 0.39}$$

$$\mathbf{C_d = 0.96}$$

Problem 11.10 *A venturimeter with throat diameter 0.065 m and coefficient of discharge 0.95 is used to calibrate a pitot static tube. Air flows through a 110 mm diameter horizontal pipe in which the venturimeter is fitted. The difference in water level in the manometer attached to the venturimeter is 50 mm. The pitot static tube is placed downstream of the venturimeter and the water manometer attached to the pitot static tube shows a reading of 7 mm.* ***Calculate the flow rate through the pipe*** *and the coefficient of velocity of the pitot static tube. Assume the density of air as 1.13 kg/m^3 and that of water as 1000 kg/m^3.*

Flow rate through the venturimeter

$$Q = C_d \frac{A_1}{\sqrt{(A_1^2 / A_2^2) - 1}} \sqrt{2gh_m \left(\frac{\rho_m}{\rho} - 1 \right)}$$

$$A_1 = \frac{\pi}{4} \times 0.11^2 = 0.0095 \text{ m}^2, \quad A_2 = \frac{\pi}{4} \times 0.065^2 = 0.00332 \text{ m}^2$$

$$\mathbf{Q} = C_d \frac{0.95 \times 0.0095}{\sqrt{(0.0095^2 / 0.00332^2) - 1}} \sqrt{2 \times 9.81 \times 0.05 \left(\frac{1000}{1.13} - 1 \right)}$$

$$= \mathbf{0.0992 \text{ m}^3/\text{s}}$$

$\therefore$ Mean velocity down stream, $V = 4 \times 0.0992/\pi \times 0.11^2 = 10.44$ m/s

Mean velocity measured by pitot static tube

$$V = C_v \sqrt{2gh_m \left(\frac{\rho_m}{\rho} - 1 \right)}$$

$$10.44 = C_v \sqrt{2 \times 9.81 \times 0.007 \left(\frac{1000}{1.13} - 1 \right)}$$

$\therefore$ $\mathbf{C_v = 0.95}$

Problem 11.11 *A venturimeter with throat diameter 5 cm and coefficient of discharge 0.96 is fitted in a pipeline which carries water in it. The pressure difference across the venturimeter is 12 N/m^2. If an orifice meter with 5 cm diameter is futted in the same pipe line,* ***determine the coefficient of discharge of*** *the orifice meter, if the pressure difference across it is 28 N/m^2.*

Flow rate in venturimeter

$$Q_v = C_d \frac{A_1}{\sqrt{(A_1^2 / A_2^2) - 1}} \sqrt{2gh}$$

$$h = \frac{P}{\rho g} = \frac{12}{1000 \times 9.81},$$

$$Q = 0.96 \frac{A_1}{\sqrt{(A_1^2 / A_2^2) - 1}} \sqrt{2 \times 9.81 \times \left(\frac{12}{1000 \times 9.81} \right)}$$

Flow rate through the orifice

$$Q_0 = C_d \frac{A_1}{\sqrt{(A_1^2 / A_2^2) - 1}} \sqrt{2gh}$$

Since the throat diameter of venturi and orifice are the same cancelling common terms

$$0.96 \sqrt{2 \times \frac{12}{1000}} = C_d \sqrt{2 \times 9.81 \times \frac{28}{1000 \times 9.81}}$$

Simplifying $0.149 = C_d \times 0.237 \quad \therefore \mathbf{C_d} = \frac{0.149}{0.237} = \mathbf{0.63}$

Problem 11.12 *Water flows in an elbowmeter creating a pressure difference of 10 kN/m² between its outer and inner wall. The elbowmeter is fitted in a vertical pipe of 15 cm diameter. If the tapping point at the outer wall of the elbowmeter is 5 cm higher than the tapping point at the innerwall.* ***Calculate the flow rate through*** *the elbowmeter.*

The pressure loss in elbowmeter is (suffices i and o indicate inside and outside)

$$\Delta P = \left(\frac{P_0}{\rho g} + y_0\right) - \left(\frac{P_i}{\rho g} + y_i\right)$$

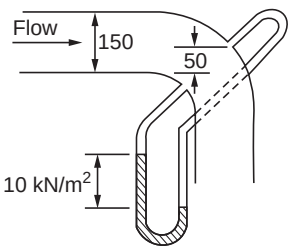


Figure P. 11.12

Also pressure loss can be expressed in terms of the flow velocity in the bend as

$$\Delta P = k \frac{V^2}{2g}$$

$$\therefore \quad k \frac{V^2}{2g} = \left(\frac{P_0}{\rho g} + y_0\right) - \left(\frac{P_i}{\rho g} + y_i\right)$$

$$V = \sqrt{\frac{2g}{k}\left[\left(\frac{P_0}{\rho g} + y_0\right) - \left(\frac{P_i}{\rho g} + y_i\right)\right]}$$

$\therefore$ Discharge through the elbowmeter

$$Q = A \times V = A \sqrt{\frac{2g}{k}\left[\left(\frac{P_0}{\rho g} + y_0\right) - \left(\frac{P_i}{\rho g} + y_i\right)\right]}$$

Assuming $k = 1$

$$\mathbf{Q} = C_d A \sqrt{2g\left[\left(\frac{P_0 - P_i}{\rho g} + (y_0 - y_i)\right)\right]}$$

$$= 0.6 \times \frac{\pi}{4} 0.15^2 \sqrt{2 \times 9.81\left[\frac{10 \times 10^3}{1000 \times 9.81} + 0.05\right]} = \mathbf{0.0486\ m^3/s}$$

Problem 11.13 *The actual velocity of a liquid issuing through a 7 cm diameter orifice fitted in an open tank is 6 m/s under a head of 3 m. If the discharge measured in a collecting tank is 0.020 m^3/s,* ***calculate the coefficient of velocity,*** *coefficient of contraction and the theoretical discharge through the orifice.*

Flow velocity in orifice $= V = C_v \sqrt{2gh}$, $6 = C_v \sqrt{2 \times 9.81 \times 3}$

Coefficient of velocity $\mathbf{C_v = 0.9124}$ $\therefore$ Actual discharge $Q = C_d \sqrt{2gh}$

$$0.020 = C_d \frac{\pi}{4} \times \left(\frac{7}{100}\right)^2 \times \sqrt{2 \times 9.81 \times 3}$$

Coefficient of discharge

$$C_d = 0.6774$$

Coefficient of contraction

$$= \mathbf{C_c =} \frac{C_d}{C_v} = \frac{0.6774}{0.9124} \mathbf{= 0.7424}$$

Problem 11.14 *Water is discharged through a 15 cm diameter orifice in the vertical side of an open tank at the rate of 190 litres per second. Water stands 15 m above the centerline of the orifice. A point on the jet measured from the vena contracta has co-ordinates 5 m horizontal and 0.5 m vertical.* ***Find the hydraulic coefficients C_v, C_c and C_d of the orifice.***

Actual discharge $Q = C_d A \sqrt{2gh}$

$$190 \times 10^{-3} = C_d \times \frac{\pi}{4} \times 0.15^2 \times \sqrt{2 \times 9.81 \times 15}, \mathbf{C_d = 0.627}$$

Let the jet travel during time t horizontally through a distance x and the jet fall by distance y during this time.

$\therefore$ $x = V \times t = C_v \sqrt{2gh}\, t$ or $x^2 = C_v^2\, 2ght^2$

$y = (1/2)gt^2$

$\therefore$ $\dfrac{x^2}{y} = 4C_v^2 h$

$\therefore$ $C_v = \sqrt{\dfrac{x^2}{4yh}}$,

Here x = 5 m, and y = 0.5 m

$\therefore$ $\mathbf{C_v} = \sqrt{\dfrac{5^2}{4 \times 0.5 \times 15}} \mathbf{= 0.913}$

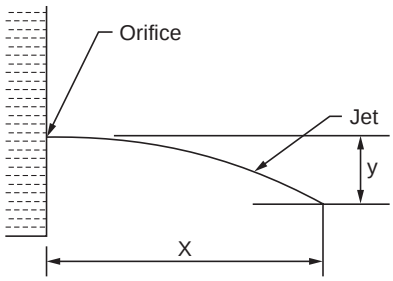


Figure P. 11.14

Coefficient of contraction,

$$\mathbf{C_c} = \frac{C_d}{C_v} = \frac{0.627}{0.913} = \mathbf{0.687}$$

Problem 11.15 *An orifice of 8 cm diameter is fitted in a 20 cm diameter pipe that carries oil of specific gravity 0.8. The mercury manometer attached to the orifice shows a reading of 0.75 m.* ***Calculate the oil flow rate*** *throught the pipe. Assume coefficient of discharge for orifice as 0.6.*

Flow rate $$Q = C_d \frac{A_1}{\sqrt{(A_1^2 / A_1^2) - 1}} \sqrt{2gh_m\left(\frac{\rho_m}{\rho} - 1\right)}$$

$$A_1 = \frac{\pi}{4} \times 0.2^2 = 0.0314 \text{ m}^2, \quad A_2 = \frac{\pi}{4} \times 0.08^2 = 0.00503 \text{ m}^2$$

$$\mathbf{Q} = \frac{0.6 \times 0.0314}{\sqrt{(0.0314^2 / 0.00503^2) - 1}} \sqrt{2 \times 9.81 \times 0.75\left(\frac{13600}{800} - 1\right)}$$

$$= \mathbf{0.047 \text{ m}^3/\text{s} = 47 \text{ l/s}}$$

Problem 11.16 *A rectangular notch is used to measure the flow rate of water in an open channel. The width of the rectangular notch is 3 m and the head of water causing the flow is 0.5 m.* ***Calculate the discharge assuming the coefficient of discharge*** *of the notch as 0.6.*

Discharge (Refer eqn 11.4.1) $$\mathbf{Q} = \frac{2}{3} C_d B \sqrt{2g}\, H^{3/2}$$

$$= \frac{2}{3} \times 0.6 \times 3\sqrt{2 \times 9.81} \times 0.5^{3/2} = \mathbf{1.88 \text{ m}^3/\text{s}}$$

Problem 11.17 ***What should be the width of a rectangular notch*** *that should be used to measure the water flow rate of 0.5 m^3/s in an open channel. The head causing the flow should not exceed half the notch width. Assume the coefficient of discharge of the rectangular notch as 0.6.*

Discharge $$Q = \frac{2}{3} C_d B \sqrt{2g}\, H^{3/2} \quad \text{or} \quad 0.5 = \frac{2}{3} \times 0.6 \times B\sqrt{2 \times 9.81}\left(\frac{B}{2}\right)^{3/2}$$

$$B^{5/2} = \frac{0.5}{0.626} = 0.799, \quad \mathbf{B = 0.914 \text{ m}}$$

Problem 11.18 *Water flow rate in an open channel is measured using a right angled V notch. The head of water over V notch is 0.15 m. Assuming the coefficient of discharge of the notch as 0.65,* ***calculate the discharge in the channel.***

As $\theta/2 = 45°$, and $\tan(\theta/2) = 1$, the flow equation 11.4.3 reduces to

Discharge, $$\mathbf{Q} = \frac{8}{15} C_d \sqrt{2g}\, H^{5/2}$$

$$= \frac{8}{15} \times 0.65 \times \sqrt{2 \times 9.81} \tan 45 \times 0.15^{5/2} = \mathbf{0.0134 \text{ m}^3/\text{s}}$$

Problem 11.19 *Water flows over a right angled V notch at the rate of 0.045 m^3/min maintaining a head of 0.048 m.* ***Calculate the coefficient of discharge*** *of the notch.*

Discharge $$Q = \frac{8}{15} C_d \sqrt{2g} \tan\frac{\theta}{2} H^{5/2}$$

$$\frac{0.045}{60} = \frac{8}{15} C_d \sqrt{2 \times 9.81} \tan 45 \times 0.048^{5/2}, \text{ Solving } \mathbf{C_d = 0.63}$$

Problem 11.20 *A right angled V notch is used to measure the flow rate in an open channel which carries water at the rate of 0.15 m^3/s. If the maximum depth of water is not to exceed 1 m,* ***find the position of the apex of the notch from*** *the bed of the channel. Assume the coefficient of discharge as 0.6.*

Discharge $$Q = \frac{8}{15} C_d \sqrt{2g} \tan\frac{\theta}{2} H^{5/2}$$

$$0.15 = \frac{8}{15} \times 0.6 \sqrt{2 \times 9.81} \tan 45 \times H^{5/2}$$

$\therefore$ **H = 0.41 m**

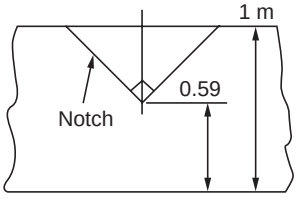


Figure P. 11.20

The distance of apex of *V* notch from the bed of channel

= Maximum depth of water – *H* = 1 – 0.41 = **0.59 m**

Problem 11.21 *An open channel is fitted with a rectangular and right angled V notch with coefficient of discharge 0.6 and 0.65 respectively. The width of rectangular notch is 0.5 m and the head causing the flow over it is 0.1 m.* ***Determine the head of water over*** *the V notch.*

Discharge through the rectangular notch,

$$Q = \frac{2}{3} C_d B \sqrt{2g} H^{3/2} = \frac{2}{3} \times 0.6 \times 0.5 \sqrt{2 \times 9.81} (0.1)^{3/2}$$

$$= 0.028 \text{ m}^3/\text{s}$$

Discharge through the *V* notch,

$$Q = \frac{8}{15} C_d \sqrt{2g} \tan\frac{\theta}{2} H^{5/2}$$

$$0.028 = \frac{8}{15} \times 0.65 \sqrt{2 \times 9.81} \tan 45 \times H^{5/2} \quad \therefore \mathbf{H = 0.202 \text{ m}}$$

REVIEW QUESTIONS

1. Explain the principle involved in measuring velocity of flow using a pitot static tube.
2. Sketch and describe the construction details of a hot wire anemometer. Describe the methods of velocity measurement using it.
3. Discuss the method of velocity measurement using (*i*) Vane anemometer and (*ii*) Turbine meter.
4. Explain the method of velocity measurement using Laser Doppler anemometers.

5. Derive the expression for computing discharge through a Venturimeter.
6. Compare the merits and demerits of flow measurement using Venturimeter, Orificemeter and nozzlemeter.
7. Define venacontracta. Derive the expression for flow measurement through an orifice in an open tank.
8. Derive an expression for the flow measurement in an open channel using rectangular notch.
9. Derive an expression for the flow measurement using a triangular notch in an open channel.
10. Compare the flow measuring devices with respect to their accuracy.

OBJECTIVE QUESTIONS

O Q.11.1 Fill in the blanks

1. Pitot static tube is used to measure _____
2. In pitot static tube the opening perpendicular to the flow direction measures _____
3. In pitot static tube the opening facing the flow direction measures _____
4. Coefficient of velocity of orifice is the ratio of _____
5. Coefficient of contraction is the ratio of _____
6. Coefficient of discharge is the ratio of _____
7. When a fluid flows though an elbow, higher pressure results at the _____
8. The rotary movement in vane anemometer and current meter is created by the _____
9. In turbine meter the rotor movement is sensed by a _____
10. In a hot-wire anemometer, an _____ is placed in a flowing stream.

Answers

1. Velocity of flow 2. Static pressure 3. Stagnation pressure 4. Actual velocity of jet at venacontracta to theoretical velocity 5. Area of the jet at venacontracta to the area of orifice 6. Actual flow/theoretical flow 7. Outerwall 8. Drag force on vanes and cones 9. Reluctance pickup coil 10. Electrically heated wire.

O Q.11.2 Select the correct answer

1. A pitot static tube is used to measure
   (*a*) Stagnation pressure
   (*b*) Static pressure
   (*c*) Dynamic pressure
   (*d*) Difference between the static pressure and dynamic pressure.
2. The more accurate flow measuring instrument is
   (*a*) Orificemeter (*b*) Venturimeter (*c*) Flowmeter (*d*) Elbow meter.
3. The range of coefficient of discharge for orifice meter is
   (*a*) 0.6 to 0.7 (*b*) 0.7 to 0.85 (*c*) 0.85 to 0.92 (*d*) 0.92 to 0.98.
4. Rotameter is used to measure
   (*a*) Viscosity (*b*) Flow (*c*) Density (*d*) Pressure.
5. Anemometer is used to measure
   (*a*) Velocity (*b*) Pressure (*c*) Viscosity (*d*) Density.
6. A *V* notch is used to measure
   (*a*) Velocity in a pipe (*b*) Wind velocity
   (*c*) Discharge of liquid in an open channel (*d*) Viscosity.

7. Flow rate in an open channel is more accurately measured using
   (*a*) Rectangular notch (*b*) Triangular notch
   (*c*) Venturi (*d*) Orifice.
8. Fluid velocity can be measured without disturbing the flow using
   (*a*) Pitot tube (*b*) Hot wire anemometer
   (*c*) Turbine meter (*d*) Laser Doppler anemometer.
9. Coefficient of discharge is the ratio of
   (*a*) Actual flow/Theoretical flow (*b*) Theoretical flow/Actual flow
   (*c*) Actual velocity/Theoretical velocity (*d*) Theoretical velocity/Actual velocity.
10. Current meter is used to measure
   (*a*) Pressure (*b*) Velocity (*c*) Density (*d*) Viscosity.

Answers

1. *c* 2. *b* 3. *a* 4. *b* 5. *a* 6. *c* 7. *b* 8. *d* 9. *a* 10. *b*

EXERCISE PROBLEMS

E 11.1 A water manometer attached to a pitot static tube used to measure air velocity shows a reading of 0.1 m. Assuming the ecoefficient of velocity as 0.98 and the density of air as 1.2 kg/m^3, calculate the air velocity. **(39.3 m/s)**

E 11.2 A pitot static tube is used to measure the velocity of water in a pipeline. If the mercury manometer attached to it shows a reading of 0.17 m, calculate the water velocity in the pipe. Assume coefficient of velocity as 0.98. **(6.3 m/s)**

E 11.3 A pitot static tube fitted in a pipe of 0.25 m diameter records the difference in stagnation and static pressure as 0.085 m of water. Assume the velocity coefficient as unity, calculate the water velocity in the pipe line. **(1.3 m/s)**

E 11.4 Air velocity in a duct is measured as 38.2 m/s by a pitot tube. Density of flowing air 1.3 kg/m^3. If the pressure difference recorded by the pitot static tube is 0.1 m of water, calculate the coefficient of velocity of the pitot static tube. **(0.98)**

E 11.5 Oil of density 800 kg/m^3 flows in a pipe of 0.2 m diameter at the rate of 0.06 m^3/s. A venturimeter with 0.1 m throat diameter is used to measure the flow rate of oil in the pipe. If the mercury manometer attached to it shows a reading of 0.18 m, calculate the coefficient of discharge of the Venturimeter. **(0.99)**

E 11.6 Oil of density 900 kg/m^3 flows through a pipe of 150 mm diameter. A venturimeter having throat of 100 mm diameter is fitted to the pipe line for measuring the flow rate of oil. A mercury manometer attached to it shows a reading of 200 mm. Assuming the coefficient of discharge for Venturimeter as 0.98, determine the oil flow rate. **(0.0064 m^3/s)**

E 11.7 A venturimeter with 0.08 m throat diameter is used to measure the flow in a pipe line of 0.16 m diameter. A mercury manometer attached to it shows deflection of 0.29 m. Assuming coefficient of discharge as 1, calculate the flow rate in the pipe. **(0.05 m^3/s)**

E 11.8 A pipe of 0.25 m diameter carries water at the rate of 7.2 m^3/s.The pressure head at the entry of the venturimeter, used to measure the flow rate in the pipe, is equivalent to 6 m of water. If the pressure head at the throat is zero, calculate the throat diameter of the venturi.

(0.12 m)

E 11.9 Water flows in a pipe of 0.3 m diameter. A venturimeter with throat diameter 0.1 m is fitted in the pipe line. If the pressure difference recorded is 19.4 m of water, calculate the discharge assuming the coefficient of the discharge as unity. **(0.15 m^3/s)**

E 11.10 Oil of density 800 kg/m^3 flows in a pipe of 0.25 m diameter. An orifice meter of 0.1 m diameter fitted in the pipe to measure the flow rate. A mercury manometer fitted across the orifice shows a reading of 0.8 m. Calculate the discharge through the pipe. Assume the coefficient of discharge of orifice as 0.65. **(0.082 m^3/s)**

E 11.11 An orifice meter of 0.15 m diameter is fitted in a 0.3 m diameter pipe to measure the flow rate of water through it. If the pressure difference across the orifice is 10 m of water head, calculate the discharge in the pipe. Assume the coefficient of discharge of the orifice meter as 0.59. **(0.15 m^3/s)**

E 11.12 An orificemeter with 5 cm diameter is used to measure the flow rate of liquid. Under a head of 4 m, the velocity of liquid at vena contracta is 7.5 m/s. If the actual discharge through the pipe is 8 l/s, calculate the coefficients of velocity, discharge and contraction. **(0.85, 0.46, 0.54)**

E 11.13 Oil of specific gravity 0.8 flows through a pipe of 0.25 m diameter. An orifice of 0.1 m diameter is fitted to the pipe to measure the flow rate. A mercury manometer fitted across the orifice records a reading of 0.8 m. Calculate the coefficient of discharge of the orifice meter if the flow rate measured by it is 0.082 m^3/s. **(0.65)**

E 11.14 A rectangular notch of 250 cm width is used to measure the flow rate of water in an open channel. If the actual flow rate is 1.16 m^3/s, under a head of 0.253 m determine the coefficient of discharge of the notch. **(0.62)**

E 11.15 Determine the coefficient of discharge of a rectangular notch of 0.8 m width used to measure the flow rate in an open channel. The head causing flow is 0.4 m. If the coefficient of discharge is 0.6, determine the flow rate. **(0.358 m^3/s)**

E 11.16 Water flows over a right angled V notch to a height of 0.2 m. Calculate the coefficient of discharge of the notch if the actual flow rate measured is 26.2 l/s. **(0.62)**

E 11.17 Water flows at the rate of 106 *l*/ sec in a open channel in which rectangular and V notches are fitted. The width of the rectangular notch is 100 cm and the head of water over it is 0.15 m. If the V notches is right angled calculate the coefficients of discharge of both rectangular and V notches. **(0.62, 0.59)**

12 Flow in Open Channels

12.0 INTRODUCTION

Flow in rivers, irrigation canals, drainage ditches and aqueducts are some examples for open channel flow. These flows occur with a free surface and the pressure over the surface is atmospheric. The surface actually represents the hydraulic grade line. In most cases water is the fluid encountered in open channel flow. While in closed conduits the flow is sustained by pressure difference, **the driving force in open channel flow is due to gravity, and is proportional to the bed slope**. The depth of flow is not restrained and this makes the analysis more complex. As most of the flow are large in scale and as viscosity of water is lower, Reynolds number are high. Hence the flow is generally turbulent. As seen in chapter 8 and 9, Froude number is the improtant parameter in the general study of open channel flow which is free surface flow. **The balance of gravity forces and surface friction forces controls the flow**. Changes in channel cross-section and changes in the slope cause changes and readjustments in the flow depth which may or not propagate upstream.

12.1.1 Characteristics of Open Channels

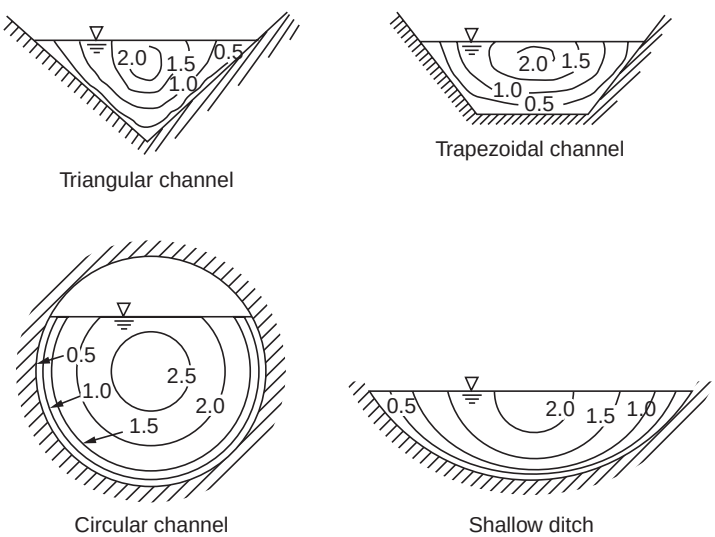


Figure 12.1.1 Velocity distribution in open channel sections

Open channels may have different cross-sections. Some of the simpler ones are **Trapezoidal, Triangular, Rectangular and Circular sections.** When the width is large, it is considered as wide flat.

Natural channels have very irregular sections and suitable approximations should be used for analysis. **The two main physical dimensions used in the analysis are the flow area, A and the wetted perimeter P. The ratio of flow area to the perimeter is defined as hydraulic radius, $\mathbf{R_h}$** (alternately *m*) and is used in all analysis to take care of all types of sections. With this definition laminar flow is limited to Reynolds number up to 500.

This is different from hydraulic mean diameter used in the analysis of flow through conduits which is four times this value. **Hydraulic depth is another term defined as the ratio of flow area to top width.** This represents the average depth of the section. For analysis purposes the average velocity of flow is used and this equals, volume flow rate/area. But actually the flow velocity varies with the depth almost logarithmically, low near the wetted surface and increasing towards the free surface. However the velocity is not maximum at the surface. The maximum velocity occurs below the free surface. **For analysis,** as mentioned earlier, the **average velocity is used.** Some examples of velocity distribution is shown in Fig. 12.1.1. The maximum velocity occurs at about 0.2 times the depth from top. The average velocity occurs between 0.4 and 0.8 times the depth from top.

12.1.2 Classification of Open Channel Flow

The common classification is based on the rate of change of free surface depth. When the **depth and velocity remain constant along the length of flow** it is called **uniform flow**. For such flow the slope and area should be uniform. When the depth changes gradually, due to area or slope changes, it is defined as g**radually varying flow.** If the slope change rapidly or suddenly, then such flow is called **rapidly varying flow.** The slope of the free surface is governed by the way in which the slope and the area change. The value of **Froude number characterises the nature of the flow** in such situations.

12.2 UNIFORM FLOW: (ALSO CALLED FLOW AT NORMAL DEPTH)

This is the simplest and common type of flow and occurs when conditions are steady and slope is not steep. This is also non accelerating flow. Consider the control volume between sections 1 and 2 shown in Fig. 12.2.1 by the dotted line. For steady uniform incompressible flow the height of the water level and area are constant, the hence velocity is constant.

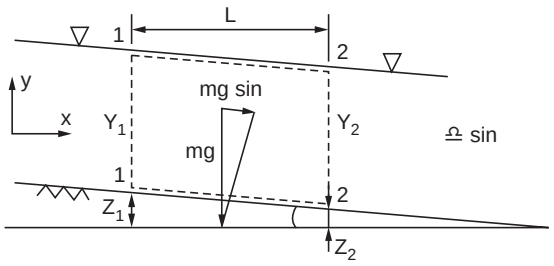


Figure 12.2.1 Flow at normal depth

(i) Continuity equation, $\rho_1 A_1 V_1 = \rho_2 A_2 V_2$ as $\rho_1 = \rho_2$ and $A_1 = A_2$, $V_1 = V_2$ or the velocity remains constant along the flow

(ii) Momentum equation. Bed slope is defined as the ratio of change in elevation over a length with the length. The bed slope S_b(= sin θ) is small and pressure distribution is hydrostatic. As there is no change in the depth of flow or velocity (no acceleration), the momentum flux through the control surface is zero. As the pressure distribution is hydrostatic the net pressure force on the control volume is zero. Only body force due to gravity and friction forces/ on the wetted surfaces, F_f, act on the control volume.

$$-F_f + mg \sin \theta = 0$$

$$\therefore \quad F_f = mg \sin \theta = mg\, S_b \tag{12.2.1}$$

mg sin θ is the component of gravity force parallel to the flow and F_f is the friction force on the wetted surface.

(iii) Energy equation. The sum of potential and kinetic heads between sections 1 and 2 should be the same if there are no losses. Assuming a loss of head of h_L,

$$\frac{V_1^2}{2g} + y_1 + z_1 = \frac{V_2^2}{2g} + y_2 + z_2 + h_L$$

For the steady flow $V_1 = V_2$, $y_1 = y_2$ and for a length L,

$$\therefore \quad h_L = z_1 - z_2 = L\, S_b \tag{12.2.2}$$

The head loss due to friction in steady flow between two sections equals the change in elevation of the bed. Defining specific energy, E, at a section by the sum $(V^2/2g) + y$, the specific energy is constant along this type of flow. The energy grade line, hydraulic grade line and the channel bed are all parallel.

12.3 CHEZY'S EQUATION FOR DISCHARGE

Considering the control volume shown in Fig. 12.2, the force balance yields frictional resistance over the wetted surface equals the component of the gravity forces along the surface, on the volume.

Gravity force component = $mg\, A\, L \sin \theta$

Frictional force $= \tau_w\, PL$

(tan θ = sin θ = θ in radians for small angles)

Equating the forces $\tau_w\, PL = \rho g\, AL \sin \theta$ (12.3.1)

or $$\tau_w = \rho g\, (A/P) \sin \theta = \rho\, g\, R_h\, S_b \tag{12.3.2}$$

This is the reason for defining hydraulic radius as A/P. For large values of Reynolds number the friction factor is independent of Reynolds number and wall shear stress is proportional to the dynamic pressure $\rho\, V^2/2g$ and is independent of viscosity. Hence

$\tau_w = K \rho\, V^2/2\, g_o$ where K is a constant of proportionality.

Substituting in equation 12.3.2,

$$K \frac{\rho V^2}{2 g_o} = \rho\, g\, R_h\, S_b$$

Denoting $2gg_o/K$ by constant C,

$$V = C\sqrt{R_h S_b} \tag{12.3.3}$$

This equation published in 1775 is known as **Chezy's** equation and the constant C is known as Chezy's constant. (for R_h and S_b other symbols like m, i etc are also used).

Flow rate $$Q = \rho A V = C \rho A (R_h S_b)^{0.5} \tag{12.3.4}$$

It may be noted that Chezy coefficient C is not dimensionless. It has a dimension of (length$^{1/2}$/ time) and hence will have different numerical values in different systems of units.

Another method of deriving the equation is as below. As in the case of closed duct flow, the friction head loss is given by

$$h_L = f L V^2/ 2 gD,$$

Substituting $D = 4R_h$, and $h_L = L S_b$,

$$V = \left(\frac{8g}{f}\right)^{0.5} (R_h S_b)^{0.5}$$

Denoting constant $$C = (8g/f)^{0.5} \tag{12.3.5}$$

$$V = C (R_h S_b)^{0.5} \tag{12.3.6}$$

Example 12.1 ***Determine the flow rate of water through a rectangular channel 3 m wide with a flow depth of 1 m. The bed slope is 1 in 2500. f = 0.038.***

wetted perimeter, $P = 3 + (2 \times 1) = 5$ m, Area $= 3 \times 1 = 3$ m^2

$\therefore$ $R_h = 3/5 = 0.6$ m, $S_b = 1/2500$, $C = \left(\frac{8g}{f}\right)^{0.5} = \left(\frac{8 \times 9.81}{0.038}\right)^{0.5} = 45.455$

$\therefore$ $V = 45.445 (0.6 \times 1/2500)^{0.5} = 0.704$ m/s,

$\therefore$ **Flow rate** $= 3 \times 1 \times 0.704 =$ **2.112 m^3/s**

12.4 DETERMINATION OF CHEZY'S CONSTANT

The Chezy's equation is simple but the determination of the constant C is rather involved. Several correlations have been suggested from experimental measurements for obtaining the value of Chezy's constant C. Out of these, three are more popular namely Bazins, Kutters and Mannings. As the first two correlations are more complex, Mannings correlation is generally used in designs.

12.4.1 Bazin's Equation for Chezy's Constant

The equation suggested by Bazin is given below.

$$C = 86.9/(1 + k/\sqrt{R_h}) \tag{12.4.1}$$

where k is known as Bazins constant. The value varies from 0.11 for smooth cement surfaces to 3.17 for earthen channel in rough condition. For Brick lined channel the value is about 0.5. This correlation is independent of bed slope.

Example 12.2 *Calculate the value of the Chezy's constants using Bazins equation in the case of a rectangular channel 3 m wide and 1 m deep for the following conditions. For a bed slope of 1/2500 find the flow rate. The value of Bazins constant, k, are given below.*

(i) Smooth cement lining : *0.06*

(ii) Smooth brick : *0.16*

(iii) Rubble masonry : *0.46*

(iv) Earthen channel is ordinary condition : *1.303*

(v) Earthen channel in rough condition : *1.75*

$R_h = 3/5 = 0.6$.

Using the above values, the constant C is calculated by the equations below:

$$C = 86.9/(1 + k/\sqrt{0.6}),\ V = C\sqrt{0.6/2500}$$

These values of Bazins constant k can be taken as typical values

S. No.	Nature of Surface	Bazins Constant	Chezy's Constant	V, m/s	Q, m^3/s
1	Smooth cement lining	0.060	**80.65**	1.250	3.75
2	Smooth brick	0.160	**72.02**	1.116	3.35
3	Rubble masonry	0.460	**54.52**	0.847	2.53
4	Earthen channel in ordinary condition	1.303	**32.40**	0.502	1.51
5	Earthen channel in rough condition	1.750	**26.66**	0.413	1.24

12.4.2 Kutter's Equation for Chezy's Constant C

$$C = \frac{23 + (0.00155/S_b) + (1/N)}{1 + (23 + 0.00155/S_b)(N + R_h^{0.5})} \qquad (12.4.2)$$

Where N is Kutter's constant, the value of which varies from 0.011 for smooth cement surface to 0.08 for poorly maintained earthen channels.

Example 12.3 *Determine* ***the flow rate*** *for a rectangular channel 3 m wide and 1 m deep with a slope of 1/2500. Using the values of Kutters constant and conditions of surface given in the tabulation.*

The results are also tabulated. Equation (12.4.2) is used for the determination of C. Equation (12.3.6) is used to calculate V. $R_h = 0.06$ m, $A = 3$ m^2, $S_b = 1/2500$

S. No.	Type of surface	Kutters const, N	Chezy's Constant	V, m/s	Q, m^3/s
1	Smooth cement lining	0.110	85.25	1.32	**3.96**
2	Smooth concrete	0.013	71.53	1.11	**3.32**
3	Rough brick	0.015	61.52	0.95	**2.86**
4	Rubble masonry/Rough concrete	0.017	53.90	0.84	**2.51**
5	Clean earthen channel	0.018	50.74	0.79	**2.36**
6	Earthen channel after weathering	0.022	41.02	0.64	**1.91**
7	Earthen channel rough with bush	0.050	1714	0.27	**0.80**

It can be seen that as N increases the flow decreases for the same slope. The values of N are available for a few more conditions, particularly for earthen channels and natural streams. Note that in examples 2.2 and 2.3 there is good agreement between the results. Due to simplicity and availability of extensive experimental support Mannings correlation is more popularly used.

12.4.3 Manning's Equation for C

In 1890 Robert Manning proposed in place of the relation given in equation (12.3.5), $C = (8g/f)^{1/2}$ that

$$C = (1/N)\, R_h^{1/6} \tag{12.4.3}$$

where N is Mannings constant established by experiments for various types of surfaces. When combined with Chezy's equation (12.3.6), this leads to

$$V = (1/N)\, R_h^{2/3}\, S_b^{1/2} \tag{12.4.4}$$

The values of N is generally a small fraction varying from 0.011 to 0.06. Sometimes the reciprocal of N is also referred to as Mannings constant. In this case the value will be in the range 16 to 90.

Example 12.4 *For a rectangular channel 3 m wide and 1 m deep with a slope of 1 to 2500 determine the* ***values of Chezy's constant and also the flow rate****. The types of surface with Mannings constant are tabulated.*

The results are obtained using equation (12.4.3) and (12.4.4) and are presented in the tabulation.

$R_h = 0.6\ m$, $A = 3\ m^2$, $S_b = 1/2500$

S. No.	Surface type	N	C	V (m/s)	Q (m^3/s)
1	Smooth cement lined surface	0.011	**83.48**	1.29	**3.88**
2	Smooth concrete	0.013	**70.64**	1.09	**3.28**
3	Rough Brick	0.015	**61.23**	0.95	**2.85**
4	Rubble masonary/Rough concrete	0.017	**54.02**	0.84	**2.51**
5	Clean earthen channel	0.018	**51.02**	0.79	**2.37**
6	Earthen channel after weathering	0.022	**41.74**	0.65	**1.94**
7	Rough earthen channel with weeds	0.050	**18.37**	0.29	**0.85**

Comparing with the result of example 12.2 and 12.3 it may be noted that there is close agreement between the results except in the case of earthen channels. Values for some more types of surfaces are given in table 12.1.

Table 12.1 Values of Manning's coefficient, N

S. No.	Surface Type	N
1	Natural channels	
	Clean and straight	0.030 ± 0.005
	Sluggish with deep pools	0.040 ± 0.010
	Major rivers	0.035 ± 0.010
2	Flood plains	
	Pasture, farm land	0.035 ± 0.010
	Light brush	0.050 ± 0.020
	Heavy brush	0.075 ± 0.025
	Trees	0.150 ± 0.050
3	Excavated eathen channels	
	Clean	0.022 ± 0.004
	Gravelly	0.025 ± 0.005
	Weedy	0.030 ± 0.005
	Stoney, cobbles	0.035 ± 0.010
4	Lined channels	
	Glass	0.010 ± 0.002
	Brass	0.011 ± 0.002
	Steel, smooth	0.012 ± 0.002
	Steel, painted	0.014 ± 0.003
	Steel, riveted	0.015 ± 0.002
	Cast iron	0.013 ± 0.003
	Concrete, finished	0.012 ± 0.002
	Concrete, Rough	0.014 ± 0.002
	Planed wood	0.012 ± 0.002
	Clay tile	0.014 ± 0.003
	Asphalt	0.016 ± 0.003
	Corrugated metal	0.022 ± 0.005
	Rubble masonry	0.025 ± 0.005
	Brick work	0.015 ± 0.002

Example 12.5 ***Determine the slope*** *required for a flow of 1500 litre of water per second for a pipe of 2 m diameter flowing half full. Use Mannings equation. The value of Mannings constant is 0.015 for the rough concrete lining used.*

Area = $\pi D^2/(4 \times 2)$, Perimeter = $\pi D/2$, $\therefore \quad R_h = D/4 = 2/4 = 0.5$ m

$$C = (1/N)R_h^{1/6} = (1/0.0015) \times 0.5^{1/6} = 59.4,$$

Velocity = Volume flow/area

$$V = 1.5 \times (4 \times 2/\pi \times 2 \times 2) = 0.955 \text{ m/s. } (1500l = 1.5 \text{ m}^3)$$

$$V^2 = C^2 R_h S_b$$

$$\therefore \quad S_b = V^2/C^2 R_h = 0.955^2/(59.4^2 \times 0.5) = 517 \times 10^{-6}$$

or 1 : 1934, $S_b = 1/1934$

12.5 ECONOMICAL CROSS-SECTION FOR OPEN CHANNELS

For a given flow rate and slope (determined by the ground slope) any one of several types of sections can be chosen. After a particular type of section, say rectangular, circular etc. is chosen, various alternatives are possible. A wider shallow section or a narrower deeper section may carry the same flow under the same slope. Hence there can exist a section which will involve minimum section in terms of cost of excavation, lining etc. This is illustrated in the problem Example 12.6, by a trial process, assuming depth and width. Derivations are given in example 12.7, 12.8 and 12.9 and solved problem Problem 12.7.

Example 12.6 *A smooth cement lined rectangular channel is proposed with a slope of 1/2500. Mannings coefficient for the surface is 0.011. For a total area of 4 m^2, assuming different ratios of depth to width* ***determine the flow rates***.

The assumed depth to width ratios and corresponding values of depth d, width b, perimeter P, hydraulic radius R_h, velocity V, and flow rates are tabulated below.

$$V = \frac{1}{N} \cdot S_b^{1/2} R_h^{2/3} = \frac{1}{0.011} \cdot \left(\frac{1}{2500}\right)^{0.5} \cdot R_h^{2/3}$$

$$= 1.8782 \, R_h^{2/3} \qquad (A)$$

$$Q = A \times V = 4 \times 1.8182 \, R_h^{2/3}$$

$$= 7.2728 \, R_h^{2/3} \qquad (B)$$

The results using equations (A) and (B) are tabulated below.

d : b	d, m	b, m	P, m	R_h, m	V, m/s	Q, m^3/s
1 : 1.5	1.6330	2.4495	5.7155	0.6999	1.4333	**5.7331**
1 : 1.75	1.5119	2.6458	5.6695	0.7055	1.4409	**5.7636**
1 : 2	1.4142	2.8284	5.6569	0.7071	1.4431	**5.7723**
1 : 2.25	1.3333	3.0000	5.6667	0.7059	1.4414	**5.7658**
1 : 2.5	1.2649	3.1622	5.6920	0.7027	1.4371	**5.7484**

It is seen that the flow rate is maximum at $d/b = 0.5$ or $b = 2d$ for rectangular channel. It is to be noted that perimeter is minimum at this value. For the square section the perimeter is maximum at bm when the flow rate is minimum at 5.55 m^3/s.

Thus in the case of rectangular channels, it appears that the economical section for a given area is the 1 : 2 section.

Example 12.7 *Derive an expression for the ratio of depth to width for open channel flow in the case of a rectangular section of a given area for economical conditions.*

This is achieved when the flow is maximum for a given area or the perimeter is minimum.

Consider a width b and depth d perimeter $P = 2d + b = 2d + (A/d)$ ...(A)

$$Q = AC\sqrt{\frac{A}{P}S_b}$$

As A,C and S_b are specified, only variable is P in this case and so $Q \propto P^{-0.5}$

i.e. **Q will be maximum when P is minimum and depth d is the independent variable.** Or $\frac{dP}{dd}$ should be zero for such a condition, using equation A

$$\frac{dP}{dd} = -\frac{A}{d^2} + 2 = 0 \quad (\text{also} \quad \frac{d^2P}{dd^2} \quad \text{is} - ve)$$

$$\therefore \qquad A = 2d^2 = d \times b \quad \therefore \quad b = 2d$$

The **depth should be one half of the width for economical rectangular section.** In this case $R_h = d/2$ (see example 12.6 also)

Using Mannings equation the flow rate for the economical rectangular section is given in terms of the depth d as

$$Q = \frac{2^{1/3}}{N} d^{8/3} S_b^{1/2} = \frac{1.26}{N} (\text{depth})^{8/3} (\text{slope})^{1/2} \qquad (12.4.5)$$

Example 12.8 *Derive an expression for the hydraulic radius for a trapezoidal channel section for maximum flow conditions. Assume uniform flow conditions. Also determine the optimum side slope.*

Refer Fig. Ex. 12.8 for variable names.

Let $$a = \cot\theta, \text{ Area } A = by + ay^2 = y(b + ay) \qquad (1)$$

$$P = b + 2w = b + 2y(1 + a^2)^{1/2} \qquad (2)$$

Eliminating b using equation 1

$$b = \frac{A}{y} - ay$$

$$\therefore \qquad P = \frac{A}{y} - ay + 2y(1 + a^2)^{1/2}$$

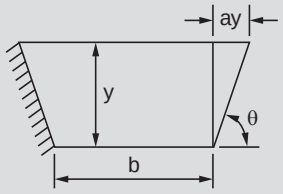


Figure Ex. 12.8

As $$Q = C\sqrt{R_h S_b} = C\, S_b^{0.5} A^{0.5}/P^{0.5} \qquad (3)$$

and as A, C and S_b are constants and as $R_h = A/P$, Q is maximum when P is minimum. Also for a given channel θ is fixed and hence a is fixed, y being the independent variables, and P is to be minimized. Using equation (3) and equating (dp/dy) to zero.

$$\frac{dp}{dy} = -\frac{A}{y^2} - a + 2(1 + a^2)^{1/2} = 0 \quad \text{Solving for } A$$

$$A = y^2 [2 (1 + a^2)^{1/2} - a], \quad \text{From equation (3),}$$

$$P = \frac{A}{y} - ay + 2y(1 + a^2)^{1/2} = \frac{A}{y} + y\,[2(1 + a^2)^{1/2} - a]$$

Substituting for A, $P = y[2(1 + a^2)^{1/2} - a] + y[2(1 + a^2)^{1/2} - a] = 2y[2\,(1 + a^2)^{1/2} - a]$

$\therefore$ $$\mathbf{R_h = \frac{A}{P} = \frac{y}{2}}$$

In the case of a rectangle $a = 0$ $\therefore$ $A = 2y^2$, $P = 4y$, and from equation 1, $b = 2y$.

In order to determine the optimum value of a (cot θ) for a given section of area A and depth y, using equation (3)

$$\frac{dP}{da} = -y + \frac{1}{2} \times 2a \times 2y(1 + a^2)^{-1/2} \quad \text{Rearranging} \quad 2a/(1 + a^2)^{1/2} = 1$$

Squaring both sides, and solving

$$a = \frac{1}{\sqrt{3}} = \cot\theta \quad \therefore \quad \theta = 60°$$

Hence the optimum condition for economical section for a given area and depth is θ = 60°

This shows that the section is half of a regular hexagon, as it can be shown that base b = side length W

$$W^2 = y^2 + a^2y^2 = y^2(1 + a^2), \text{ as } \quad a = \cot 60 = 0.5774$$

$$W^2 = 1.333\ y^2 = (4/3)\ y^2 \quad \therefore \quad W = (2/\sqrt{3}\,)y$$

$$b = \frac{A}{y} - ay,\ A = y^2\,[2(1 + a^2)^{1/2} - a]$$

$\therefore$ $$b = 2y[(1 + a^2)^{1/2} - a], \quad \therefore \quad b = \frac{2}{\sqrt{3}}y \quad \therefore \quad b = W$$

Example 12.9 *Derive an expression for the optimum angle for a triangular channel section of given area.*

Refer Fig. Ex. 12.9

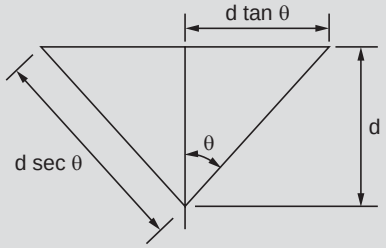


Figure Ex. 12.9

Area, $A = d^2 \tan\theta \quad \therefore \quad d = \left(\frac{A}{\tan\theta}\right)^{0.5}$

Perimeter, $P = 2d \sec\theta$, Substituting for d

$$P = 2 \sec\theta \left(\frac{A}{\tan\theta}\right)^{0.5}$$

$$= 2\left(\frac{A \sec^2\theta}{\tan\theta}\right)^{0.5} = 2\left(\frac{A(1 + \tan^2\theta)}{\tan\theta}\right)^{0.5}$$

$$= 2\{A(\tan\theta + \cot\theta)\}^{0.5} \quad \text{As} \quad Q = AV = AC\sqrt{\frac{A}{P}\,S_b}$$

$\therefore$ For maximum of Q, P should be minimized,

Considering θ as independent variable, and minimizing P^2,

$$\frac{dP}{d\theta} = \frac{d}{d\theta}\,[\tan\theta + \cot\theta] = \sec^2\theta - \operatorname{cosec}^2\theta = 0$$

$\therefore$ $\cos^2\theta = \sin^2\theta \quad \therefore \quad \theta = 45°$, The optimum half angle is 45°

$d = (A/\tan 45)^{0.5} = A^{0.5}$ or $A = d^2$

$\therefore$ $\mathbf{P = 2d \sec 45 = 2\sqrt{2}\ d = 2.828\ d}$

$\therefore$ $R_h = A/P = d/2^{3/2} = 0.353\ d$

12.5.1 Examples 12.6 to 12.9 provide the optimum geometry for some sections. Table 12.2 gives the conditions for optimum geometry for various sections together with normal depth d and cross sectional area in terms of flow rate, Mannings coefficient and bed slope.

$$(Q/A) = R_h^{2/3}\, S_b^{1/2}/N, \text{ and } R_h = A/P, \quad \therefore \quad A = \left[\frac{QN}{S_b^{1/2}}\right]^{3/5} P^{2/5}$$

Table 12.2. Optimum geometry, normal depth and area when flow rate and bed slope are fixed.

Section	Optimum Geometry	Normal depth	Cross sectional area
Rectangular, width, b depth, d	$\mathbf{b = 2d}$	$0.917\left[\frac{QN}{S_b^{1/2}}\right]^{3/8}$	$1.682\left[\frac{QN}{S_b^{1/2}}\right]^{3/4}$
Trapezoidal width, b depth, d	$\theta = 60°$ $\mathbf{b = 2d/\sqrt{3}}$	$0.968\left[\frac{QN}{S_b^{1/2}}\right]^{3/8}$	$1.622\left[\frac{QN}{S_b^{1/2}}\right]^{3/4}$
Circular, diameter, D depth, d	$\mathbf{D = 2d}$	$1.00\left[\frac{QN}{S_b^{1/2}}\right]^{3/8}$	$1.583\left[\frac{QN}{S_b^{1/2}}\right]^{3/4}$
Wide flat, width b, depth d q, flow/m	$\mathbf{b>>d}$	$1.00\left[\frac{qN}{S_b^{1/2}}\right]^{3/8}$	---

Example 12.10 *For a given slope of 1/2500 and flow rate of 4m³/s,* ***determine the depth of flow and area of cross-section at optimum conditions*** *for (i) Rectangular, (ii) Trapezoidal, (iii) Triangular and (iv) Circular sections. Mannings coefficient, N = 0.011.*

Using equations in table 12.2, the values are calculated as below.

(i) Rectangular: Normal depth

$$d = 0.917\left(\frac{QN}{S_b^{1/2}}\right)^{3/8} = 0.917\left[\frac{4\times 0.011}{(1/2500)^{1/2}}\right]^{3/8}$$

$$= 0.917\ (2.2)^{3/8} = 1.2347\ \text{m},$$

$\therefore$ $b = 2.4649$ m

$A = d \times b = 3.04\ \text{m}^2$, Also $A = 1.682(2.2)^{0.75} = 3.04\ m^2$

(ii) Trapezoidal (Note: $QN/S_b^{1/2} = 2.2$ from previous calculation is used in the following calculations)

$$d = 0.968\ (2.2)^{3/8} = 1.301\ \text{m},\ b = 2 \times 1.301/\sqrt{3} = 1.502\ \text{m}$$

$\therefore$ $A = 1.622(2.2)^{0.75} = 2.93\ \text{m}^2$

check (1.502 + 1.301 cot 60) 1.301 = **2.93 m²**

(iii) Triangular: $d = 1.297\ (2.2)^{3/8} = 1.743$ m, top width = 2 × 1.743 = 3.468 m

$A = 3.039\ \text{m}^2$, check: $1.682\ (2.2)^{0.75}$ = **3.04 m²**

(iv) Circular: $d = 1.0(2.2)^{3/8} = 1.344$ m, Diameter $D = 2 \times 1.344 = 2.688$ m

$A = \pi\ 2.688^2/(2 \times 4) = 2.84\ \text{m}^2$, check: $1.583\ (2.2)^{3/4}$ = **2.86 m²**

Note that the minimum area is in the case of the circular section. Next comes trapezoidal one. The triangular and rectangular sections need the same areas, but the depth is more for the triangular section. The velocity will be maximum in the circular section.

Example 12.11 *Using the results of Example 12.10,* ***determine the flow velocity*** *in each case and check the same using Mannings equation.*

$$V = \frac{1}{N}\ R_h^{2/3} S_b^{1/2} = \frac{1}{0.011} \times \left(\frac{1}{2500}\right)^{0.5}\ R_h^{2/3} = 1.8182 \times R_h^{2/3}$$

The result are tabulated below: Flow rate = 4 m³/s

Section	R_h	Area m²	Velocity using Area m/s	Velocity using Manning's m/s	Fr
Rectangular	$(d/2) = 1.2347/2$	3.04	**1.32**	**1.3168**	0.38
Tapezoidal	$(d/2) = 1.301/2$	2.93	**1.37**	**1.3650**	0.42
Triangular	$(d/2\sqrt{2}) = (1.743/2\sqrt{2})$	3.04	**1.32**	**1.3166**	0.45
Circular	$(d/2) = (1.344/2)$	2.84	**1.41**	**1.3950**	0.44

Froude number is calculated on the basis of hydraulic depth = Area/top width

$Fr = V/\sqrt{yg}$ where y is the average depth given by, Area/top width.

Trapezoidal $$y = \frac{A}{\text{Topwidth}} = \frac{d(b + d \sin 30)}{b + 2d \sin 30} = \frac{1.301(1.502 + 0.65)}{1.502 + 1.301} = 0.9991$$

Triangular $$y = \frac{3.04}{2 \times 1.743} = 0.872 \text{ m, circular } \quad y = \frac{2.84}{2.688} = 1.057 \text{ m}$$

Note : Velocity is maximum in the case of circular section.

Example 12.12 ***Determine the height of flow*** *in the case of a rectangular channel of 3 m width and slope of 1/2500 for a flow rate of 4 m^3/s. Consider well finished concrete surface*

The value of Mannings coefficient for well finished concrete surface is 0.011.

Assume that the depth is d as $p = 3$ m

$\therefore$ $$A = b \times d = 3d, P = b + 2d = 3 + 2d, R_h = 3d/(3 + 2d)$$

$$V = \frac{1}{N} R_h^{2/3} S_b^{1/2}, Q = AV = V \times 3d, \quad \text{Substituting the values,}$$

$$4 = \frac{1}{0.011} \times \left(\frac{1}{2500}\right)^{0.5} \left(\frac{3d}{3 + 2d}\right)^{2/3} \times 3d = 5.4545 \times d \times \left(\frac{3d}{3 + 2d}\right)^{2/3}$$

$$\left(\frac{d^{5/2}}{3 + 2d}\right) = 0.2093, \text{ Solving by trial } \mathbf{d = 1.021 \text{ m}}$$

Check:

$R_h = (3 \times 1.021)/(3 + (2 \times 1.021)) = 0.6033$ m

$Q = [(3 \times 1.021)/(0.011)] \times (1/2500)^{0.5} \times (0.6033)^{2/3} = 3.976 \text{ m}^3/\text{s}$

Note: Determination of depth or width for specified values of flow rate, slope etc. involves polynomials of more than degree 2. Hence iterative working is necessary.

12.6 FLOW WITH VARYING SLOPES AND AREAS

It is found almost impossible to have open channels of uniform slope and or uniform area all over the length of the channel. When the terrain changes the slope has to change, sometimes gradually and sometimes steeply, and hence the flow area also should change. In such conditions the flow depth readjusts in some cases gradually and in some cases, suddenly. Certain situations like in spillways the energy in the flow has to be dissipated without damage to the surfaces. The study of such flows is an important and practical aspect of open channel flow.

12.6.1 Velocity of Wave Propagation in Open Surface Flow

The nature of readjustment of flow level due to slope change or sudden drop in bed level is found to depend on the velocity of propagation of any disturbance or wave velocity in the flow. In stagnant surface any disturbance will spread uniformly around the point of disturbance. An example is dropping of a small stone in a stagnant pool. In case water is flowing with a velocity V and if the wave velocity is c then if $V < c$, the disturbance will travel upstream at a

lower speed. In case $V \geq c$ then disturbance cannot travel upstream and only the down steam flow will change. The change is mainly in the form of change in height of flow.

Referring to Fig. 12.6.1, let a disturbance be created by moving the vertical plate slightly along the x direction. This will cause a small ripple or wave and let its velocity or propagation be c. In order to facilitate analysis, the wave can be brought to rest by imposing a velocity c in the opposite direction to the wave movement.

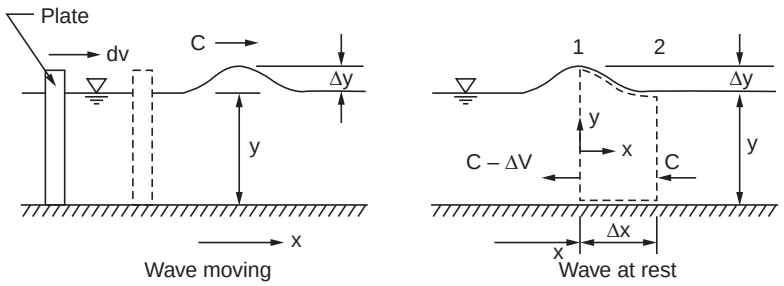


Figure 12.6.1 Wave celerity

The flow is assumed steady, incompressible and the section is constant across the x direction with a width b.

Considering sections 1 and 2 and applying continuity conditions *i.e.* flow is equal at sections 1 and 2

$$\rho\, b(y + \Delta y)\,(c - \Delta V) - \rho\, byc = 0 \quad \text{Solving}$$

$$\Delta V = c\,\frac{\Delta y}{y + \Delta y},$$

as Δy is small this can be approximately as $\Delta V = c\dfrac{\Delta y}{y}$ (*A*)

Applying momentum equation to the sections, as the pressure is hydrostatic at any section, the pressure force is given by (for unit width) $\rho\, g\, y^2/2$

At section 1, Force = $\rho\, g\, (y + \Delta y)^2/2$

At section 2, Force = $\rho g\, y^2/2$

Net pressure force between the sections equals the difference between these two,

$$\rho g y\, \Delta y + g\,\frac{\Delta y^2}{2}$$

Neglecting the second order term, net force equals $\rho g\, y\, \Delta y$

The rate of change of momentum is $dV \times$ flow rate $\quad \therefore \quad dV\, \rho cy$

Subsituting for dV from equation (*A*) and equating the force and change of momentum,

$$\rho\, gy\, \Delta y = c^2\, \rho\, \Delta y$$

$$c^2 = gy \quad \text{or} \quad \mathbf{c = \sqrt{gy}} \qquad (12.6.1)$$

The wave velocity c is also called wave celerity.

The main assumption is that $\Delta y << y$ *i.e.* the wave height is small compared to the depth. The local depth at the crest of the wave is more compared to the trailing edge. This causes accumulation of water in the crest and as the wave travels further the height increases, finally leading to the breaking of the waves, which is seen near beaches. Note that the velocity calculated is that of surface wave and not that of propagation at depths.

Example 12.13 *Compare the celerity of waves for depths of 0.1, 0.5, 100, 1000 and 4000 m.*

Using equation (12.6.1), $c = \sqrt{gy}$, the results obtained are tabulated below :

depth, *m*	0.1	0.5	1	10	100	1000	4000
velocity, *m/s*	0.99	2.21	3.13	9.91	31.32	99.05	198.09

The depth of deep ocean is about 4 km. For this depth the disturbance will travel at 198 m/s which is about 713 km/hr. Such a disturbance can be produced by earthquake or volcanic activity. The resulting tidal waves may travel at this speed and cause heavy damage.

12.6.2 Froude Number

Froude number in connection with open surface flow is defined as $V/\sqrt{gl}$. In the case of open channel flow, the characteristic length, l, is the depth y and V is the flow velocity. Hence Froude number can be represented by the ratio, Flow velocity/wave velocity.

As already indicated, there are three possible flow situations namely $(V/c) < 1$, $(V/c) = 1$ and $(V/c) > 1$ or the Froude number for the flow is less than or equal to or greater than 1.

Case (i) If Fr < 1, then V < c and any disturbance can travel upstream. The downstream conditions can change the flow conditions upstream. Such a flow is called subcritical or **tranquil** flow. Only gradual changes occur in such a situation.

Case (ii) Fr = 1. The flow is called critical flow. Disturbances cannot travel upstream. A standing wave may generally result.

Case (iii) Fr > 1, such flows are called supercritical or rapid or shooting flows. Disturbances cannot travel upstream. Downstream conditions cannot be felt upstream. Changes occur only in the downstream flow. These are similar to subsonic, sonic and supersonic flows in the case of flow of compressible fluids where Match number is the governing factor also defined as V/c, where c is the sonic speed or velocity of propagation of small disturbance in the fluid.

12.6.3 Energy Equation for Steady Flow and Specific Energy

Assumptions in the case are

(*i*) steady incompressible and uniform flow

(*ii*) pressure distribution is hydrostatic

(*iii*) small bed slope, $(\sin\theta \approx \tan\theta \approx \theta)$

(*iv*) Shear work term negligible,

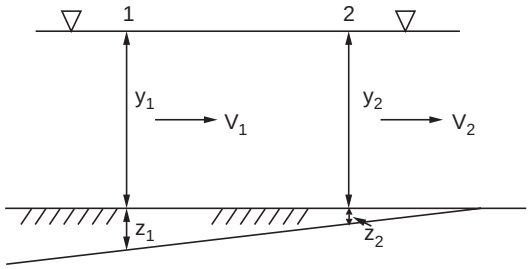


Figure 12.6.2

Considering sections 1 and 2 in the flow as shown in Figure 12.6.2, Bernoulli equation is written including head loss due friction h_L.

$$\frac{V_1^2}{2g} + y_1 + z_1 = \frac{V_2^2}{2g} + y_2 + z_2 + h_L \tag{12.6.1}$$

As in pipe flow the pressure loss in due to friction in the open channel flow. The head due to change in level namely $(z_1 - z_2)$ equals the friction head h_L. The term $\mathbf{\frac{V^2}{2g} + y}$ **is found to be an important parameter is open channel flow. This quantity is defined as specific energy or specific head. The symbol used is E.** The variation of depth and velocity for a given specific energy provides an idea about the type of flow.

To illustrate the idea the case of a rectangular section is analysed in the following sections.

For a flow rate Q and sectional area A,

$$V = \frac{Q}{A} \quad \text{and} \quad A = b \times y \text{ where } b \text{ is the width and } y \text{ is the depth.}$$

$$\therefore \qquad \frac{V^2}{2g} = \frac{Q^2}{2g\, b^2 y^2}.$$

The specific energy can now be expressed as below

$$E = y + \frac{Q^2}{2g\, b^2 y^2} \tag{12.6.2}$$

For unit width, $q = Q/b$

$$\therefore \qquad E = y + \frac{q^2}{2gy^2} \tag{12.6.3}$$

It will be useful to investigate the variation of depth and velocity for a given flow rate. In this process the value of **minimum energy for a given flow** is found as follows.

Differentiating the equation (12.6.3) and equating the result to zero, we get

$$\frac{dE}{dy} = -\frac{q^2}{gy^3} + 1 = 0.$$

$$\therefore \qquad y^3 = \left(\frac{q^2}{g}\right) \quad \text{or} \quad \mathbf{y = \left(\frac{q^2}{g}\right)^{1/3}} \tag{12.6.4}$$

The value of y for the minimum every for a given flow rate is formed as critical depty y_c.

$$\therefore \qquad y_c = \left(\frac{q^2}{g}\right)^{1/3} \tag{12.6.4(a)}$$

Substituting the value in the general equation (12.6.3).

$$\mathbf{E_{min}} = y_c + \frac{{y_c}^3}{2\,{y_c}^2} = \mathbf{3/2\ y_c}. \tag{12.6.5}$$

or $$\mathbf{y_c} = \frac{2}{3}\,\mathbf{E_{min}}. \tag{12.6.5}a$$

From the definition of velocity, and eqn. 12.6.4a for unit width at the critical condition,

$$V_c^2 = \frac{q^2}{{y_c}^2} = \frac{g \cdot {y_c}^3}{{y_c}^2} = gy \quad \text{or} \quad \mathbf{V_c} = \sqrt{\mathbf{gy}} \tag{12.6.6}$$

From the expression of wave velocity, $\sqrt{gy} = c$

$\therefore \quad V_c = c$, and $\frac{V_c}{c} = \frac{V_c}{\sqrt{gy}} = 1$ or Froude number is unity.

The flow rate at this condition is given by (for unit width)

$$q_{max} = V_c y_c = \sqrt{g{y_c}^3} \tag{12.6.7}$$

At this critical depth condition the following relations hold.

$$y_c = \left(\frac{q^2}{g}\right)^{1/3},\ V_c = \sqrt{gy_c},\ E_c = \frac{3}{2}\,y_c,\ q_c = \sqrt{g{y_c}^3}$$

and $$A_c = by_c = b\left(\frac{q^2}{g}\right)^{1/3}$$

These relations specify only a single condition in the flow. To investigate the complete variation the general equation (12.6.3) is modified and written as

$$y^3 - Ey^2 + \frac{q^2}{2g} = 0. \tag{12.6.8}$$

This equation leads to two positive values of y for a given E and q. The third solution is negative and has no significance.

This means that **For a given value of E and q two combinations of depth and velocity exist. These are called alternate depths and velocities.** As two quantities are involved with depth of flow two separate plots are used to illustrate the flow variation. One plot is for a constant energy and the head variation with flow rate. This is shown in Figure 12.6.3 The other is for constant flow rate and head variation with energy. This is shown in Figure 12.6.4.

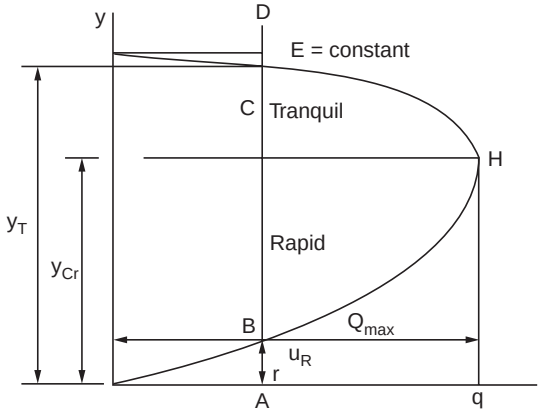


Figure 12.6.3 Variation of depth with specific energy for a given flow rate

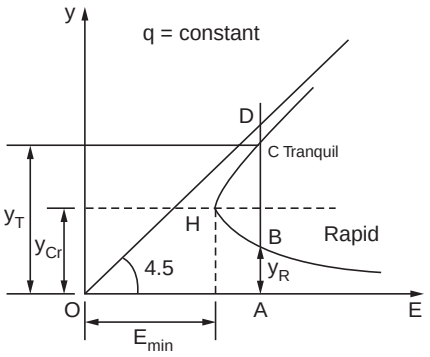


Figure 12.6.4 Variation of discharge with depth for a given specific energy

In figure 12.6.3 the variation of depth with flow rate is shown plotted for a constant value of E. Similar curves will result for other value of specific energy E. The resulting curve shows that at any flow less than q_{max}, there are two possible depths of flow, one greater than y_c and the other smaller than y_c. The condition y_c divides the flow into two regions namely tranuil and rapid flows. In the tranquil flow the depth is larger and the velocity is smaller. In the rapid flow the depth is smaller and the velocity is more. For example for the tranquil flow of the given q at A, the depth is AC and the dynamic head is CD. In the rapid flow region, the depth is AB and the dynamic head is BD. At q_{max} or the critical condition there is only one solution for the depth.

In the tranquil flow region, the Froude number is less than 1 and any disturbance downstream will be felt upstream and the flow upstream will be readjusted by the disturbing wave. The flow velocity V will be less than V_c.

In the rapid flow region, the Froude number is greater than one and down stream disturbances like changed slope will not be felt upstream. and $V > V_c$.

At the critical condition a standing wave will be generated.

Figure 12.6.4 is a plot of depth against specific energy for a given flow rate. At $q = 0$, $y = E$. This is represented by the line at 45°. A curve for a given flow rate is shown ploted. Here also very similar conditions are seen. For a given flow rate at any energy greater than E min, the flow can exist at two different combinations of depth and velocity. This is shown by the line $ABCD$. When the depth is equal to $y_R = AB$, the kinetic energy is given by BD. This is in the rapid flow region. When the depth is y_T (AC), the kinetic energy is CD. For different values of flow rate different curves will result.

12.6.4 Non Dimensional Representation of Specific Energy Curve

By one dimensional representation a single curve will result for all values of specific energy. The specific energy curve can be presented in a non dimensional form by dividing the terms of equation (12.6.3) by q_{max}.

$$\frac{E}{{q_{max}}^2} = \frac{y}{{q_{max}}^2} + \frac{1}{2g{y}^2}\left(\frac{q}{q_{max}}\right)^2$$

As $q_{max} = (g\, {y_c}^3)^{1/2}$, this reduces to

$$\frac{E}{gy_c^3} = \frac{y}{gy_c^3} + \frac{1}{2gy^2}\left(\frac{q}{q_{\max}}\right)^2$$

or
$$\left(\frac{q}{q_{\max}}\right)^2 = \frac{2E}{y_c}\left(\frac{y}{y_c}\right)^2 - 2\left(\frac{y}{y_c}\right)^3$$

For a rectangular channel $E_{\min} = (3/2)\, y_c$

$$\therefore \quad \left(\frac{q}{q_{\max}}\right)^2 = 3\left(\frac{y}{y_c}\right)^3 - 2\left(\frac{y}{y_c}\right)^3 \tag{12.6.6}$$

This will result in a single curve for all values of E when q is plotted against y.
Similarly for given values of q, the equation below will result in a single curve.
This is obtained by dividing the general eqaution by y_c and then simplifying

$$\frac{E}{y_c} = \frac{y}{y_c} + \frac{1}{2}\left(\frac{y_c}{y}\right)^2 \tag{12.6.7}$$

This will result in a single curve for all values of q when E is plotted against y.

Example 12.14 *Water flows in a rectangular channel at the rate of 3 m^3/s per m width, the depth being 1.5 m, Determine whether the flow is subcritical or supercritical. Also determine the alternate depth and Froude numbers in both cases.*

Considering 1 m width

Velocity = $3/(1.5 \times 1) = 2$ m/s, Specific energy = $(V^2/2g) + y = 1.7039$ m.

$$Fr = \frac{V}{\sqrt{gy}} = \frac{2}{\sqrt{9.81 \times 1.5}} = 0.5214.$$ The flow is subcritical

Critical height is given by $y_c = (q^2/g)^{1/3} = 0.9717$ m

$E_{\min} = (3/2) \times y_c = 1.4575$ m, $V_c = (g \times y_c)^{1/2} = 3.0874$ m/s

check: $q = V_c \times y_c = 3.0874 \times 0.9717 = 3\ m^3/s/m$

$$Fr = \frac{3.0874}{\sqrt{9.81 \times 0.9717}} = 1.0$$

The alternate depth is obtained using equation (12.6.3), $y^3 - Ey^2 + q^2/2g = 0$

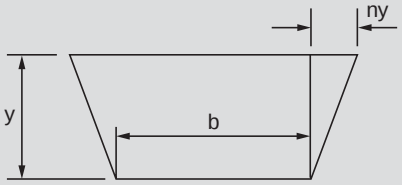
As q and E are known, solving by trial, $y_{al} = 0.6643$ m; $\therefore$ $V = 4.516$ m/s

$$\textbf{Fr} = \frac{4.516}{\sqrt{9.81 \times 0.6643}} = \textbf{1.7961, The flow is supercritical}$$

check $$\frac{V^2}{2g} + y = \frac{4.516^2}{2 \times 9.81} + 0.6643 = 1.7038 \text{ m.}$$

Example 12.15 ***Derive an expression for critical depth and critical velocity*** *of a trapezoidal channel in terms of specific energy E. Assume a bottom width b and side slope 1 : n.*

Area of section, $A = (b + ny)\, y$,



specific energy is given by

$$E = \frac{V^2}{2g} + y \quad \therefore \quad V = \sqrt{2g(E - y)}$$

Volume flow, $Q = A\,V = y(b + ny)\,\sqrt{2g(E - y)}$ ...(*A*)

For constant value of E, Q is maximum when $(dQ/dy) = 0$

Taking natural log on both sides of equation (*A*).

$$\ln Q = \ln y + \ln (b + ny) + (1/2) \ln (2g) + (1/2) \ln (E - y)$$

differentiating with respect to y,

$$\frac{1}{Q}\frac{dQ}{dy} = \frac{1}{y} + \frac{n}{b + ny} - \frac{1}{2(E - y)}, \text{ from section 12.6.3 when } \frac{dQ}{dy} = 0, y = y_c$$

$$\therefore \quad \frac{1}{y_c} + \frac{n}{b + ny_c} - \frac{1}{2(E - y_c)} = 0$$

Rearranging after summing up and considering the numerator to be zero,

$$5ny_c^{\,2} + (3b - 4nE)\, y_c - 2bE = 0 \qquad (B)$$

Solving for y_c

$$\mathbf{y_c} = \frac{-(3b - 4nE) \pm \sqrt{9b^2 - 24bnE + 16n^2E^2 + 40bnE}}{10n}$$

$$= \frac{(4nE - 3b) \pm \sqrt{16n^2E^2 + 16nEb + 9b^2}}{10n} \qquad (C)$$

This is a general solution. Chart solutions are available for various values of b and n.

In the case of triangle, $b = 0$ and $y_c = (4/5)\, E$ (D)

From equation (B) for a rectangular section as $n = 0$, $y_c = (2/3)\, E$, as was established earlier. Critical velocity is obtained by substituting y_c in the equation

$$\mathbf{V_c = \{2g[E - y_c]\}^{0.5}}$$

Example 12.16 ***Determine the critical depth*** *of a channel with trapezoidal cross-section with a flow of (1/3) m^3/s. The base width is 0.6 m. and the side slope is 45°*

In the case of sections other than rectangles, the hydraulic depth is given by Area/top width. In this case for critical flow, Hydraulic depth = y_c , i.e. *A*/top width

From section 12.6.3,

$$A_c = \left(\frac{b_{top}Q^2}{g}\right)^{1/3}; \text{ also } A_c = y_c\,(b + 2y_c) = y_c\,(0.6 + 2y_c)$$

$Q = (1/3)$ m^3/s, $b_{top} = 2y_c + b$, Substituting

$$y_c(b + 2y_c) = \left(\frac{(b + 2y_c) \times Q^2}{g}\right)^{1/3} \quad \therefore \quad y_c^3\,(0.6 + 2y_c)^2 = \frac{Q^2}{g} = \left(\frac{1}{9 \times 9.81}\right)$$

$$\therefore \qquad \mathbf{y_c}\,(0.6 + 2y_c)^{2/3} = \left(\frac{1}{9 \times 9.81}\right)^{1/3} = \mathbf{0.2246\ m}$$

Solving by trial $\mathbf{y_c = 0.2244\ m}$

The actual depth is different from hydraulic depth. Let it be equal to y

Area = $(y + b)\,y = (b + 2y_c)\,y_c$, Solving $y = 0.265$ m

$\mathbf{V_c} = (gy_c)^{0.5} = (9.81 \times 0.2244)^{0.5} = 1.4867$ m/s

check flow rate: $1.4837 \times 0.265\,(0.6 + 0.265) = 0.34\ m^3/s$

Example 12.17 *A rectangular channel 6 m wide is to carry a flow of 22.5 m^3/s. For depth of 3 m and 0.6 m* ***determine the slope required.*** *Also determine the* ***Froude number*** *and* ***alternate depth*** *for the specific energy conditions. Calculate the critical depth also. Take Mannings coefficient as 0.012.*

Critical depth is given by the equation (12.6.4*a*) (for rectangular section)

$$y_c = \left(\frac{Q^2}{gb^2}\right)^{1/3} = \left(\frac{22.5^2}{9.81 \times 6^2}\right)^{1/3} = 1.1275\ \text{m}$$

Hence for a depth of 3 m, the flow is subcritical. For a depth of 0.6 m, the flow is supercritical.

Case (1) depth = 3 m, $\mathbf{R_h} = 6 \times 3/(6 + 2 \times 3) = \mathbf{1.5\ m}$

To determine the slope, the Manning flow equation is used.

$Q = A \times R_h^{2/3}\,S_b^{1/2}/N$, Substituting the values,

$$22.5 = (6 \times 3) \times \frac{1.5^{2/3}}{0.012}\,S_b^{1/2}, \text{ solving} \quad \mathbf{S_b = 1/7631}$$

Velocity = $22.5/6 \times 3 = 1.25$ m/s

$$\therefore \qquad E = (V^2/2g) + y = \{1.25^2/(2 \times 9.81)\} + 3 = 3.0796\ \text{m}$$

Froude number: $1.25/\sqrt{3 \times 9.81} = \mathbf{0.2304}$

To determine alternate depth, the equation used is (12.6.8)

$$y^3 - Ey^2 + (q^2/2g) = 0$$

Substituting values for $E = 3.0796$, $q = (22.5/6)\ m^3/s/m$

$y^3 = 3.0796\,y^2 + 0.7167 = 0$, solving by trial the alternate value of $\mathbf{y = 0.53\ m}$

$V = 22.5/(6 \times 0.53) = 7.07$ m/s, $\mathbf{Fr = 7.075/\sqrt{0.53 \times 9.81} = 3.1}$

Specific energy calculated using this velocity is 3.081 m (check)

Case (2) 0.6 m depth, $R_h = 6 \times 0.6/(6 + 2 \times 0.6) = 0.5$ m

$22.5 = (6 \times 0.6) \times (1/0.012)\,(0.5)^{2/3}\,S_b^{1/2}$, solving $S_b = 1/70.5$

$$\mathbf{Fr} = \frac{22.5}{6 \times 0.6} \times \frac{1}{\sqrt{9.81 \times 0.6}} = 2.58,$$

Alternate depth is obtained using

$$y^3 - Ey^2 + q^2/2g = 0, \quad \mathbf{E} = 0.6 + \left(\frac{22.5}{6 \times 0.6}\right)^2 \times \frac{1}{2 \times 9.81} = \mathbf{1.991\ m}$$

Substituting and solving by trial, alternate depth is 1.7595 m, V = 2.137 m/s, Fr = 0.513
At the critical flow condition, as calculated y_c = 1.1275 m

$$E = 1.1275 + \left(\frac{22.5}{6 \times 1.1275}\right)^2 \times \frac{1}{2 \times 9.81} = 1.691 \text{ m, which is } 1.5 \times y_c \text{, checks.}$$

12.7 EFFECT OF AREA CHANGE

When changes in flow area occurs over short distances like over a short bump or flow under a sluice gate, the effect of area change is more compared to frictional effects. Assuming channel bed to be horizontal, these cases can be analysed. As friction is neglected Bernoulli equation applies for steady flow conditions.

12.7.1 Flow Over a Bump

The flow along a horizontal rectangular channel of constant width, b, is considered. There is a bump in the channel bed as shown in Fig. 12.7.1. The height of the bump at location x is h, and the water depth is y measured from the bed level at the location 2. The flow is assumed to be uniform at each section. As the pressure on the free surface is the same at all locations, Bernoulli equation reduces to (taking bed level as datum)

$$\frac{V_1^2}{2g} + y_1 = \frac{V_2^2}{2g} + y_2 + h = \text{constant over the bump.}$$

$$\frac{Q}{b} = V_1 y_1 = V_2 y_2, \text{ Substituting for } V_1 \text{ and } V_2.$$

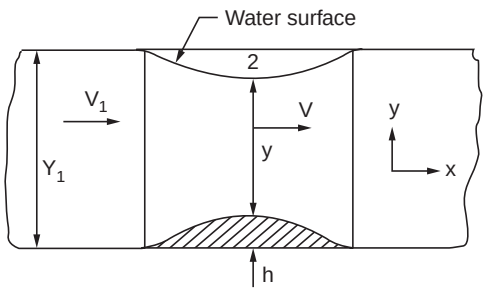


Figure 12.7.1 Flow over a bump

$$\frac{Q^2}{2gb^2 y_1^2} + y_1 = \frac{Q^2}{2gb^2 y^2} + y + h = \text{constant} \qquad (12.7.1)$$

In order to determine the variation of y along the flow, the expression on the R.H.S. is differentiated with respect to x.

$$-\frac{Q^2}{2gb^2y^3}\frac{dy}{dx}+\frac{dy}{dx}+\frac{dh}{dx}=0, \quad \text{Solving for } (dy/dx),$$

$$\frac{dy}{dx}=\frac{(dh/dx)}{[(Q^2/gb^2y^3)-1]}=\frac{(dh/dx)}{[(V^2/gy)-1)}=\frac{(dh/dx)}{Fr^2-1} \tag{12.7.2}$$

As dh/dx is specified, dy/dx can be determined. It is seen that the variation depends on the flow Froude number.

(1) There are two possible variations in the bed level, namely a positive bump and a negative bump or depression.

(2) There are two possible regimes of flow namely subcritical and supercritical.

The type of surface variations are listed below:

1. For subcritical flow + ve bump decreases the flow height, (similar to subsonic nozzle $M < 1$)

2. For subcritical flow – ve bump increases the flow height, (similar to subsonic diffuser $M < 1$)

3. For supercritical flow + ve bump increases the flow height, (similar to supersonic nozzle $M > 1$)

4. Supercritical flow – ve bump decreases the flow height, (similar to supersonic diffuser $M > 1$)

The case of $Fr = 1$ is more complex and other factors have to considered to determine the flow downstream.

Example 12.18 *In a rectangular channel, the flow height is 0.6 m and the flow velocity is 0.6 m/s. A smooth bump with a peak height of 0.06 m exists on the bed surface.* ***Determine the flow velocity and depth over the peak of the bump.***

Refer Fig. Ex. 12.18. Consider sections 1 and 2

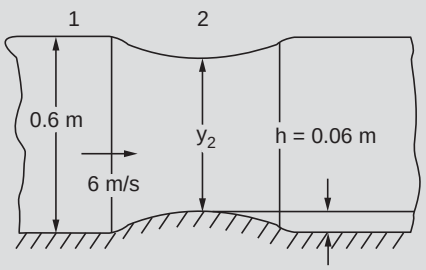


Figure Ex. 12.18

Under steady flow conditions, Bernoulli equation reduces to $\frac{V_1^2}{2g}+y_1=\frac{V_2^2}{2g}+y_2+h, V_2=(V_1y_1/y_2)$

Continuity equation is $\therefore \quad V_1y_1 = V_2y_2$

specific energy at section (1), $E_1 = \frac{V_1^2}{2g} + y_1 = \frac{0.6^2}{2 \times 9.81} + 0.6 = 0.61835$ m

specific energy at section 2, $E_2 = \frac{V_2^2}{2g} + y_2 = E_1 - h$

$= 0.61835 - 0.06 = 0.5583$ m,

substituting for V_2, $= \frac{(V_1 y_1)^2}{2g} \times \frac{1}{y_2^2} + y_2 = 0.5583$ m,

solving by trial, $\mathbf{y_2 = 0.5353\ m}$. $\therefore$ $\mathbf{V_2 = 0.6725\ m/s}$

change in surface level $= 0.5353 - 0.6 + 0.06 = -0.0047$ m

or 4.7 mm, decrease in level.

$$\mathbf{Fr} = \frac{0.6}{\sqrt{9.81 \times 0.6}} = \mathbf{0.2473}$$

In case $Fr = 1$ to occur at the bump, then $y_c = (q^2/g)^{1/3} = 0.3323$ m. The flow depth at section 2 should be 0.3323 m. The size of the bump corresponding to this can be worked out by the above procedure.

12.7.2 Flow Through Sluice Gate, from Stagnant Condition

The reservoir is assumed to be large or y_0 = constant, $V_0 \simeq 0$.

Applying Bernoulli equation between section 0 and 1

$$y_0 = \frac{V_1^2}{2g} + y_1,$$

Since $V = Q/A$ and $A = by$, $y_0 = \frac{Q^2}{2gb^2 y_1^2} + y_1$ (12.7.3)

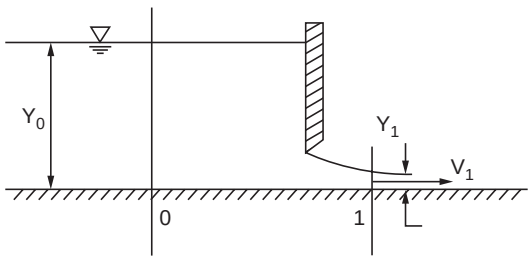


Figure 12.7.3 Flow through sluice gate

Rearranging

$$y_1^2 (y_0 - y_1) = Q^2/2gb^2 \quad (12.7.4)$$

The flow rate Q fixes the value of y_1 and as the equation is a quadratic, there can be two solutions. One will be in the subcritical flow and the other in supercritical flow region.

The same is shown plotted in non dimensional form in Fig. 12.7.4.

At maximum flow rate there is a single value for y_1.

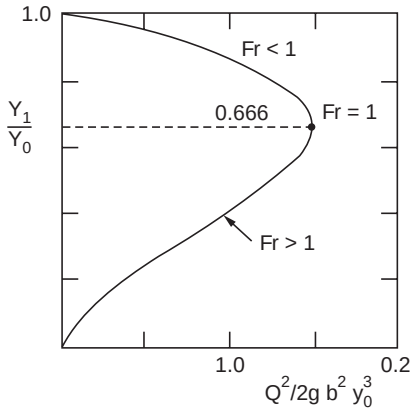


Figure 12.7.4

Taking the derivative of equation (12.7.4) with respect to y_1

$$\frac{d}{dy_1}\left[\frac{Q^2}{2gb^2}\right] = 0 \quad i.e., \quad 2y_1y_0 - 3y_1^2 = 0$$

$$\therefore \qquad y_1 = (2/3)\, y_0 \tag{12.7.5}$$

Substituting this value in flow rate term,

$$\frac{Q_{max}^2}{gb^2} = 2\left[\frac{2}{3}y_0\right]^2\left[y_0 - \frac{2}{3}y_0\right] = \frac{8}{27}\,y_0^3$$

Also $Q/b = q$, then $q_{max}^2/g = (8/27)\, y_0^3$ (12.7.6)

$$V_{max} = \frac{q_{max}}{A} = \frac{q_{max}}{1\times y_1},\ Fr^2 = \frac{V_{max}^2}{gy_1^8} = \frac{q_{max}^2}{gy_1^3} = \frac{8}{27}\,y_0^3\left(\frac{3}{2y_0}\right)^3 = 1$$

Maximum flow corresponds to $Fr = 1$

There is no correlation available relating y_1 and the gate opening.

12.7.3 Flow Under a Sluice Gate in a Channel

In this case water flows with velocity V_1 at section 1 and the level before the gate is y_1. The velocity at section 2 is V_2 and depth is y_2. Steady, incompressible uniform flow is assumed. Also specific energy at section 1 equals specific energy at section 2.

$$E_1 = \frac{V_1^2}{2g} + y_1 \text{ Replacing } V \text{ by } q, \text{ (flow/unit width)},$$

$$E_1 = \frac{q^2}{2gy_2^2} + y_2$$

The depth y_2 should be the alternative depth. As y_1 and V_1 are specified, y_2 is determined by trial using specific energy value. $V_2 = V_1 y_1/y_2$ and so V_2 can be determined. In case velocity $V_1 = 0$, then the condition will give y_0. Maximum flow rate can be obtained by the condition that $y_2 = (2/3)\ y_0$. When the sluice is opened to obtain this maximum flow, the condition at section 1 will change. But y_0 will remain the same.

Example 12.19 *Water is let off from a large reservoir through a sluice gate. The water level in the dam above the level of the sluice is 6 m.* ***Calculate the flow rate*** *for values of $y_1/y_0 = 0.8$ and 0.4 (y_1 is the flow depth downstream). Also determine the* ***maximum flow rate*** *and the minimum depth of flow down stream.*

$y_0 = 6$ m, Considering unit width, From equation 12.7.3. and 12.6.3.

Case (i) $y_1 = 0.8 \times 6 = 4.8$ m, $y_0 = \dfrac{q^2}{2g{y_1}^2} + y_1$

$\therefore$ $q^2 = 2g\ {y_1}^2\ (y_0 - y_1) = 4.8^2\ (6 - 4.8) \times 2 \times 9.81 = 542\ .45$

$\therefore$ **$q = 23.29\ m^3/s/m$, $V_1 = 23.29/4.8 = 4.852$ m/s**

$Fr = V/\sqrt{gy} = 4.852/\sqrt{9.81 \times 4.8} = 0.7071$, (subcritical)

Case (ii) $y_1/y_0 = 0.4 \quad \therefore \quad y_1 = 6 \times 0.4 = 2.4$ m,

$q^2 = 2.4^2\ (6 - 2.4) \times 2 \times 9.81 = 406.84$

$q = 20.17\ m^3/s/m$, $V = 20.17/2.4 = 8.404$ m/s

$Fr = 8.404/\sqrt{9.81 \times 2.4} = 1.732$, (supercritical)

Maximum flow occurs at $y_1 = (2/3)\ y_0 = 4$ m

$\therefore$ $q^2 = 4^2(6 - 4) \times 2 \times 9.81 = 627.84$, **$q = 25.06\ m^3/s/m$**

V $= 25.06/4 =$ **6.264**, m/s, $Fr = 6.264/\sqrt{9.81 \times 4} = 1$

Example 12.20 *Water flows at the upstream of a channel at 2m/s and the depth is 2 m, when the sluice is opened and flow is steady.* ***Determine the depth at the downstream side.*** *Also determine the* ***maximum flow rate*** *conditions i.e. depth velocity and flow rate downstream.*

Steady, incompressible uniform flow is assumed to prevail.

Specific energy upstream $= \dfrac{{V_1}^2}{2g} + y_1 = \dfrac{2^2}{2g} + 2 = 2.2039$ m

$Fr = 2/\sqrt{9.81 \times 2} = 0.452$ (subcritical)

In case the velocity upstream is negligible, this will be the value of y_0 *i.e.* $y_0 = 2.2039$ m

As the downstream flow is the same, specific energy remains constant. Expressing it in terms of q

$q^2/(2g{y_2}^2) + y_2 = 2.2039$

$q = 2 \times 2 =$ 4 $m^3/s/m$

This depth will be the alternate depth for the flow.

${y_2}^3 - 2.2039\ {y_2}^2 + \dfrac{4^2}{2 \times 9.81} = 0$, Solving by trial **$y_2 = 0.7485$ m**

$\mathbf{V_2}$ = 4/0.7485 = **5.344 m/s** (supercritical)

Maximum flow will occur when

$y_2 = (2/3)\, y_0 \quad \therefore \quad \mathbf{y_{max}} = 2.2039 \times (2/3) = \mathbf{1.4692}$

As $Fr = 1 \quad \mathbf{V_{max}} = \sqrt{g y_{max}} = \sqrt{1.4692 \times 9.81} = \mathbf{3.796\ m/s}$

$\mathbf{q_{max}} = 7.796 \times 1.469 = \mathbf{5.58\ m^3/s/m}$

When the sluice position corresponds to this flow rate, the upstream condition at section 1 will change.

12.8 FLOW WITH GRADUALLY VARYING DEPTH

When open channel flow encounters a change in bed slope or is approaching normal depth, flow depth changes gradually. As the change is continuous, the analysis should take into consideration a differential control volume instead of sections upstream and downstream. Water depth and channel bed height are assumed to change slowly. The velocity at any section is assumed to be uniform. Refer Fig. 12.8.1.

In this case the energy grade line and free surface are not parallel. The slope of the bed is S_b. The slope of the energy grade line is S.

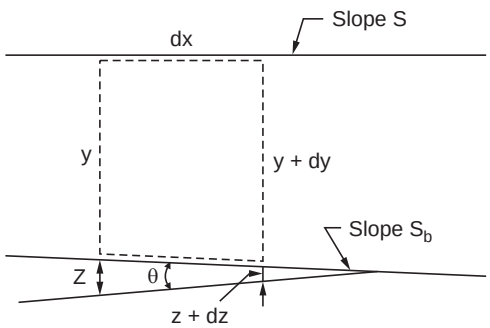


Figure 12.8.1 Flow with gradually varying depth

The specific energy flowing in at location x is $\dfrac{V^2}{2g} + y + z$,

The energy flow out at location $x + dx$ is

$$\frac{V^2}{2g} + d\left[\frac{V^2}{2g}\right] + y + dy + z + dz + dh_L$$

where dh_L is the head loss. The change in bed elevation can be expressed in terms of bed slope as $- S_b\, dx . dh_L = S dx$ where S is the slope of the energy grade line. Taking the net flow and equating to the gravity drap,

$$d\left[\frac{V^2}{2g}\right] + dy = (S_b - S)\, dx \qquad (12.8.1)$$

$$\frac{d}{dx}\left[\frac{V^2}{2g}\right]+\frac{dy}{dx}=S_b-S \tag{12.8.2}$$

expressing V in terms of flow q

$$\frac{d}{dx}\left[\frac{V^2}{2g}\right]=\frac{d}{dx}\left[\frac{q^2}{2gy^2}\right]=-2\,\frac{q^2}{2gy^3}\,\frac{dy}{dx}=-\frac{V^2}{gy}\,\frac{dy}{dx}=-Fr^2\,\frac{dy}{dx}$$

Substituting in 12.8.2.

$$\frac{dy}{dx}=\frac{S_b-S}{1-Fr^2} \tag{12.8.3}$$

The change in bed level dy along the flow direction for length dx is given by this equation. The gradient of the energy grade line can be obtained by assuming that loss is equal to the loss in steady uniform flow at normal depth. This is obtained from Manning's equation

$$\mathbf{S=\frac{N^2V^2}{R_h^{4/3}}} \tag{12.8.4}$$

Substituting for S in equation (12.8.3), results in a first order nonlinear ordinary differential equation that describes the variation of water surface profile. The sign of dy depends on the value of Froude number and the relative magnitudes of S and S_b

12.8.1 Classification of Surface Variations

The study is somewhat simplified when applied to a wide rectangular channel of depth y, when $R_h = y$. The equation (12.8.3) can be applied to this situation.

$$\frac{dy}{dx}=\frac{S_b-S}{1-Fr^2}=\frac{S_b[1-(S/S_b]}{1-Fr^2} \tag{12.8.5}$$

Using equation (12.8.4) we can show that, where y_n is the depth for normal flow

$$\frac{S}{S_b}=\left(\frac{y_n}{y}\right)^{10/3} \tag{12.8.6}$$

Also $$Fr=\left(\frac{y_c}{y}\right)^{3/2}$$

Substituting in (12.8.5).

$$\frac{dy}{dx}=S_b\,\frac{[1-(y_n/y)^{10/3}]}{[1-(y_c/y)^3]} \tag{12.8.7}$$

Also

On the basis of relative values of $\mathbf{y_n}$ and $\mathbf{y_c}$ the flow can be classified as mild slope, critical slope and steep slope types.

Then on the basis relative values of y, y_c and y_n for these types there can be three types of profiles.

In addition, horizontal bed slope and adverse bed slope type of flows will also lead to different shapes.

More often, the surface profiles are to be calculated by numerical methods. Now computer software's are also available to show the profiles graphically.

12.9 THE HYDRAULIC JUMP (RAPIDLY VARIED FLOW)

In subcritical flow due to any change in bed slope or cross-section, the disturbance produced will move upstream and downstream resulting in smooth adjustment of the flow depth. In supercritical or shooting flow such a disturbance cannot move upstream. The flow upstream will remain unchanged. The adjustment is sudden and can be only in the downstream side. **Change from supercritical to subcritical conditions thus cannot be smooth. Such a change occurs by increase of flow depth downstream. This is called Hydraulic jump.** The abrupt change involves loss of mechanical energy due to turbulent mixing. The heat produced does not significantly affect the temperature of the stream.

Hydraulic jump is used to dissipate mechanical energy into heat in various hydraulic structures. Fig. 12.9.1 shows a typical hydraulic jump. Horizontal bed condition is assumed.

Considering the control volume, the net momentum flow is equal to the net force (b = width).

Net momentum flow $= \rho\, Q\,(V_2 - V_1) = \rho\, V_1 y_1 b(V_2 - V_1)$ (12.9.1)

The pressure distribution in the flow depth is hydrostatic and acts at the centroids

$$F_1 = \rho\, gb\, y_1^2/2,\ F_2 = \rho\, gb\, y_2^2/2 \tag{12.9.2}$$

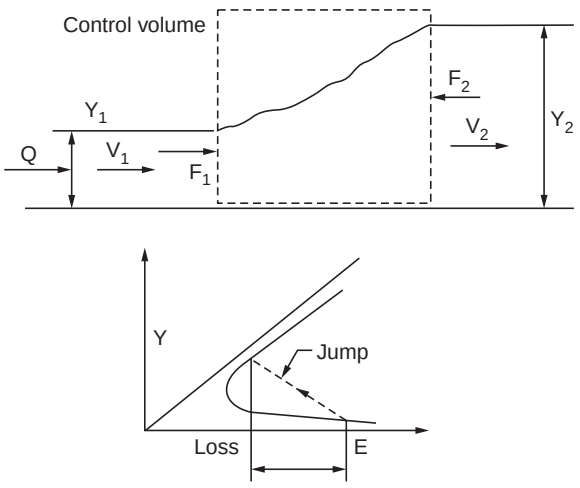


Figure 12.9.1 Hydraulic jump

$$\therefore \qquad F_1 - F_2 = \rho\, gb \left(\frac{y_1^2}{2} - \frac{y_2^2}{2} \right) \quad \text{Equating}$$

$$\rho\, gb \left(\frac{y_1^2}{2} - \frac{y_2^2}{2} \right) = \rho\, bV_1 y_1 (V_2 - V_1) \quad \text{Rearranging}$$

$$(y_1^2 - y_2^2)/2 = \frac{V_1 y_1}{g} (V_2 - V_1) \tag{12.9.3}$$

Continiuity equation is $y_1\, b\, V_1 = y_2 b V_2 = Q$

Energy equation is

$$y_1 + \frac{V_1^2}{2g} = y_2 + \frac{V_2^2}{2g} + h_L \tag{12.9.4}$$

Wall shear is neglected. Bed is assumed to be horizontal. The loss is due of violent mixing. Using momentum and continuity equations, to eliminate V_2

$$(y_1^2 - y_2^2)/2 = \frac{V_1^2 y_1}{g y_2} (y_1 - y_2)$$

or
$$(1/2)\,(y_1 + y_2)\,(y_1 - y_2) = \frac{V_1^2 y_1}{g y_2} (y_1 - y_2) \tag{12.9.5}$$

Cancelling $(y_1 - y_2)$ and multiplying by (y_2/y_1^2) and nothing $Fr_1 = V_1 \sqrt{g y_1}$

$$\left(\frac{y_2}{y_1} \right)^2 + \left(\frac{y_2}{y_1} \right) - 2Fr_1^2 = 0 \tag{12.9.6}$$

Solving for (y_2/y_1), from the quadratic equation above and nothing that – ve sign for the root is not possible

$$\mathbf{\frac{y_2}{y_1} = (1/2) \left[\sqrt{1 + 8Fr_1^2} - 1 \right]} \tag{12.9.7}$$

The depth ratio is tabulated below for various values of upstream Froude number. As hydraulic jump is possible only in supercritical flow conditions, $Fr > 1$ alone is considered. Note that if $Fr_1 < 1$ then $y_2 < y_1$. This leads to increase in specific energy which is not possible see Fig. 12.9.1, *y Vs E* diagram.

Fr	1	1.5	2	2.5	3	3.5	4	4.5	5	10
$\mathbf{y_2/y_1}$	1.0	1.67	2.37	3.07	3.77	4.48	5.18	5.88	6.59	13.7

The depth downstream is always higher than that at the upstream. Depth upstream should be smaller than critical depth. Depth down stream will be higher than critical depth.

From Equation (12.9.5), loss of head can be directly determined as

$$h_L = (y_1 - y_2) + (V_1^2 - V_2^2)/2g$$

$$\frac{h_L}{y_1} = [1 - (y_2/y_1)] + \frac{Fr_1^{\,2}}{2}[1 - (y_1/y_2)^2] \tag{12.9.8}$$

(y_2/y_1) is obtained from Equation (12.9.7). This equation can be also simplified (using a rather long algebraic work) as

$$\frac{\mathbf{h_L}}{\mathbf{y_1}} = \frac{\mathbf{y_1}}{\mathbf{4y_2}}\left[\frac{\mathbf{y_2}}{\mathbf{y_1}} - \mathbf{1}\right]^{\mathbf{3}} \tag{12.9.9}$$

As h_L is positive (y_2/y_1) should be greater than one.

Expressing (y_2/y_1) in terms Froude number, the loss can be expressed as a fraction of specific energy at section 1.

$$\frac{h_L}{E_1} = \frac{(a-3)^3}{[8(a-1)(2+Fr_1^{\,2})]} \text{ where } a = \sqrt{1+8\,Fr_1^{\,2}} \tag{12.9.10}$$

The ratio h_L/E_1 is shown tabulated below as a percentage for various values of Froude number.

Fr	1.5	2	3	4	5	6	8	10
h_L/E_1	2.2	9.1	25.7	39.1	49.1	56.4	66.4	72.7

It can be seen that for streaming flows with $Fr > 5$, a larger fraction of the mechanical energy is dissipated by eddies. This idea is used to dissipate the energy of water flowing over spillways. By proper design, energy may be dissipated without damage to the structure.

Hydraulic jump is similar to normal shock in compressible flow. In flow through supersonic nozzles, back pressure changes will not pass the supersonic region. In hydraulic jump also the disturbance downstream does not pass upstream. Pressure adjustment is automatic with normal shock and so also in hydraulic jump. Normal shock occurs in supersonic flow. Hydraulic jump occurs in supercritical flow. There is a pressure rise across normal shock. There is head increase in hydraulic jump.

Example 12.21 *In the flow through a sluice in a large reservoir, the velocity downstream is 5.33 m/s while the flow depth is 0.0563 m.* ***Determine the downstream conditions if a hydraulic jump*** *takes place downstream. Calculate the energy dissipated by eddies in the jump.*

$$Fr = V/\sqrt{gy} = 5.33/\sqrt{9.81 \times 0.0563} = 7.172$$

$$\frac{y_2}{y_1} = (1/2)\left(\sqrt{1+8\,Fr_1^{\,2}} - 1\right) = (1/2)\left(\sqrt{1+8\times 7.172^2} - 1\right) = 9.655$$

$\therefore$ $\quad \mathbf{y_2} = 9.655 \times 0.0563 = \mathbf{0.5436\ m}$

$\therefore$ $\quad \mathbf{V_2} = \dfrac{0.0563}{0.5436} \times 5.33 = \mathbf{0.552\ m/s} \quad Fr_2 = 0.239$

$$\frac{h_L}{y_1} = \frac{y_1}{4y_2}\left[\frac{y_2}{y_1} - 1\right]^3 = \frac{1}{4} \times \frac{1}{9.655}[9.655 - 1]^3 = 16.78$$

$\therefore$ $\mathbf{h_L} = 16.78 \times 0.053 = \mathbf{0.945\ m}$

Specific energy at inlet $= \dfrac{V_1^2}{2g} + y_1 = \dfrac{5.33^2}{2 \times 9.81} + 0.0563 = 1.504$ m

$\therefore$ **% dissipation** = $(0.945/1.504) \times 100 = \mathbf{62.82\ \%}$, check using

$$\frac{h_L}{E_1} = \frac{\left[\sqrt{1 + 8 \times Fr_1^2} - 3\right]^3}{8\left[\left(\sqrt{1 + 8 \times Fr_1^2} - 1\right)(Fr^2 + 2)\right]},$$

Substituting $Fr = 7.172$,

$$\frac{h_L}{E_1} = 0.6283$$

About 63% of mechanical energy is dissipated by the hydraulic jump. This can be shown on the specific energy diagram in Figure 12.9.1. The jump does not proceed along the specific energy curve, rather it jumps from 1 to 2 directly. The solutions for hydraulic jumps in other than rectangular channels is similar to that of rectangular channel. The phenomenon is similar. Froude number should be calculated using hydraulic depth.

12.10 FLOW OVER BROAD CRESTED WEIR

A broad crested weir consists of an obstruction in the form of raised portion of bed extending across the full width of the channel with a flat upper surface sufficiently broad in the direction of flow. Fig. 12.10.1 shows a broad crested weir with free fall. As the crest is broad, the flow surface becomes parallel to the crest. The upstream edge is well rounded to avoid losses.

The flow upstream will be subcritical and the downstream allows free fall. As there is no restraint downstream the flow will be maximum or the depth above the weir surface will be the critical depth y_c. Considering rectangular channel, expression for flow rate is derived

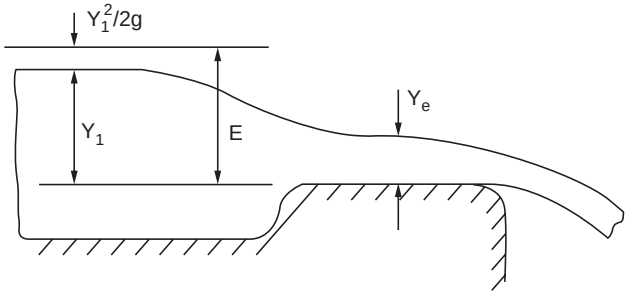


Figure 12.10.1 Broad crested weir

$$y_c = (Q^2/gb^2)^{1/3} = (q^2/g)^{1/3}$$

$\therefore$ $Q = b(gy_c^3)^{0.5}$ or $q = (gy_c^3)^{0.5}$

As $y_c = (2/3)E$, $\mathbf{Q} = b\ \{g \times (8/27)\ E^3\} = \mathbf{1.705\ bE^{3/2}}$ (12.10.1)

$$E = y_1 + (V_1^2/2g)$$

In the tranquil flow upstream $\mathbf{V_1}$ **is small.** In that case $\mathbf{Q = 1.705 \times b \times y^{3/2}}$

The measurement of h (over the crest of the weir) would be sufficient to determine the discharge. The height of the weir will not affect the flow over the crest. The upstream level will adjust as per the height of the weir.

In case the bed level downstream is equal to the bed level upstream, the level downstream will rise.

In this case though the level over the crest will decrease, it may not fall to the critical value y_c. The rate of flow is calculated using Bernoulli equation and continuity equation.

12.11 EFFECT OF LATERAL CONTRACTION

The channel width may be reduced keeping the bed horizontal. **If the upstream flow is subcritical, the depth in the reduced section will decrease. If the upstream flow is supercritical the level in the reduced section will increase.** Flumes with lateral contraction followed by expansion can be used for flow measurement. If the free surface in the section does not pass through critical depth, the arrangement is called **Venturi Flume,** similar to venturimeter.

The details are shown in Fig. 12.11.1

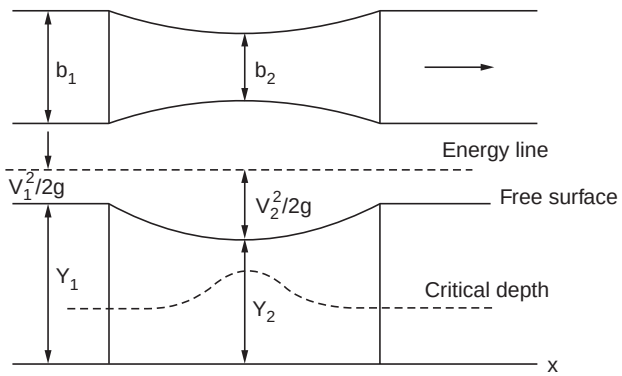


Figure 12.11.1 Lateral contraction

Using continuity and energy relations, it can be shown that

$$V_2 = \left[\frac{2gh}{1-(b_2y_2/b_1y_1)^2}\right]^{0.5} \quad \text{and} \quad Q = b_2y_2\left[\frac{2gh}{1-(b_2y_2/b_1y_1)^2}\right]^{0.5} \tag{12.11.1}$$

The flow rate $Q = C_d A_2 V_2$ where C_d is the coefficient of discharge having values in the range 0.95 to 0.99. In case the upstream flow is subcritical, and if critical conditions occur at the throat, then, the flow rate is given by, where E is the specific energy and b is the channel width.

$$Q = 1.705\, b_2\, E^{3/2} \tag{12.11.2}$$

In case the flow velocity upstream is small, then,

$$Q = 1.705\, b_2\, H^{3/2} \tag{12.11.3}$$

where H is the difference between the level at the throat and upstream water level. In this case if the bed slope down stream is the same as in upstream, the flow will revert to subcritical condition downstream by means of a hydraulic jump. Such a flume is known as standing wave flume.

Example 12.22 *A venture flume is formed in a horizontal channel of 2 m width by constructing the width to 1.3 m and raising floor level in the constricted section by 0.2 m above that of the channel. If the difference in level between the throat and downstream is 25 mm and both upstream and downstream depths are 0.6 m,* ***determine the rate of flow****.*

In the case the downstream conditions are changed such that a standing wave forms after the throat, calculate the flow assuming upstream depth is still 0.6 m.

The flow rate is given by the equation [Refer equation 12.11.1]

$$Q = b_2 y_2 \left[\frac{2gh}{1-(b_2 y_2 / b_1 y_1)^2}\right]^{0.5}$$

$b_1 = 2$ m, $b_2 = 0.9$ m, $y_2 = 0.6 - 0.2 - 0.025 = 0.375$ m, $h = 0.025$ m,

$$\mathbf{Q} = 1.3 \times 0.375 \left[\frac{2 \times 9.81 \times 0.025}{1-(1.3 \times 0.375 / 2 \times 0.6)^2}\right]^{0.5} = \mathbf{0.3736\ m^3/s}$$

When standing wave forms consider the equation (12.11.2)

$Q = 1.705\, b_2\, E^{3/2}$,

E = the total head at construction which is $0.6 - 0.2 = 0.4$ *m*

$\therefore$ $Q = 1.705 \times 1.3 \times 0.4^3 = 0.50607$ m^3/s

The velocity head portion of energy upstream is neglected in this case.

The velocity upstream, $V_1 = 0.5607/(2 \times 0.6) = 0.4673$ m/s

$\therefore$ $V_1^2/2g = 0.011$ m,

Compared to 0.4 m, this is 2.8% and hence may be neglected.

SOLVED PROBLEMS

Problem 12.1 ***Determine the flow rate of water*** *in a rectangular channel of 3 m width when the depth of flow is 1 m. The bed slope is 1 in 2500. Friction factor f = 0.038.*

As friction factor is given Chezy' constant can be determined. Chezy's constant,

$$C = \sqrt{8g/f} = [8 \times 9.81/0.038]^{0.5} = \mathbf{45.45}$$

To determine the hydraulic depth,

$A = 3 \times 1 = 3$ m^2 , $P = 3 + 1 + 1 = 5$ m,

$\therefore$ $R_h = 3/5 = 0.6$, bed slope = 1/2500

Using Chezy's equation, indicating velocity by, V

$$V = C\sqrt{R_h S_b} = 45.45\ \sqrt{0.6/2500} = 0.704 \text{ m/s},$$

Flow rate = $VA = 0.704 \times 3 =$ **2.11 m³/s**

Froude number = $V/\sqrt{gy} = 0.704/\sqrt{9.81 \times 1} = 0.225$

So the flow is in the subcritical region.

Problem 12.2 *Analyse the flow in the channel of problem 12.1. Considering combination of depths of 0.5, 1, 2 and 2.5 m with bed slopes of 1/500, 1/1000, 1/1500, 1/2000 and 1/2500.*

The values calculated using Chezy's equation are tabulated below. Velocity V, m/s, Flow rate Q, m³/s. Depth and R_h are given in m.

Slope		1/500		1/1000		1/1500		1/2000		1/2500	
Depth	**R_h**	**V**	**Q**	**V**	**Q**	**V**	**Q**	**V**	**Q**	**V**	**Q**
0.5	0.38	1.24	1.87	0.88	1.32	0.7	1.1	0.6	0.9	0.6	0.8
1.0	0.60	1.57	4.72	1.11	3.34	0.9	2.7	0.8	2.4	0.7	2.1
1.5	0.75	1.76	7.92	1.24	5.60	1.0	4.6	0.9	4.0	0.8	3.5
2.0	0.86	1.88	11.3	1.33	7.98	1.1	6.5	0.9	5.6	0.8	5.1
2.5	0.94	1.97	14.8	1.39	10.4	1.1	8.5	1.0	7.4	0.9	6.6

Note. 1. **As slope becomes less steep the velocity and flow rates decrease,** but not in direct proportion.

2. **As depth increases the flow increases more rapidly** because both velocity and area increase with depth.

Problem 12.3 *In problem 12.2 determine the Froude number in each of the cases.*

$Fr = V/\sqrt{gy}$, Depth and R_h are given in m.

Slope		1/500		1/1000		1/1500		1/2000		1/2500	
Depth	**R_h**	**V**	**Fr**	**V**	**Fr**	**V**	**Fr**	**V**	**Fr**	**V**	**Fr**
0.5	0.38	1.24	0.56	0.88	0.40	0.7	0.3	0.6	0.3	0.6	0.3
1.0	0.60	1.57	0.50	1.11	0.35	0.9	0.3	0.8	0.3	0.7	0.2
1.5	0.75	1.76	0.46	1.24	0.32	1.0	0.3	0.9	0.2	0.8	0.2
2.0	0.86	1.88	0.42	1.33	0.30	1.1	0.3	0.9	0.2	0.8	0.2
2.5	0.94	1.97	0.40	1.39	0.28	1.1	0.2	1.0	0.2	0.9	0.2

As depth increases for a given slope, velocity increases but Froude number decreases and flow is subcritical. As slope becomes steeper Froude number increases for the same depth due to velocity increase.

Problem 12.4 ***Determine the slope*** *with which a waste water pipe of 2 m diameter is to be laid for carrying water at the rate of 1500 l/s. The depth of flow is to be half the diameter. Assume CI pipe with Manning constant N = 0.05.*

Manning equation for discharge is

$$Q = \frac{A}{N} R_h^{2/3} S_b^{1/2}, \quad A = \frac{\pi \times 2 \times 2}{4 \times 2} = \frac{\pi}{2} \text{m}^2, P = \pi D/2 = \pi \text{ m}$$

$$\therefore \quad R_h = (\pi/2)/\pi = 0.5 \text{ m}, \; Q = 1500l/\text{s} = 1.5 \text{ m}^3/\text{s}$$

$$\therefore \quad 1.5 = \frac{\pi}{2} \times \frac{1}{0.015} (0.5)^{2/3} (S_b)^{1/2} \quad \therefore \quad \mathbf{S_b = 1/1934}$$

The flow rate and depth uniquely define the slope. Similarity flow rate and slope uniquely define the depth, called normal depth.

Problem 12.5 ***Determine the maximum discharge*** *through a rectangular open channel of area 8 m ^{2} with a bed slope of 1/2000. Assume Mannings constant as 0.022.*

For rectangular section the optimum depth equals half the width and maximum discharge occurs for this condition.

$$A = y \times b = y \times 2y = 2y^2 = 8 \quad \therefore \quad y = 2, b = 4, \quad \therefore \quad R_h = 8/8 = 1 \text{ m}$$

$$\mathbf{Q} = \frac{A}{N} R_h^{2/3} S_b^{1/2} = \frac{8}{0.022} (1)^{2/3} \left(\frac{1}{2000} \right)^{0.5} = \mathbf{8.13 \text{ m}^3/\text{s}}$$

This is maximum can be verified by calculating the flows for different depths like 1.8, 1.9, 2.1 and 2.2 for which the flows are 8.11, 8.12, 8.12 and 8.11 m^3/s-very near the optimum value but is still less.

Problem 12.6 ***Determine the economical cross-section*** *for an open channel of trapezoidal section with side slopes of 1 vertical to 2 horizontal, to carry 10 m^3/s, the bed slope being 1/2000. Assume Manning coefficient as 0.022.*

For economical section perimeter should be minimum. The condition for the same is (n-side slope) Refer example 12.8.

$$\frac{b + 2ny}{2} = y\sqrt{n^2 + 1}. \text{ Here } n = 2 \text{ and } b + 4y = 2y\sqrt{5},$$

$$b = y \; (\sqrt{20} - 4) = 0.4721 \, y, A = (b + 2y) \, y = 2.4721 \, y^2$$

Hydraulic mean depth (A/P) for economical section:

$$P = b + 2y \sqrt{n^2 + 1} = b + 2 \, y\sqrt{5} = 4.9443y$$

$$\therefore \quad R_h = A/P = 2.4721 \, y^2/4.9443 \, y = y/2$$

$$Q = 10 \text{ m}^3/\text{s}, N = 0.022, S_b = 1/2000$$

$$\therefore \quad 10 = \frac{1}{0.022} \times 2.4721 y^2 \left(\frac{y}{2} \right)^{3/2} (S_b)^{1/2}$$

$$10 = \frac{1}{0.022} \times \frac{2.4721 y^2}{2^{3/2}} (y)^{3/2} \left(\frac{1}{2000}\right)^{1/2} = 0.8884\, y^{7/2}$$

$\therefore$ **y = 1.9971 m, b = 0.9429 m** (check for flow)

Problem 12.7 ***Derive the expression for depth of flow*** *in a channel of circular section for maximum flow.*

Refer figure.

Let the flow depth be y, and Let the angle subtended be θ.

Wetted perimeter $P = 2\, R\, \theta$

Flow area $= (2\,\theta/2\,\pi)\,\pi R^2 - R \sin\theta\, R \cos\theta = R^2\theta - (R^2/2)\sin 2\theta$

$= (R^2/2)\,(2\theta - \sin 2\theta)$

Using Chezy's equation,

$$Q = AV = AC\sqrt{\frac{A}{P} S_b} = C\left[\frac{A^3}{P} S_b\right]^{0.5}$$

Q will be maximum if A^3/P is maximum. The condition is determined, using

$$\left[\frac{d(A^3/P)}{d\theta}\right] = 0 \quad i.e., \quad \frac{P \times 3 \times A^2 \frac{dA}{d\theta} - A^3 \frac{dP}{d\theta}}{P^2} = 0$$

or $$3P \frac{dA}{d\theta} = A \frac{dP}{d\theta}$$

$$P = 2\, R\, \theta, \quad \therefore \quad \frac{dP}{d\theta} = 2\, R,$$

$$A = (R^2/2)\,[2\theta - \sin 2\theta]$$

$$\frac{dA}{d\theta} = \frac{R^2}{2}(2 - 2\cos 2\theta), \text{ Substituting}$$

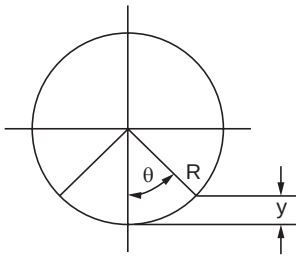


Figure P. 12.7

$$3 \times 2R\theta \frac{R^2}{2}(2 - 2\cos 2\theta) = \frac{R^2}{2}(2 - 2\sin 2\theta)\, 2R$$

$$3\theta(1 - \cos 2\theta) = [\theta - (\sin 2\theta/2)]$$

$$2\theta - 3\theta\cos 2\theta + \frac{\sin 2\theta}{2} = 0$$

This is a transcendental equation to be solved by trial, $\theta = 2.69$ radian or about 154°. (check using radian mode in the calculator)

The area for flow $= R^2\theta - \frac{R^2}{2}\sin 2\theta = 3.0818\, R^2$

Perimeter $= 2R\theta = 5.3756\, R \quad \therefore \quad R_h = 0.5733\, R$

Depth of flow $= R - R\cos\theta = \mathbf{R(1 - \cos\theta)} = 1.8988\,R$ or $0.9494\,D$

Note: This uses Chezy equation:

In case Manning equation is used, then the flow,

$$Q = \frac{A}{N}\left(\frac{A}{P}\right)^{3/2} S_b^{1/2} = \frac{1}{N}\frac{A^{5/2}}{P^{3/2}} S_b^{1/2}$$

$\frac{A^{5/2}}{P^{3/2}}$ or $\frac{A^5}{P^3}$ should be maximized with θ as independent variable

$$P^3 \times 5 \times A^4 \frac{dA}{d\theta} - A^5\, 3P^2 \frac{dP}{d\theta} = 0, \text{ Substituting,}$$

$$5P\frac{dA}{d\theta} - 3A\frac{dP}{d\theta} = 0 \quad \text{and} \quad \text{Substituting for } \frac{dA}{d\theta} \text{ and } \frac{dP}{d\theta}$$

$$5 \times 2R\theta \frac{R^2}{2}(2 - 2\cos 2\theta) = 3\frac{R^2}{2}(2\theta - \sin 2\theta) \times 2R$$

$$5\theta(2 - 2\cos 2\theta) - 3(2\theta - \sin 2\theta) = 0, \quad \text{or} \quad 4\theta - 10\theta\cos 2\theta + 3\sin 2\theta = 0$$

Solving, $\theta = 2.5007$ **radian or 143.28°**

In this case depth $R - R\cos\theta = R(1 - \cos\theta) = 1.802\,R$ or $0.901\,D$

Problem 12.8 Derive the condition for maximum velocity of flow in a channel of circular section.

Refer problem Problem 12.7.

$$A = \frac{R^2}{2}(2\,\theta - 2\sin 2\theta),\ P = 2R\theta,\ V = C\sqrt{R_h S_b}\,,\ R_h = \frac{A}{P}$$

For V to be maximum, A/P should be maximized,

$$\frac{d(A/P)}{d\,\theta}\left(P\frac{dA}{d\theta} - A\frac{dP}{d\theta}\right)/P^2 = 0 \quad \therefore \quad P\frac{dA}{d\,\theta} = A\frac{dP}{d\theta} \tag{A}$$

$$\frac{dA}{d\theta} = \frac{R^2}{2}(2 - 2\cos 2\theta),\ \frac{dP}{d\theta} = 2R, \text{ Substituting}$$

$$2R\theta\frac{R^2}{2}(2 - 2\cos 2\theta) = \frac{R^2}{2}(2\theta - 2\cos 2\theta)\,2R$$

$$2\theta(1 - \cos 2\theta) = 2\theta - \sin 2\theta$$

$$-2\theta\cos 2\theta + \sin 2\theta = 0 \quad \text{or} \quad \tan 2\theta = 2\theta$$

$\therefore$ Solving $\theta = 2.247$ radians, or 128.75°

Depth for maximum velocity y = R(1 – cos θ) $= 1.626\,R$ or $0.813\,D$

Note: This is using Chezy equation, In case Manning equation is used, the same result is obtained as in equation *A*.

Problem 12.9 ***Determine the maximum discharge*** *through a circular pipe of 2 m diameter with a bed slope of 1/1000. Also determine the depth for maximum velocity and the corresponding discharge Chezy's constant C = 60*

Adopting Chezy equation (Refer problem 12.7)

$$A = \frac{R^2}{2}(2\theta - 2\sin 2\theta),\ P = 2R\theta,\ R = 1,\ Q = 2.69 \text{ radians}$$

$$\therefore \quad A = \frac{1}{2}[2 \times 2.69 - \sin(2 \times 2.69)] = 3.083 \text{ m}^2,$$

$$P = 2 \times 1 \times 2.69 = 5.38 \text{ m}$$

$$R_h = A/P = 3.083/5.38 = 0.573 \text{ m}$$

$$Q = AC\sqrt{R_h S_b} = 3.083 \times 60\sqrt{0.573 \times \left(\frac{1}{1000}\right)} = 4.43 \text{ m}^3/\text{s}$$

For maximum velocity: θ = 2.247 radians

$$A = \frac{R^2}{2}(2\theta - 2\sin 2\theta) = \frac{1}{2}(2 \times 2.247 - \sin(2 \times 2.247))$$

$$= 2.7351 \text{ m}^2$$

$$P = 2R\theta = 2 \times 1 \times 2.247 = 4.494 \text{ m}$$

$$R_h = A/P = 2.7351/4.494 = 0.6086 \text{ m}$$

$$Q = 2.7351 \times 60 \times \sqrt{0.6086 \times \frac{1}{1000}} = 4.049 \text{ m}^3/\text{s}$$

In case of full area flow: $A = \pi \times 1^2 = \pi \text{m}^2$

$$P = \pi D \quad \therefore \quad R_h = \pi/\pi D = 1/2 = 0.5$$

$$\therefore \quad \mathbf{Q} = \pi \times 60\sqrt{\frac{0.5}{1000}} = \mathbf{4.215\ m^3/s}$$

(**Note:** Compare the flows for maximum discharge, maximum velocity and full flow)

Problem 12.10 *In a rectangular open channel of 5 m width the flow rate is 10 m^3/s and depth of flow is 1.0 m.* ***Determine the critical depth and the alternate depth.***

Flow velocity, $V = 10/5 \times 1 = 2$ m/s, Froude number $= V/\sqrt{gy} = 2/\sqrt{9.81 \times 1} = 0.64$

Flow is in the subcritical region.

$$\text{Specific energy} = \frac{Q^2}{2gA^2} + y = \frac{10^2}{2 \times 9.81 \times 5^2} + 1 = 1.2039 \text{ m}$$

$$\text{At critical condition, } \mathbf{y_c} = \left(\frac{Q^2}{gb^2}\right)^{1/3} = \left(\frac{10^2}{5^2 \times 9.81}\right)^{1/3} = \mathbf{0.7415\ m}$$

Velocity at this condition = $10/(5 \times 0.745) = 2.6971$ m/s

Froude number = $2.6971/\sqrt{9.81 \times 0.7415} = 1.0$ (checks)

Minimum energy = $(3/2)\, y_c = 1.11225$ m

To determine the alternate depth,

$$\frac{V_2^2}{2g} + y_2 = 1.2039,\ V_2^2 = \left(\frac{Q}{A_2}\right)^2 = \frac{Q^2}{b^2 y^2}$$

$$\frac{Q^2}{2gb^2 y_2^2} + y_2 = 1.2039,\ \frac{10^2}{2 \times 9.81 \times 5^2}\frac{1}{y_2^2} + y_2 = 1.2039$$

$0.2039 + y_2^3 = 1.2039\, y_2^2$ or $y_2^3 - 1.2039\, y_2^2 + 0.2039 = 0$

Solving by trail y_2 = **0.565 m** and $V_2 = 3.5398$ m/s, $Fr = 1.5036$,

$\therefore$ Supercritical region. Check for energy :

$\therefore$ $(V^2/2g) + y = 1.2036$ (checks).

Problem 12.11 ***Calculate the bed slope of a trapezoidal channel*** *of bed width 6 m and horizontal to vertical side slope of 1:3. Water flows at the rate of 10 m^3/s at a depth of 2 m. Assume Chezy's constant as 50.*

Discharge $Q = AC\sqrt{R_h S_b}$

$A = 2 \times [6 + (2/3)] = 13.3333$ m^2,

$$P = 6 + \left(\sqrt{2^2} + (2/3)^2\right)^2 = 10.2164 \text{ m}$$

$R_h = A/P = 13.3333/10.2164 = 1.3051$ m

$10 = 13.3333 \times 50\sqrt{1.3051 \times S_b}$. Solving, $\mathbf{S_b = 1/5800}$

Problem 12.12 ***Calculate the water discharge through an open channel*** *shown in figure. Assume Chezy's constant as 60 and bed slope as 1 in 2000. Also calculate the Mannings constant for the flow.*

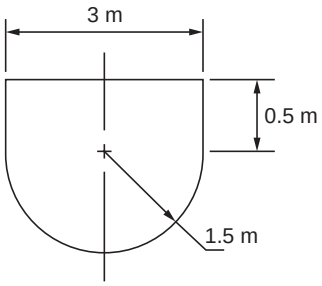


Figure P. 12.12

Discharge $Q = AC\sqrt{R_h S_b}$, $A = (3 \times 0.5) + \dfrac{\pi \times 1.5^2}{2}$

$= 5.0343$ m^2

$P = 0.5 + 0.5 + (\pi \times 1.5) = 5.7123$ m,

$R_h = 5.0343/5.7123 = 0.8813$ m

$\boldsymbol{Q} = 5.0343 \times 60\ [0.8813/2000]^{0.5}$ = **6.341 m^3/s**

To determine Mannings constant

$$6.341 = \frac{5.0343}{N}\left[0.8813^{2/3}\left(\frac{1}{2000}\right)^{1/2}\right]$$

$\therefore \quad N = 0.0163$

Problem 12.13 *Using Bazins formula,* ***determine the discharge through*** *a rectangular ordinary earthen channel 2 m wide and 0.5 m deep with a slope of l in 2500. Assume Bazins constant k = 1.303. If Manning constant for this type is 0.025, determine and compare the flow.*

$$\text{Discharge } Q = AC\sqrt{R_h S_b}\,,\ \text{C} = \frac{86.9}{1 + k/\sqrt{R_h}}$$

$$A = 2 \times 0.5 = 1\ \text{m}^2,\ P = 0.5 + 2 + 0.5 = 3\ \text{m},$$

$$R_h = A/P = 1/3 = 0.333$$

$$C = \frac{86.9}{1 + 1.303/\sqrt{0.3333}} = 26.682$$

$\therefore \quad \mathbf{Q} = 1 \times 26.682\sqrt{0.333 \times \left(\frac{1}{2500}\right)} = \mathbf{0.3081\ m^3/s}$

By Mannings equation,

$$\mathbf{Q} = \frac{1}{0.025}(0.333)^{2/3}\left(\frac{1}{2500}\right)^{1/2} = \mathbf{0.385\ m^3/s}$$

Problem 12.14 *A rectangular open channel having 4 m depth and 8 m width is concrete lined with a bed slope of 1 in 2000. Use Kutter's formula and* ***calculate the discharge through*** *it. Kutters constant = 0.012*

Discharge $\quad Q = AC\sqrt{R_h\, S_b}$

$$A = 32\ \text{m}^2,\ P = 16\ \text{m},\ R_h = 2\ \text{m},\ S_b = 1/2000 = 0.0005$$

$$C = \frac{23 + (0.00155/S_b) + \dfrac{1}{N}}{1 + (23 + (0.00155/S_b))\dfrac{N}{\sqrt{R_h}}}$$

$$= \frac{23 + (0.00155/0.0005) + (1/0.012)}{1 + (23 + (0.00155/0.0005))(0.012/\sqrt{2}} = 89.59$$

$$\mathbf{Q} = 32 \times 89.59\sqrt{2 \times 0.0005} = \mathbf{90.66\ m^3/s},$$

For concrete, taking **Manning constant** as 0.012

$$\mathbf{Q} = (32/0.012)2^{2/3}\,(0.0005)^{1/5} = \mathbf{94.65\ m^3/s}$$

Problem 12.15 ***Determine the slope for a V-shaped concrete*** *lined channel with total included angle of 80° and a depth of 3 m if the discharge is 10.7 m^3/s. Manning constant N = 0.012*

$$\text{Area} = \frac{2 \times 3 \tan 40 \times 3}{2} = 7.5519 \text{ m}^2,$$

Wetted perimeter $= 2 \times 3/\cos 40 = 7.8324$ m

Hydraulic mean depth $= R_h = A/P = 7.552/7.8324 = 0.9642$ m

$$Q = 10.7 = \frac{7.5519}{0.012}(0.9642)^{2/3} S_b^{1/2} \quad \therefore \mathbf{S_b = 1/3295}$$

Calculating the corresponding Chezy constant, C

$$10.7 = 7.5519 \times C\left(0.9642 \times \frac{1}{3295}\right)^{1/2} \quad \therefore C = 82.83$$

Problem 12.16 ***Estimate the discharge of water in*** *an open channel of trapezoidal section with bottom width of 1 m and side slope of 1:1 with a flow depth of 1 m. The bed slope is 1 in 2000. Use Manning formula with constant N = 0.05*

Discharge $Q = \frac{A}{N} \times R_h^{2/3} \times S_b^{1/2}$, $A = \{(1 + 3)/2\} \times 1 = 2 \text{ m}^2$;

$P = 1 + 2\sqrt{1^2 + 1^2} = 3.83$ m, $R_h = A/P = 2/3.83 = 0.5224$ m

$\therefore$ $\mathbf{Q} = (2/0.05) \times (0.5224)^{2/3} \times (1/2000)^{1/2} = 0.58 \text{ m}^3/\text{s} =$ **580 *l/s***

Problem 12.17 ***Determine the most economical cross-section*** *of a rectangular channel of width b and depth y to carry 1000 litres of water per second with a bed slope of 1 in 500. assume Mannings constant, N = 0.022*

$$\text{Area} = A = b \times y$$

For most economical cross-section

$$b = 2y, A = 2y^2, R_h = y/2$$

$$Q = \frac{A}{N} R_h^{2/3} S_b^{1/2}, \text{ flow rate} = 1000 \, l/s \text{ or } 1 \text{ m}^3/\text{s}$$

$$1 = \frac{2y^2}{0.022}\left(\frac{y}{2}\right)^{2/3}\left(\frac{1}{500}\right)^{1/2},$$

Solving depth **y = 0.7028 m, width b = 1.4056 m**

Problem 12.18 *The most economical cross-section of a trapezoidal open channel is 5 m^2.* ***Find the discharge in the channel*** *for a depth of flow of 0.5 m. Assume the Chezy constant C = 50 and bed slope as 1 in 1000.*

For most economical cross-section of the trapezoidal channel

$$R_h = y/2 = 0.5/2 = 0.25 \text{ m},$$

Discharge, $\mathbf{Q} = AC\sqrt{R_h S_b} = 5 \times 50 \sqrt{0.25/1000} =$ **3.95 $\mathbf{m^3/s}$**

Problem 12.19 ***Design the bed slope*** *for the most economical cross-section for a trapezoidal earthen open channel with a flow velocity of 2 m/s and discharge of 5 m^3/s. The side slope vertical to horizontal may be taken as 1 in 2 and Chezy constant C = 50*

$$A = \frac{Q}{V} = \frac{5}{2} = 2.5 \text{ m}^2, \text{ For most economical cross-section}$$

$$A = y(b + ny) = y(b + 2y),\ 2.5 = b \times y + 2y^2,\ b \times y = 2.5 - 2y \qquad (1)$$

Also $R_h = y/2$

Perimeter $P = b + 2y\sqrt{n^2 + 1} = b + 2y\sqrt{5}$, $R_h = y/2 = A/P = 2.5/(b + 2y\sqrt{5})$

$$b \times y + 4.472\, y^2 = 5 \qquad (2)$$

Substituting equation (1) in equation (2)

$$2.5 - 2y^2 + 4.472y^2 = 5, \quad 2.472\, y^2 = 2.5, \quad y = \sqrt{\frac{2.5}{2.472}} = 1.006 \text{ m}$$

Substituting y = 1.006 in equation (1)

$$1.006\, b = 2.5 - 2 \times (1.006)^2 \quad b = 0.473 \text{ m}$$

Velocity, $V = C\sqrt{R_h S_b}$ *i.e.* $2 = 50\sqrt{\frac{1.006}{2} \times S_b}$ $\therefore$ **Slope, S_b = 1/316**

Problem 12.20 *A rectangular channel of 5 m width discharges water at the rate of 1.5 m^3/s into a 5 m wide apron with 1/3000 slope at a velocity of 5 m/s.* ***Determine the height of the hydraulic jump*** *and energy loss.*

Depth of water on the upstream side of the jump,

$$y = \frac{Q}{V_1 \times b} = \frac{1.5}{5 \times 5} = 0.06 \text{ m}$$

$\therefore$ $$Fr = \frac{5}{\sqrt{9.81 \times 0.06}} = 6.52.$$ Hence flow is supercritical.

Hence hydraulic jump is possible. Depth of water on the downstream side of jump

$$y_2 = -\frac{y_1}{2} + \sqrt{\frac{y_1^2}{4} + 2y_1 V_1^2 / g}$$

$$= -\frac{0.06}{2} + \sqrt{\frac{0.06^2}{4} + \frac{2 \times 0.06 \times 5^2}{9.81}} = 0.5238 \text{ m}$$

Height of hydraulic jump = $y_2 - y_1$ = 0.5238 – 0.06 = **0.4638 m**

$$V_2 = \frac{V_1 y_1}{y_2} = \frac{5 \times 0.06}{0.5238} = 0.5757 \text{ m/s}$$

$$\textbf{Energy loss} = E_1 - E_2 = \left(y_1 + \frac{V_1^2}{2g}\right) - \left(y_2 + \frac{V_2^2}{2g}\right)$$

$$= \left(0.06 + \frac{5^2}{2 \times 9.81}\right) - \left(0.5238 + \frac{0.5727^2}{2 \times 9.81}\right)$$

$$= \textbf{0.7937 m head of water.}$$

Problem 12.21 *Water is discharged at a velocity of 8 m/s with a depth of 0.7 m in a horizontal rectangular open channel of constant width when the sluice gate is opened upwards.* ***Determine the height of the hydraulic jump and the loss of energy.***

Depth of water on the downstream of the jump (modified equation 12.9.7)

$$\mathbf{y_2} = -\frac{y_1}{2} + \sqrt{\frac{y_1^2}{4} + \frac{2y_1V_1^2}{g}} = -\frac{0.7}{2} + \sqrt{\frac{0.7^2}{4} + \frac{2 \times 0.7 \times 8^2}{9.81}} = \textbf{2.69 m}$$

$$V_2 = \frac{V_1 h_1}{y_2} = \frac{8 \times 0.7}{2.69} = 2.082 \text{ m/s}$$

Energy loss = $\mathbf{E_1 - E_2}$

$$= \left(y_1 + \frac{V_1^2}{2g}\right) - \left(y_2 + \frac{V_2^2}{2g}\right) = \left(0.7 + \frac{8^2}{2 \times 9.81}\right) - \left(2.69 + \frac{2.082^2}{2 \times 9.81}\right)$$

$$= 1.05 \text{ m head of water.}$$

Problem 12.22. *A Wide channel of uniform rectangular section with a slope of 1/95 has a flow rate of 3.75 $m^3/s/m$. The Manning constant is 0.013. Suddenly the slope changes to 1/1420.* ***Determine the normal depths for*** *each case. Show that a hydraulic jump has to occur and calculate the downstream flow height.*

As the channel is said to be wide the hydraulic mean depth will be equal to the depth of flow.

The normal depth is obtained from the Manning equation

$$Q = \frac{A}{N} R_h^{2/3} S_b^{1/2}$$

Here $A = 1 \times y$, $R_h = y$, $S_b = 1/95$. Substituting the values

$$Q = \frac{S_b^{1/2}}{N} y^{5/3}$$

$$\therefore \quad 3.75 = \frac{(1/95)^{1/2}}{0.013} y_1^{5/3}, \text{ Solving } \quad y_1 = \textbf{0.6399,}$$

$$V_1 = 3.75/0.6399 = \textbf{5.86 m/s, } \mathbf{Fr_1 = 2.339, Supercritical flow}$$

For slope 1/1420, Normal depth is

$$3.75 = \frac{(1/1420)^{1/2}}{0.013}\, y_2^{5/3}.\ \text{Solving } \mathbf{y_2 = 1.4404\ m}$$

$$\mathbf{V_2} = 3.75/1.4404 = \mathbf{2.6035\ m/s}$$

$$\mathbf{Fr_2} = 2.6035/\sqrt{9.81 \times 1.4404} = \mathbf{0.6926} \quad \therefore \text{ Subcritical}$$

As flow is from supercritical to subcritical flow, hydraulic jump should occur. (equation 12.9.7)

$$y_2 = \frac{y_1}{2}\left[-1+\sqrt{1+8Fr_1^2}\right]$$

$$= \frac{0.6399}{2}\left[-1+\sqrt{1+8/2.339^2}\right] = 1.8208\ \text{m}$$

Problem 12.23 *A trapezoidal channel has a bed slope of 1/2500. The channel is to carry 2 m^3/s.* ***Determine the optimum dimensions****. Side slope is 1:1. Chezy' constant = 50 $m^{1/2}\ s^{-1}$.*

The conditions for the most optimum section are

$$b = 2y\ [(n^2+1)^{0.5} - n] \text{ and } n = 1 \text{ As slope 1:1}$$

$\therefore$ $$b = 2y\ [\sqrt{2} - 1] = 0.8284y,\ A = (b+y)\,y = y^2 + 0.8284\,y^2 = 1.8284\,y^2$$

$$P = b + 2\sqrt{(y^2+y^2)} = 0.8284y + 2.82y = 3.6568y$$

$$R_h = \frac{1.8284y^2}{3.6568y} = 0.5\,y$$

$$Q = 2 = 18284\,y^2 \times 50\sqrt{(0.5y)(1/2500)} = 1.29287y^{5/2}$$

Solving **y = 1.1907 m, b = 0.9863 m**

Problem 12.24 *A rectangular channel of 6 m width has a flow rate of 22.5 m^3/s when the depth is 3 m.* ***Determine the alternate depth and the critical depth.***

Refer section 12.9

Velocity $V = 22.5/(6 \times 3) = 1.25$ m/s, $Fr = 1.25/\sqrt{3 \times 9.81} = 0.23$

Flow is subcritical.

$$\text{Specific energy} = \frac{V_2^2}{2g} + y = \frac{1.25^2}{2 \times 9.81} + 3 = 3.0796\ \text{m}$$

The specific energy is the same at the alternate depth.

$$\frac{V^2}{2g} + y_2 = 3.0796.\ \text{Expressing } V_2 \text{ in terms of flow,}$$

$$\frac{Q^2}{2gb^2 y_2^2} + y_2 = 3.0796.\ \text{or } y_2^3 - 3.0796y^2 + 0.726 = 0.\ \text{Solving by trial,}$$

$\mathbf{y_2 = 0.5302\ m}$, $V_2 = 7.07$ m, $Fr = 7.07/\sqrt{9.81 \times 0.5302} = 3.33$

Hence supercritical

To find the critical depth,

$$\mathbf{y_c} = \left(\frac{Q^2}{gb^2}\right)^{1/3} = \left(\frac{22.5^2}{9.81 \times 6^2}\right)^{1/3} = \mathbf{1.1275\ m}$$

$$E_{min} = (3/2)\, y_c = 1.691 \text{ m}, V_c = \frac{22.5}{6 \times 1.1275} = 3.32 \text{ m/s},$$

$$Fr = 3.32/\sqrt{9.81 \times 1.1275} = 1$$

Problem 12.25 *Water flows across a broad crested weir in a rectangular channel 0.4 m wide. The depth of water upstream is 0.07 m and the crest of the weir is 40 mm above the channel bed.* ***Determine the fall in the surface level*** *and the discharge over the weir. Assume negligible velocity of approach.*

As there is free fall over the weir, the flow will be maximum. The discharge is given by the equation (12.10.1),

$$Q = 1.705\ b\ H^{3/2},$$

where H the upstream level above the crest in the channel.

Given $b = 0.4$ m, $H = 0.03$ m

$\therefore$ $\mathbf{Q} = 1.705 \times 0.4 \times 0.03^{3/2} = \mathbf{3.54 \times 10^{-3}\ m^3/s}$

As the flow is maximum the level above the crest should equal critical depth.

$$\mathbf{y_c} = (Q^2/gb^2)^{1/3} = \left[\left(\frac{3.54 \times 10^{-3}}{0.4}\right)^2 \frac{1}{9.81}\right]^{1/3} = \mathbf{0.020\ m}$$

$\therefore$ drop in depth $= 0.03 - 0.02 = 0.01$ m or 10 mm

Problem 12.26 *A venturi flume with level bed is 12 m wide and the depth of flow upstream is 1.5 m. The throat is 6 m wide. In case a standing wave forms downstream,* ***calculate the rate of flow of water,*** *correcting for the velocity of approach. Assume* $C_d = 0.94$.

Refer section 12.11

For venturi flume with standing wave downstream $Q = C_d 1.705\ b_2\ E^{3/2}$

where b_2 is the throat area and E is the specific energy. In this case the first assumption is $E = y$ upstream.

$\therefore$ $\mathbf{Q} = 0.94 \times 1.705 \times 6 \times 1.5^{3/2} = \mathbf{17.666\ m^3/s}$

Velocity upstream $= 17.666/12 \times 1.5 = 0.981$ m/s

$\therefore$ $E = 1.5 + 0.9814^2/(2 \times 9.81) = 1.549$ m

Now using the corrected value of E

$$\mathbf{Q} = 0.94 \times 1.705 \times 6 \times 1.549^{3/2} = \mathbf{18.54\ m^3/s}$$

Further iteration can be made using this flow.

Problem 12.27 *A venturi flume is placed in a rectangular channel of 2 m width in which the throat width is 1.2 m. The upstream flow depth is 1 m and the depth at the throat is 0.9 m. Determine the flow rate. If a standing wave forms downstream of throat* ***determine the flow.*** *Assume the bed to be horizontal.*

In the first case the flow Q is given by equation (12.11.1)

$$Q = b_2 y_2 \left[\frac{2gh}{1-(b_2 y_2 / b_1 y_1)^2}\right]^{0.5}$$

where h is the difference in levels between upstream and throat

$$\mathbf{Q} = 1.2 \times 0.9 \left[\frac{2 \times 9.81 \times 0.1}{1-(1.2 \times 0.9 / 2 \times 1)^2}\right]^{0.5} = \mathbf{1.797\ m^3/s}$$

In case standing wave forms, considering upstream velocity. Specific energy

$E = 0.9 + (V^2/2g)$, As $V = 1.797/(1.2 \times 0.9) = 1.663$ m/s,

$E = 1.041$ m.

$Q = 1.705\, b_2\, E^{3/2} = 1.705 \times 1.2 \times 1.041^{1.5} = 2.17\ m^3/s$

In case upstream velocity is neglected, then

$Q = 1.705 \times 1.2 \times 1^{3/2} = 2.046\ m^3/s.$

Problem 12.28 *Water enters a channel at a velocity of 4 m/s. The Froude number is 1.4. Calculate the depth of flow after the jump. Also calculate the loss of specific energy.*

The depth of water after the jump is given by equation (12.9.7)

$$y_2 = \frac{y_1}{2}\left[-1+\sqrt{1+8Fr_1^2}\right] \qquad (A)$$

To determine, $Fr = V/\sqrt{gy}$, $1.4 = 4/\sqrt{9.81}\sqrt{y}$, Solving for y, $y = 0.8321$ m

Substituting in equation (A),

$$y_2 = \frac{0.8321}{2}\left[-1+\sqrt{1+8\times 1.4^2}\right]$$

$$= 1.2832 \text{ m}, V_2 = \frac{y_1 V_1}{y_2} = \frac{0.8321 \times 4}{1.2832} = 2.5939 \text{ m/s}$$

$$\textbf{Loss} = \left(y_1 + \frac{V_1^2}{2g}\right) - \left(y_2 + \frac{V_2^2}{2g}\right)$$

$$= \left(0.8321 + \frac{4^2}{2\times 9.81}\right) - \left(1.2832 + \frac{2.5939^2}{2\times 9.81}\right)$$

$= 1.6476 - 1.6261 =$ **0.021 m head.**

REVIEW QUESTIONS

1. Define open channel flow.
2. State Chezy, Kutter and Manning formula for uniform flow through an open channel.
3. Distinguish between uniform and non uniform open channel flow.
4. Derive the condition for the most economical section of a rectangular channel.
5. Derive the condition for the best side slope of the most economical trapezoidal channel.
6. Prove that for a channel of circular section for flow to be maximum, the depth of flow, $h = 0.95 \times$ diameter of circular section.
7. Explain the terms (*i*) specific energy (*ii*) critical depth and (*iii*) Critical velocity.
8. Derive an expression for critical depth and critical velocity.
9. Explain the term hydraulic jump. Derive an expression for the downstream depth of hydraulic jump.
10. Show that in a rectangular open channel the critical depth is two thirds of specific energy.
11. Derive an expression for wave celerity.

OBJECTIVE QUESTIONS

O Q. 12.1 Define the following terms

(1) Froude number (2) Wetted perimeter (3) Hydraulic radius (4) Wave celerity (5) Subcritical, critical and supercritical flow (6) Specific energy (7) Alternate depths (8) Normal depth (9) Energy grade line (10) Hydraulic grade line (11) Manning roughness coefficient (12) Optimum cross section of channels (13) Critical depth (14) Hydraulic jump (15) Venturi flume (16) Broad crested weir (17) Standing wave (18) Bed slope.

O Q. 12.2 Fill in the blanks

(1) Depth of flow is the vertical distance between the free liquid surface to the _____

(2) The channel lining (side and base of the channel) which comes in direct contact with the liquid stream is called _____

(3) The hydraulic mean depth represents the ratio of the flow area to the _____

(4) The slope of the total energy line is referred to as _____

(5) The flow is laminar in an open channel if the Reynolds number is less than _____

(6) The flow is certainly turbulent in an open channel if the Reynolds number is greater than _____

(7) The Chezy coefficient C is a variable with its value depending on the flow _____ number and the _____

(8) The phenomenon of sudden increase in depth of flow in a channel is referred to as _____

(9) The energy loss through a hydraulic jump equals the difference between the _____ at the upstream and downstream sections.

(10) A strong hydraulic jump causes about _____ of energy dissipation.

Answers

(1) Lowest point of the channel section (2) Wetted perimeter (3) Wetted perimeter (4) Energy gradient (5) 500 (6) 2000 (7) Reynolds, Surface roughness (8) Hydraulic jump (9) Specific energies (10) 95%

O Q. 12.3 Fill in the blanks :

1. For a supercritical flow Froude number should be _____
2. For a given slope and flow rate in a channel, the depth is called _____
3. When there is a change in slope in supercritical flow _____ will form.
4. Across a hydraulic jump specific energy will _____
5. For a given specific energy as flow depth increases Froude number will _____
6. For a given specific energy if one flow depth is supercritical the other will be _____
7. The flow depth in subcritical flow will be _____ compared to the flow depth a critical flow.
8. For a given specific energy the two possible depths of flow are called _____
9. As roughness increases Mannings coefficient will _____
10. For a given area _____ section gives the maximum flow.
11. The flow depth _____ across hydraulic jump.

Answers

1. Higher than one 2. Normal depth 3. Hydraulic jump 4. Decrease 5. Decrease 6. Subcritical 7. Higher 8. Alternate depths 9. Increase 10. Semicircular 11. Increases

O Q. 12.4 State Correct or Incorrect

1. Hydraulic radius is the ratio of wetted perimeter to area or (P/A).
2. Hydraulic depth is the average depth (Area/Topwidth).
3. Chezy's constant is a dimensional constant.
4. Bazin formula does not relate chezy coefficient C to bed slope S_b.
5. The dimension of Chezy constant is $\sqrt{L} / T$.
6. Kutters formula considers bed slope.
7. Manning proposed that C varied as $R_h^{1/3}$.
8. Flow rate through open channels is inversely proportional to square root of bed slope ($S_b^{1/2}$).
9. The slope is unique for a given flow rate and depth of flow in a given rectangular channel.
10. As Mannings constant increases the flow will increases.
11. As roughness increases, Mannings constant will decrease.
12. Specific energy is the sum of kinetic head and flow depth.
13. As depth increases the wave velocity decreases.

Answers

Correct : 2, 3, 4, 5, 6, 9, 12 **Incorrect :** 1, 7, 8, 10, 11, 13

O Q. 12.5 State Correct or Incorrect :

1. At critical depth of flow Froude number should be equal to one.
2. If one depth of flow for a given specific energy is at supercritical condition the alternate depth also should be at supercritical condition.
3. Disturbance in supercritical flow will not be communicated upstream.
4. Across the hydraulic jump, specific energy remains constant.
5. At critical flow, the specific energy is minimum.
6. Hydraulic jump helps to dissipate energy without damage to surfaces/structures.
7. When supercritical flow meets a bump in the bed, the level at that point will decrease.
8. Hydraulic gradient line represents the depth of flow.

9. Energy gradient line represents specific energy.
10. As velocity drops hydraulic gradient line will rise.
11. The distance between energy gradient line and hydraulic line represents velocity head.
12. For optimum area in a rectangular channel, the depth should be twice the width.

Answers

Correct 1, 3, 5, 6, 8, 9, 10, 11 **Incorrect :** 2, 4, 7, 12

O Q. 12.6 State Correct or Incorrect

1. The included angle at optimum flow in a triangular channel is 45°.
2. In the case of circular section, the flow height for maximum flow is the same as per maximum velocity.
3. Optimum circular section for a given flow is a semicircle.
4. For a given area, maximum value for hydraulic mean depth will be for semi circular shape.
5. For maximum flow through a sluice gate the downstream depth should be 2/3 of upstream depth.
6. In subcritical flow level will increase for negative bump.
7. At minimum specific energy condition for flow through a rectangular channel, the kinetic head will be 0.5 times the flow depth.

Answers

Correct : 3, 4, 5, 7 **Incorrect :** 1, 2, 6

EXERCISE PROBLEMS

E12.1 A rectangular open channel has 5 m width and 1.5 m depth. The bed slope is 1:1000. Assuming Chezy constant C = 50, determine the flow rate. **(11.48 m^3/s)**

E12.2 A triangular open channel with 0.25 m depth and 60° angle conveys water. If the bed slope is 1:137 and Chezy constant C = 52, determine the flow rate. **(40 liters/s)**

E12.3 A semicircular open channel of diameter 1 m conveys water at the rate of 1.83 m^3/s. If the slope of the bed is 1:950, fine the Chezy constant.

E12.4 Determine the bed slope of a circular pipe that should carry 2.47 m^3/s of water at half full condition. Pipe diameter is 256 cm. Assume Chezy constant as 60. **(1:2500)**

E12.5 A rectangular channel with most economical cross-section carries 8000 l/s of water with an average velocity of 2 m/s. If Chezy constant is 65, determine it's cross-section and slope. **(2.8 × 1.4 m, 1:746)**

E12.6 The area of cross-section of a trapezoidal channel is 30 m^2. Find the base width and flow depth for most economical design if the slope of the bed is 1 in 1500. Side slope is 1 vertical 2 horizontal. **(h = 4.16 m, w = 5.138 m)**

E12.7 Determine the dimensions of a trapezoidal section for a discharge of 40 m^3/s with a bed slope of 1:2500 and Manning's constant N = 1/50. **(h = 3.86 m, w = 4.459 m)**

E12.8 Water flows at the rate of 5 m^3/s in a rectangular open channel of 3 m width. Assuming Manning's constant N = 1/50, calculate the slope required to maintain a depth of 2 m. **(1:736)**

E12.9 A rectangular channel of 5 m width carries water at the rate of 15 m^3/s. Calculate the critical depth and velocity. **(h_c = 0.972 m, V_c = 3.69 m/s)**

E12.10 Water flows through a rectangular open channel at the rate of 2 m^3/s. If the width of the channel is 2 m, what would be the critical depth of the channel ? If a standing wave is to be formed at a point where the upstream depth is 0.2 m , what would be the rise in water level.

(h_c = 0.467 m, 0.715 m)

E12.11 In a pensive mood a boy throws a stone in a mountain stream 1.3 m deep. It is observed that the waves created do not travel upstream. Calculate the minimum velocity of the stream.

(3.571 m/s)

E12.12 Water flows in a wide rectangular channel with a flow rate of 2.5 m^3/s per m width. If the specific energy is 2.2 m, determine the two depths of flow possible. **(2.12 m, 0.426 m)**

E12.13 Water flows over a smooth bump in a wide rectangular channel. The height of the bump at any location is $h(x)$. Neglecting energy losses, **show that the slope of the water surface $dy/dx = -(dh/dx)/[1 - (u^2/gy)]$** where u is the velocity and y is the depth at location x where the height of the bump is h.

E12.14 The measurement of the parameters of a small stream shows that A = 26 m^2, P = 16 m and S_b = 1/3100. Determine the average shear stress on the wetted perimeter of the channel.

(5.14 N/m^2)

E12.15 Determine the percentage reduction in flow in a rectangular channel if a thin partition in the middle divides it into two equal widths along the flow. **(23.71)**

E12.16 A trapezoidal channel with side slopes of 45° and bottom width of 8 m is to carry a flow of 20 m^3/s. The slope is 1/1796. Considering Mannings coefficient as 0.03, determine the width at the water line. **(12 m)**

E12.17 Compare the perimeter length for a (*i*) 90° triangle and (*ii*) square section to carry water at 2 m^3/s with a slope of 1/80.

E12.18 Water flows in a rectangular channel of width b and depth $b/3$. Considering same slope and Manning coefficient, determine the diameter of a circular channel that will carry the same flow when (*i*) half full (*ii*) Maximum flow condition and (*iii*) Maximum velocity condition.

E12.19 Determine the percentage reduction flow in an equilateral triangular section, flowing full if the top is closed, with water wetting the surface. **(23.7%)**

E12.20 Determine the critical depth in a rectangular channel 10 m wide when the flow rate is 200 m^3/s. **(3.44 m)**

E12.21 Show that for a hydraulic jump in a rectangular channel, the Froude numbers upstream and downstream are related by

$$F_{r2}^2 = \frac{8F_{r1}^2}{[(1 + 8F_{r1}^2)^{1/2} - 1]^3} .$$

E12.22 In a hydraulic jump in a spillway the upstream and downstream depths are 0.7 m and 3.6 m, determine the flow rate per m width of the spillway. Also calculate the Froude numbers.

(7.29 m^3/s, 3.97, 0.34)

E12.23 Water flows over a broad crested weir with a height of 1.5 m above the bed. If the free surface well upstream is at 0.5 m above the weir surface. Determine the flow rate. Also determine the minimum depth above the weir. **(0.3395 m^3/s, 0.333 m)**

E12.24 In a horizontal rectangular channel there is a small bump of height 30 mm on the bed. At upstream the depth is 0.3 m and velocity is 0.5 m/s. Determine the height of flow above the bump and the flow speed at this section. **(0.267 m, 0.563 m/s)**

E12.25 Water flows under a sluice gate. Upstream the depth is 1.5 m and the velocity is 0.2 m/s. Assuming that specific energy remains constant determine the depth downstream, if the flow rate is 0.3 m^3/s/m. Also calculate the maximum flow rate. **(0.0563 m, 3.14 m^3/s/m)**

E12.26 Compare the flow rates of square and semicircular channels of 2 m water surface, when slope is 1/1000 and Manning coefficient N = 0.015. **(6.44, 2.09 m^3/s)**

E12.27 At the exit flow under a sluice gate the depth of flow was 56.3 mm and the velocity was 5.33 m/s. Calculate the downstream depth and head loss across the jump. **(0.543 m, 0.942 m)**

E12.28 In a venturi flume the bed is horizontal. The width reduces from 600 mm to 300 mm at the throat while the flow depth changes from 300 mm to 225 mm at the throat. Determine the flow rate. **(0.0883 m^3/s)**

E12.29 In a trapezoidal channel of 2.4 m bottom width and 45° side slope the flow rate is 7.1 m^3/s. with normal depth of flow of 1.2 m. Determine the bed slope. N = 0.022 **(1.93/1000)**

E12.30 In the above problem with the same conditions as mentioned if the flow rate is increases to 15 m^3/s, determine the normal depth. **(1.79 m)**

E12.31 A hydraulic jump occurs on a horizontal apron downstream from a spillway at a location where the depth is 0.9 m and speed is 25 m/s. Estimate the depth and velocity downstream. Also calculate the percentage loss of head. **(10.3 m, 2.18 m/s, 67.9%)**

13 Dynamics of Fluid Flow

13.0 INTRODUCTION

In chapter 3 the forces exerted by static fluid on the containment surfaces was discussed. In this chapter the forces exerted by fluid particles on the surfaces over which they flow, is discussed. In case the surfaces cause a change in the magnitude and direction of the velocity of the fluid particles, the fluid particles exert a force on the surface. In turn the surfaces exert an equal and opposite force on the fluid particles. **The force exerted by moving fluid particles on the surface is called dynamic force.** Dynamic force always involves a change in the magnitude and direction of the velocity of the fluid. Forces due to viscous resistance is excluded in the discussions in this chapter to reduce complexity in the analysis.

13.1 IMPULSE MOMENTUM PRINCIPLE

When applied to a single body Newtons second law can be started as "The sum of forces on the body equals the rate of change of momentum of the body in the direction of the force. In equation from (F and V are in the same direction)

$$\Sigma F = \frac{d\,(mV)}{dt} \tag{13.1.1}$$

This can also be written as

$$\Sigma F\,dt = d\,(mV) \tag{13.1.1 a}$$

where m is the mass of the body and V is the velocity of the body and t is the time. This also means the impulse **Fdt equals the change in momentum of the body during the time dt.**

When applied to control volume, through which the fluid is flowing, the principle can be stated as "The sum of forces on the fluid equals the difference between the momentum flowing in and momentum flowing out and the change in momentum of the fluid inside the control volume under steady flow condition the last term vanishes. So the forces in the fluid is given by

$$\Sigma F = \frac{d\,(mV)_{out}}{dt} - \frac{d\,(mV)_{in}}{dt} \tag{13.1.2}$$

In other words, the net force on the fluid mass is equal to the net rate of out flow of momentum across the control surface.

This can also be written as

$$\Sigma F = \rho_2\, Q_2 V_2 - \rho_1 Q_1 V_1 \tag{13.1.3}$$

If the fluid is in compressible, then

$$\Sigma F = \rho Q\,(\Delta V) \tag{13.1.3a}$$

In this case **ΔV should be taken as the vectorial addition of V_1 and V_2 and the force will be in the direction of the resultant of V_1 and V_2 .**

In case the forces in the cartesian co-ordinate directions is required, the equation in scalar form is written as

$$\Sigma F_x = \rho Q\, \Delta u \tag{13.1.4}$$

$$\Sigma F_y = \rho Q\, \Delta v \tag{13.1.4a}$$

$$\Sigma F_z = \rho Q\, \Delta w \tag{13.1.4b}$$

where u, v and w are the components of velocity in the x, y and z directions.

When calculating the momentum flowing in or out, if the velocity over the section is not uniform a correction has to applied. The correction factor β is given by

$$\beta = \frac{1}{AV^2} \int_A u^2 dA \tag{13.1.5}$$

where V is the average velocity.

13.1.1 Forces Exerted on Pressure Conduits

Consider the reducer section shown in Figure 13.1.1 (a). The free body diagram is given in Figure 13.1.1 (b) :

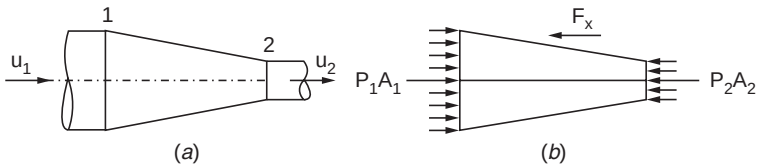


Figure 13.1.1

Assuming ideal fluid flow,

$$\Sigma F_x = P_1A_1 - P_2A_2 - F_x = \rho Q\,(u_2 - u_1) \tag{13.1.6}$$

or the force on the fluid is given by

$$F_x = P_1A_1 - P_2A_2 - \rho Q\,(u_2 - u_1) \tag{13.1.6a}$$

This force is the force exerted by the reducer on the fluid in the x direction. This force acts towards the left as assumed in the figure. The negative sign in the LHS of 13.1.6 is due to this assumption. The numerical value will show the actual sign. The force exerted by the fluid

on the reducer will be equal and opposite to this force F_x. The plus and minus signs used in the equations depend in the + ve or – ve directions of the co-ordinate system along which the force is assumed to act.

In case both the magnitude and direction of the velocity is changed by a reducer bend then the force exerted by the bend on the fluid, the turning angle being θ, (suffix *F* indicates as on the fluid)

$$F_{xF} = P_1A_1 - P_2A_2 \cos\theta - \dot{m}\ (V_2 \cos\theta - V_1) \tag{13.1.7}$$

$$F_{yF} = P_2A_2\ \sin\theta + \dot{m}\, V_2 \sin\theta \tag{13.1.8}$$

The forces on the bend will be equal and opposite to these forces.

Example 13.1 *A reducer in the horizontal plane has an inlet area of 0.02 m^2 and the outlet area is 0.01 m^2. The velocity at the inlet is 4 m/s. The pressures are 40 kPa at inlet and 10 kPa of outlet.* ***Determine the force exerted by the reducer on the fluid.***

As the flow is in the horizontal plane body forces are neglected. Refer Figure 13.1.1

As $A_1/A_2 = 2, \quad V_2 = 2V_1 = 8$ m/s

Using equation (13.1.6*a*)

$$F_x = P_1A_1 - P_2A_2 - m\,(u_2 - u_1)$$

$$m = 4 \times 0.01 \times 1000 = 40 \text{ kg/s}$$

$$F_x = 40 \times 10^3 \times 0.02 - 10 \times 10^3 \times 0.01 - 40\,(8 - 4)$$

$$= \textbf{540 N on the fluid towards left.}$$

On the reducer 540 N along positive *x* direction.

Example 13.2 *A 45° bend in the horizontal plane is shown in figure. The inlet area is 1.2 m^2 and the outlet area is 0.6 m^2. The velocity of water at inlet is 12 m/s. The pressures at inlet and cutlet are 40 and 30 kPa respectively.* ***Calculate the*** *magnitude and direction of the* ***resultant force*** *on the bend.*

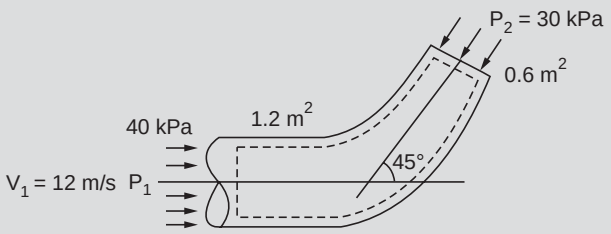


Figure Ex. 13.2

For convenience the control volume should be chosen such that the inlet and outlet areas are normal to the velocities at these sections. In this case the force on the bend is required. It is convenient to calculate the forces in the *x* and *y* directions separately.

$$u_1 = 12 \text{ m/s} \quad \therefore \quad u_2 = 24 \text{ m/s}$$

Mass flow $= 12 \times 1.2 \times 1000 = 14.4 \times 10^3$ kg/s

Using equations (13.1.7) and (13.1.8)

$$F_x = P_1A_1 - P_2A_2 \cos\theta - \dot{m}\ (V_2 \cos\theta - V_1)$$

$$F_y = P_2A_2\ \sin\theta + \dot{m}\ V_2 \sin\theta$$

Substituting the values, Force m the fluid is

$$\mathbf{F_x} = 40 \times 10^3 \times 1.2 - 30 \times 10^3 \times 0.6 \times \cos 45 - 14.4 \times 10^3 (24 \cos 45 - 12)$$

$$= \mathbf{-36.3\ kN.}\quad \textbf{in the –ve } \boldsymbol{x} \text{ direction}$$

$$\mathbf{F_y} = 30 \times 10^3 \times 0.6 \times \sin 45 + 14.4 \times 10^3 \times 24 \times \sin 25$$

$$= \mathbf{257.1\ kN}\quad \textbf{in the +ve y direction.}$$

The forces on the bend will be 36.3 kN along x and 257.1 kN downwards.

The resultant is $\sqrt{257.1^2 + (-36.3)^2}$ = **259.65 N.**

The direction is $\theta = \tan^{-1} \dfrac{36.3}{257.1} = \mathbf{8.04°}$

with the negative y direction

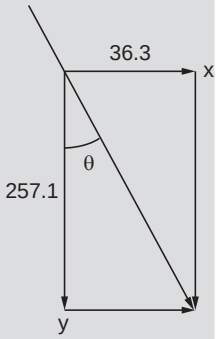


Figure Ex. 13.2

13.1.2 FORCE EXERTED ON A STATIONARY VANE OR BLADE

In the case of turbomachines fluid passes over blades and in this context, the force on a vane due to the fluid flowing over it is discussed. In turbomachines the blades are in motion. To start the analysis force on stationary vane is considered. Here the direction of the velocity is changed. There is negligible change in the magnitude. In the case considered pressure forces are equal both at inlet and outlet. The flow is assumed to occur in the horizontal plane.

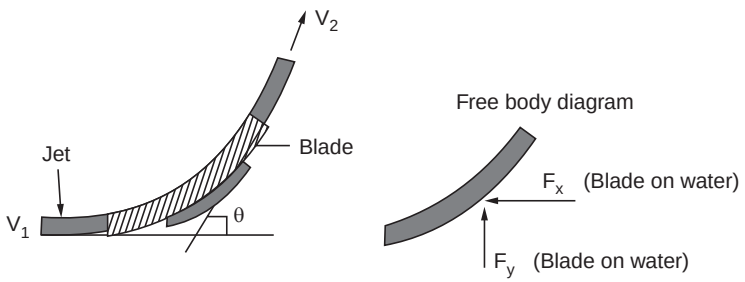


Figure 13.1.2

Force along x direction by the blade on fluid, with the assumed direction : Assuming $V_2 = V_1$ as no other energy transfer occurs,

$$-F_x = \dot{m}\ (V_{2u} - V_1) = \dot{m}\ (V_2 \cos\theta - V_1) = \dot{m}\ (V_1 \cos\theta - V_1) \qquad (13.1.8)$$

$$F_y = \dot{m}\ (V_{2y} - V_{1y}) = \dot{m}\ V_1 \sin\theta \qquad (13.1.9)$$

Example 13.3 *A blade turns the jet of diameter 3 cm at a velocity of 20 m/s by 60°.* ***Determine the force exerted by the blade*** *on the fluid.*

Rate of flow $\dot{m} = \dfrac{\pi \times 0.03^2}{4} \times 20 \times 1000 = 14.14$ kg/s

$$-F_x = 14.14\ (20 \cos 60 - 20) = -141.4\ \text{N}$$

or $\mathbf{F_x = 141.4\ N.}$ in the assumed direction

$\mathbf{F_y} = 14.14\ (20 \times \sin 60) = \mathbf{244.9\ N}$

The forces on the blade are 141.4 N along x direction and 244.9 in the –ve y direction.

$$\textbf{Resultant} = (244.9^2 + 141.4^2)^{0.5} = \mathbf{282.8\ N}$$

$$\boldsymbol{\theta} = \tan^{-1}\frac{141.4}{244.9} = \mathbf{30°}$$

30° with the negative y direction as in figure.

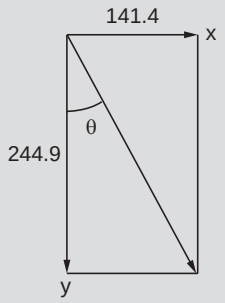


Figure Ex. 13.3

13.2 ABSOLUTE AND RELATIVE VELOCITY RELATIONS

In order to determine the force on moving blades and the energy transfer between the blades and the fluid the relative velocity between the fluid and the blade becomes an important factor. The blade may move in a direction at an angle to the velocity of the fluid. **The relative velocity of a body is its velocity relative to a second body which may in turn be in motion relative to the earth.**

The absolute velocity V of the first body, is the vector sum of its velocity relative to the second body v, and the absolute velocity of the latter, u

Vectorially $\mathbf{V = u + v}$

This is easily determined by vector diagram called as velocity triangle. Some possible diagrams are shown in Figure 13.2.1.

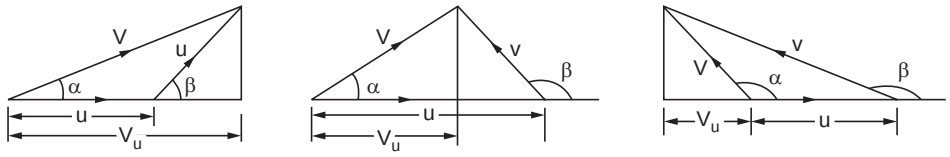


Figure 13.2.1 Sample Velocity diagrams

Some of the general relations are

$$V \sin \alpha = v \sin \beta \qquad (13.2.1)$$

$$V_u = V \cos \alpha = u + v \cos \beta \qquad (13.2.2)$$

V_u is the component of the absolute velocity of the first body in the direction of the velocity u of the second body.

13.3 FORCE ON A MOVING VANE OR BLADE

The force on a single moving vane is rarely met with. But this forms the basis for the calculation of force and torque on a series of moving vanes fixed on a rotor. There are two main differences between the action of the fluid on a stationary vane and a moving vane in the direction of the

fluid motion. In the case of the moving vane it is necessary to consider both the absolute and relative velocities. The other difference is that the amount of fluid that strikes a moving vane at any time interval is different from that which strikes the stationary vane. If a jet of area A with a velocity V_1 strikes a stationary vane, the mass impinging per unit time on the vane equals $8AV_1$ kg/s. But when the vane moves away from the direction of the jet with a velocity u, then the mass of water striking the vane equals $\rho A(V_1 - u)$ kg/s. $(V - u)$ is the relative velocity between the jet and the vane. This can be realised when the consider the velocity of the vane to be equal to that of the jet. In this case no water will strike the vane, obviously. Consider the flow as shown in figure 13.3.1.

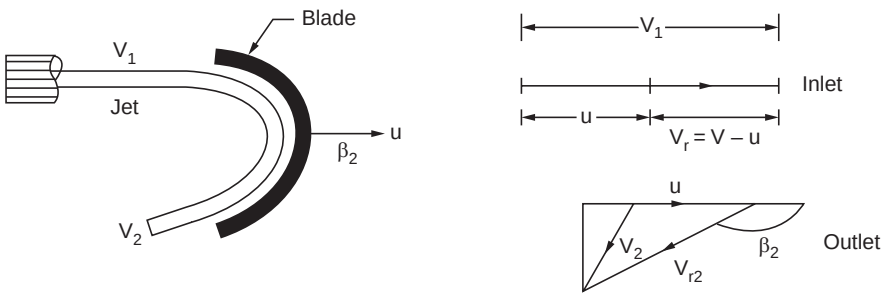


Figure 13.3.1

The velocity diagram with as inlet and outlet are shown in the figure. Considering the force on the fluid in the direction of blade velocity (can be considered as x direction)

$$F_u = \rho A\,(V_1 - u)\,(V_{u_2} - V_{u_1}) \tag{13.3.1}$$

$$V_{u2} = (V_{r_2}\cos\beta_2 - u)\text{, denoting } V_r \text{ as relative velocity}$$

$$\therefore \qquad F_u = \rho A\,(V_1 - u)\,(V_{r_2}\cos\beta_2 - u - V_{u_1}) \tag{13.3.2}$$

In the case shown, $V_{u_1} = V$ itself

It is possible that $(V_{r_2}\cos\beta_2 - u)$ or V_{u_2} is negative depending upon the relative values of u and V_r *i.e.* $u > V_{r_2}\cos\beta_2$.

It is to be noted that the vane angle at the inlet should be in the direction of the relative velocity of the water when it touches the vane. Otherwise loss will occur due to the jet hitting the vane at an angle and then turning the follow on the vane surface.

It was assumed that the relative velocity at inlet and at outlet are equal as no work was done by the vane on the fluid. In case of friction, $V_{r_2} = cV_{r_1}$ where c is a fraction. In case the vane moves at a direction different from that of the jet velocity say at an angle α, then force on the fluid on the vane will be at an angle.

In such a case, $$F_x = \rho A\,(V_1\cos\alpha_{1i} - V_{r_2}\cos\beta_2 - u)\,(V_1\cos\alpha_1 - u)$$

$$= \rho A\,(V_1\cos\alpha_1 - V_2\cos\alpha_2)$$

As it was already mentioned, a single moving vane is not of practical importance when a series of vanes fixed on the periphery of a well is struck by the jet, then the mass of fluid striking the when will be ρAV itself.

Work or energy transfer between the fluid and the water will be $F \times u$.

Example 13.5 *A 4 cm diameter water jet with a velocity of 35 m/s impinges on a single vane moving in the same direction at a velocity of 20 m/s. The jet enters the vane tangentially along the x direction. The vane deflects the jet by 150°.* ***Calculate the force exerted by the water on the vane.***

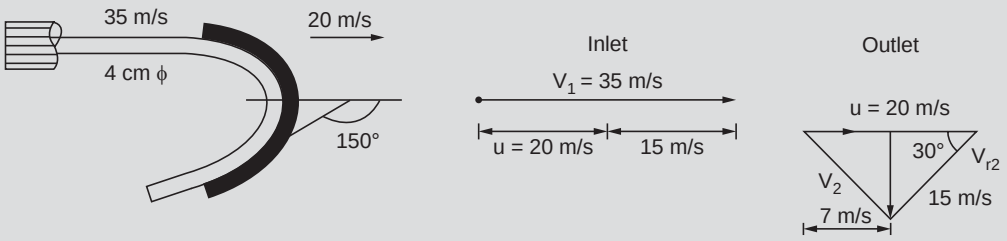


Figure Ex. 13.5

The relative velocity is given by

$$V_r = 35 - 20 = 15 \text{ m/s.}$$

V_{u_1} = 30 m/s itself in the direction of blade velocity.

From exit velocity triangle

$$V_{u_2} = u - V_{r_2} \cos 30 = 20 - 15 \cos 30 = 7 \text{ m/s}$$

This is in the same direction as V_{u_1}

$\therefore$ $\Delta V_u = 35 - 7 = 28$ m/s.

$$\mathbf{F_u} = \frac{1000 \times 0.04^2 \times \pi}{4} \times (35 - 20)\,(28) = \mathbf{527.8\ N.}$$

Energy transfer rate $= \mathbf{F \times u} = 527.8 \times 20 = \mathbf{10556\ Nm/s\ or\ W.}$

$$\mathbf{F_y} = \frac{1000 \times 0.04^2 \times \pi}{4} \times (35 - 20) \times (15 \sin 30 - 0) = \mathbf{141.37\ N}$$

(Note $V_2 \sin \alpha_2 = V_{r_2} \sin \beta_2$).

In case series of vanes have been used,

$$\mathbf{F_x} = \frac{1000 \times 0.04^2 \times \pi}{4} \times 35 \times 28 = \mathbf{1231.5\ N}$$

Energy transfer $= 1231.5 \times 20 = \mathbf{24630\ W}$

In case there is friction for the flow over the blade, $V_{r_2} = k V_{r_1}$

In case the water jet direction and blade velocity direction are at an angle α_1,then at the inlet $V_u \neq V_1$ but will be $V_u = V_1 \cos \alpha_1$. This is illustated by the following example.

Example 13.6 *A water jet 20 mm in diameter and having a velocity of 90 m/s strikes series of moving blades in a wheel. The direction of the jet makes 20° with the direction of movement of the blade. The blade angle at inlet is 35°. If the jet should enter the blade without striking, what should be the blade velocity. If the outlet angle of the blade is 30°,* ***determine the force on the blade.*** *Assume that there is no friction involved in the flow over the blade.*

This problem has to be solved using the velocity diagram.

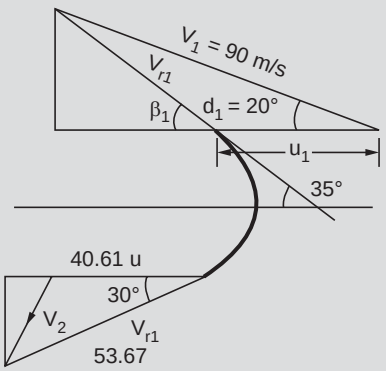


Figure Ex. 13.6

$$u = V_1 \cos \alpha_1 - V_{r_1} \cos \beta_1$$

$$V_1 \sin \alpha_1 = V_{r_1} \sin \beta_1$$

$$\therefore \quad V_{r_1} = \frac{V_1 \sin \alpha_1}{\sin \beta_1} = \frac{90 \times \sin 20}{\sin 35} = 53.67 \text{ m/s}$$

$$\therefore \quad u = 90 \cos 20 - 53.67 \cos 35 = 40.61 \text{ m/s}$$

$$V_{u_1} = V_1 \cos \alpha_1 = 90 \times \cos 20 = 84.57 \text{ m/s}$$

$$V_{u_2} = V_{r_2} \cos \beta_2 - u = 53.67 \times \cos 30 - 40.61$$

$$= 5.87 \text{ m/s (opposite direction to } V_{u_1})$$

$$\therefore \quad \Delta V_u = 84.57 + 5.87 = 90.44 \text{ m/s}$$

Series of blades : Mass flow $= \frac{\pi \times 0.02^2}{4} \times 1000 \times 90 = 28.274$ kg/s

$\therefore$ **Force F_x** = 28.274 × 90.44 = **2557 N**

Energy transfer rate = 2557 × 40.61 = **103845 Nm/s or W**

Energy in the jet $= \frac{mV_1^2}{2} = \frac{28.274 \times 90^2}{2} =$ **1145097 W**

$$F_y = (90 \sin 20 - 53.67 \sin 30)\, 28.274 = \mathbf{111.6\ N}$$

13.4 TORQUE ON ROTATING WHEEL

Blades or vanes **may be fixed at the periphery** of the wheel in which case the radius at which fluid enters will be the same as at fluid exit. There are cases where the blades are **fixed at the sides of a disc** such that the radius at which the fluid enters the vane will be different from the radius at which it exits. The former type is known **axial blading** and the later is known **as radial blading.** In the former case the blade velocity will be constant and in the latter case the blade velocity will very with radius. Thus the force on the blade will very with the radius and the previous method cannot be used to find the fluid force on the blade. In this case the **moment of momentum theorem is used to determine the torque on the wheel.** The theorem states that **torque on the wheel equals the rate of change of moment of momentum of the fluid as it flows over the blades.** Thus it is necessary to determine the moment of momentum at the inlet and outlet to determine the torque. Torque can be produced only by the velocity component along the periphery.

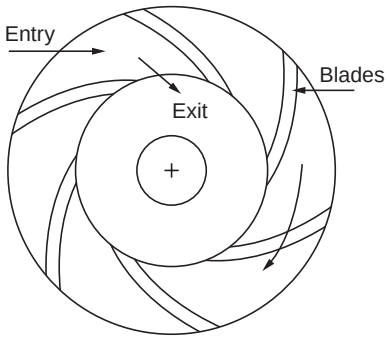


Figure 13.4.1 Radial blading

The components of the velocity in the tangential direction are V_{u_1} and V_{u_2} equal to $V_1 \cos \alpha_1$ and $V_2 \cos \alpha_2$. Momentum at entry = $\dot{m} V_1 \cos \alpha_1$. Moment of momentum of entry = $\dot{m} V_1 \cos \alpha_1 \times r_1$

Similarly moment of momentum at exit = $\dot{m} V_2 \cos \alpha_2 \times r_2$

$$T = \dot{m} \ (r_1 V_1 \cos \alpha_1 - r_2 V_2 \cos \alpha_2)$$

$$\text{Power} = \omega T. \quad \omega = \frac{2\pi N}{60}$$

Substituting : $\dfrac{2\pi r_1 N}{60} = u_1$ tangential velocity at entry

$$\frac{2\pi r_2 N}{60} = u_2 \text{ tangential at exit}$$

$$\therefore \qquad P = \dot{m} (V_{u_1} u_1 - V_{u_2} u_2).$$

where V_{u_1} and V_{u_2} are the components of the absolute velocities of the fluid in the tangential direction. In this case the direction of blade velocity is the tangential direction to the wheel on which the blades are fixed.

Example 13.7 *Blades are fixed in a disc with outer and inner diameters of 0.8 m and 0.4 m. The disc rotates at 390 rpm. The flow rate through blades is 4000 kg/s. The inlet angle of the blade is 80°. The blade width is 0.25 m.*

If the flow at outlet is radial, determine ***the blade outlet angle.*** *Determine the angle at which the water should flow for smooth entry. Determine* ***the torque exerted*** *and the* ***power resulting therefrom.***

$$u_1 = \frac{\pi DN}{60} = \frac{\pi \times 0.8 \times 390}{60} = 16.34 \text{ m/s}$$

$$\therefore \quad u_2 = u_1 \times \frac{D_2}{D_1} = \frac{16.34 \times 0.4}{0.8} = 8.17 \text{ m/s}$$

From continuity $Q = \pi D b V_f$

where V_f is the flow velocity along the radius.

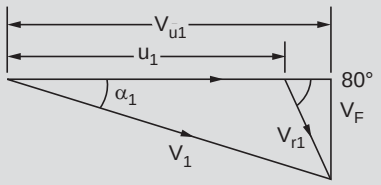


Figure Ex. 13.7a

$$\frac{4000}{1000} = \pi \times 0.8 \times 0.25 \times V_f$$

$$\therefore \quad V_f = 6.37 \text{ m/s.}$$

$$V_{u_1} = u_1 + V_f/\tan 80$$

$$= 16.34 + \frac{6.37}{\tan 80} = 16.52 \text{ m/s}$$

$$\frac{V_{f_1}}{V_{u_1}} = \tan \alpha_1, \quad \frac{6.37}{16.52} = \tan \alpha_1, \quad \alpha_1 = \mathbf{21.09°}$$

The jet should be inclined at this angle to the periphery of the wheel

$$\therefore \quad V_1 \tan \alpha_1 = V_{f_1} \quad \therefore \quad \mathbf{V_1} = \frac{6.37}{\tan 21.09} = \mathbf{16.51 \text{ m/s.}}$$

As the blade width is constant, the flow velocity at exit is

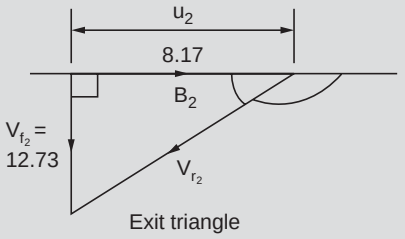


Figure Ex. 13.7b

$$4 = \pi \times 0.4 \times 0.25 \times V_{f_2}$$

$$\therefore \quad V_{f_2} = 12.73 \text{ m/s}$$

$$\therefore \quad \tan \beta_2 = \frac{12.13}{8.17}$$

$$\therefore \quad \beta_2 = \mathbf{57.3°}$$

As exit is radial, $V_{w_2} = 0$ as $V_{f_2} = V_2$

$$\mathbf{T} = \dot{m}\ (r_1 V_{w_1} - 0) = 4000 \times \frac{0.8}{2} \times 16.52 = \mathbf{26432 \text{ mN.}}$$

$$\mathbf{P} = \omega T = \frac{2\pi N}{60} \times \frac{26432}{1000}$$

$$= \mathbf{1079.5 \text{ kW.}} \text{ Also equal to } \dot{m}\ V_{w_1} V_{u_1} \text{ (check)}$$

We can also determine V_{r_1} and V_{r_2} if required.

SOLVED PROBLEMS

Problem 13.1 *A pipe line of 150 mm ID branches into two pipes which delivers the water at atmospheric pressure. The diameter of the branch 1 which is at 30° anti clockwise to the pipe axis is 75 mm. and the velocity at outlet is 12 m/s. The branch 2 is at 15° with the pipe centre line in the clockwise direction has a diameter of 100 mm. The outlet velocity is 12 m/s. The pipes lie in a horizental plane.* ***Determine the magnitude and direction of the forces on the pipes.***

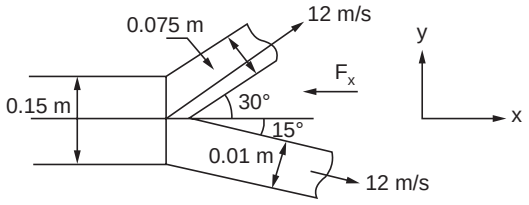


Figure P. 13.1

The flow rates in the pipes are

Branch 1 : $\dfrac{\pi \times 0.075^2}{4} \times 12 \times 1000 = 53$ kg/s

Branch 2 = $\dfrac{\pi \times 0.1^2 \times 12 \times 1000}{4} = 94$ kg/s

Flow in the pipe = 94 + 53 = 147 kg/s

Velocity in the pipe : $V_1 = 0.147 / \dfrac{\pi\, 0.15^2}{4} = 8.333$ m/s

To determine the pressure in the pipe

$$\frac{P_1}{p} = \frac{P_0}{S} + \frac{V_2^2}{2} - \frac{V_1^2}{2},$$

Assuming zero gauge pressure at exit.

$$P_1 = \rho \left[\frac{V_2^2 - V_1^2}{2}\right] = 1000 \left[\frac{12^2 - 8.33^2}{2}\right]$$

$$= 37.3 \times 10^3 \text{ N/m}^2$$

The x directional force assuming it to act in the – ve x direction

$$\mathbf{F_x} = 37.3 \times 10^3 \times \frac{\pi \times 0.15^2}{4} - 94 \times 12 \cos 15 - 53 \times 12 \cos 30 + 147 \times 8.333$$

$$= \mathbf{242\ N}$$

To determine F_y : Assuming to act in the + ve y direction

$$\mathbf{F_y} = 53 \times 12 \sin 30 - 94 \times 12 \sin 15$$
$$= \mathbf{26\ N}$$

Problem 13.2 *A jet 30 mm diameter with velocity of 10 m/s strikes a vertical plate in the normal direction.* ***Determine the force on the plate if*** *(i) The plate is stationary (ii) If it moves with a velocity of 4 m/s towards the jet and (iii) If the plate moves away from the plate at a velocity of 4 m/s.*

Case (*i*) The total x directional velocity is lost.

$$\therefore \qquad F = \dot{m}\,\mathrm{V}, \quad \dot{m} = \rho AV$$

$$\therefore \qquad \mathbf{F} = \frac{\pi \times 0.03^2}{4} \times 10 \times 10 \times 1000 = \mathbf{70.7\ N}$$

Case (*ii*) $\qquad \dot{m} = \rho A(V_r), \quad V_r = V + u = 14$

$$\therefore \qquad \mathbf{F} = \frac{\pi \times 0.032}{4} \times 14 \times 1000 \times 10 = \mathbf{99\ N}$$

Case (*iii*) $\qquad F\,V_r = V - u = 6$ m/s

$$\mathbf{F} = \frac{\pi \times 0.03^2}{4} \times 6 \times 1000 \times 10 = \mathbf{42.4\ N}$$

Problem 13.3 *A jet of water at a velocity of 100 m/s strikes a series of moving vanes fixed at the periphery of a wheel, 5 at the rate of kg/s.*

The jet is inclined at 20° to the direction of motion of the vane. The blade speed is 50 m/s. The water leaves the blades at an angle of 130° to the direction of motion.

Calculate the blade angles at the forces on the wheel in the axial and tangential direction.

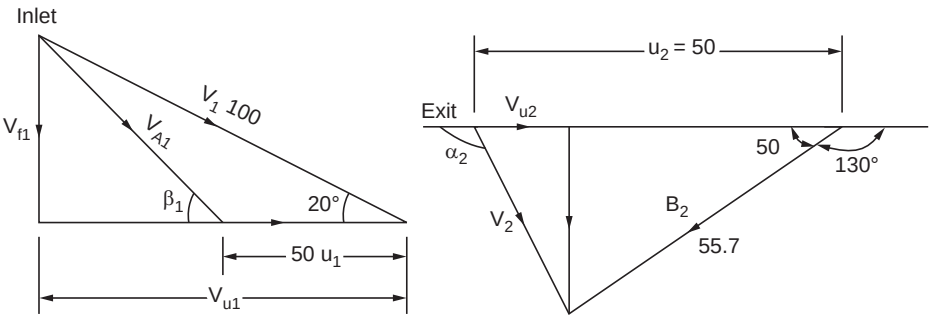


Figure P. 13.3

$$\tan \beta_1 = \frac{V_1 \sin \alpha_1}{V_1 \cos \alpha_1 - u} = \frac{100 \times \sin 20}{100 \cos 20 - 50}$$

Blade angle at inlet $\quad \therefore \quad \beta_1 = \mathbf{37\ .88°}$

$$\sin \beta_1 = \frac{V_1 \sin \alpha_1}{V_{r_1}}$$

$$\therefore \quad V_{r_1} = \frac{100 \sin 20}{\sin 37.88} = 55.7 \text{ m/s}$$

In this type of blade fixing

$$V_{r_2} = V_{r_1} \quad \text{and} \quad u_2 = u_1$$

Referring to the exit triangle

$$V_{r_2} \cos 50 < u = 50$$

Hence this shape

$$V_{r_2} \cos 50 = 35.8. \quad \therefore \quad Vu_2 = 50 - 35.8$$

$$= \textbf{14 .2 m/s} \text{ in the same direction as } V_{u_1}$$

$\therefore$ Tangential force $= 500 \times (V_{u_1} - V_{u_2})$

$$V_{u_1} = 100 \cos 20 = 93.97 \text{ m/s}$$

$\therefore$ **Tangential force** $= 5\ (93.97 - 14.2) =$ **3488 N**

Axial force $\mathbf{F} = \dot{m}\,[V_1 \sin \alpha - V_{r_1} \sin \beta_2]$

$= 5\ [100 \sin 20 - 55.7 \text{ . } \sin 50] =$ **– 8.5 N**

Problem 13.4 *Water jet at the rate of 10 kg/s strikes the series of moving blades at a velocity of 50 m/s. The blade angles with respect to the direction of motion are 35° and 140°. If the peripheral speed is 25 m/s,* ***determine the inclination of the jet*** *so that water enters the blades without shock. Also* ***calculate the power developed and the efficiency*** *of the system. Assume blades an mounting on the periphery of the wheel.*

In this type of mounting u remains the same so also relative velocity. β_1, V_1 and u are known :

Refer figure

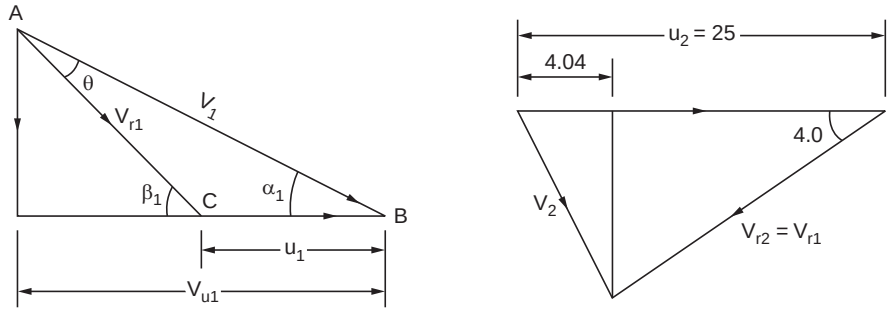


Figure P. 13.4

$$\frac{V_1}{\sin(180-\beta_1)} = \frac{u}{\sin\theta}$$

$$\therefore \quad \frac{50}{\sin(180-35)} = \frac{25}{\sin\theta}$$

Solving $\theta = 16.7°$. $\therefore \quad \alpha_1 = 180 - (180 - 35) - 16.7 = \mathbf{18.3°}$

Direction of the jet is 18.3° to the direction of motion.

$$V_{u_1} = 50 \times \cos 18.3 = 47.47 \text{ m/s},$$

$$V_{r_1} = \frac{50 \sin 18.3}{\sin 35} = 27.37 \text{ m/s}$$

$\beta_2 = (180 - 140) = 40°$, $V_{r_2} \cos 40 = 20.96 < 25\ (u)$

$\therefore$ The shape of the exit triangle will be as in figure

$$V_{u_2} = u - V_{r_2} \cos \beta_2 = 25 - 20.96 = 4.04 \text{ m/s}$$

Tangential force $= m\ (V_{u_1} - V_{u_2}) = 10\ (47.47 - 4.04) = \mathbf{434.3\ N}$

Power $= F \times u = 434.3 \times 25 = \mathbf{10.86 \times 10^3\ W}$

Energy in jet $= \dfrac{10 \times 50^2}{2} = 12.5 \times 10^3\ W$

$$\therefore \quad \eta = \frac{10.86 \times 10^3}{12.5 \times 10^3} = 0.8686 \text{ or } \mathbf{86.86\ \%}$$

Problem 13.5 *Curved vanes fixed on a wheel on the surface receive water at angle of 20° to the tangent of the wheel. The inner and outer diameter of the wheel are 0.9 and 1.6 m respectively.*

The speed of rotation of the wheel is 7 revolutions per second. The velocity of water at entry is 75 m/s. The water leaves the blades with an absolute velocity of 21 m/s at an angle of 120° with the wheel tangent at outlet. The flow rate is 400 kg/s. ***Determine the blade angles*** *for shockless entry and exit.* ***Determine the torque and power.*** *A also determine the radial force.*

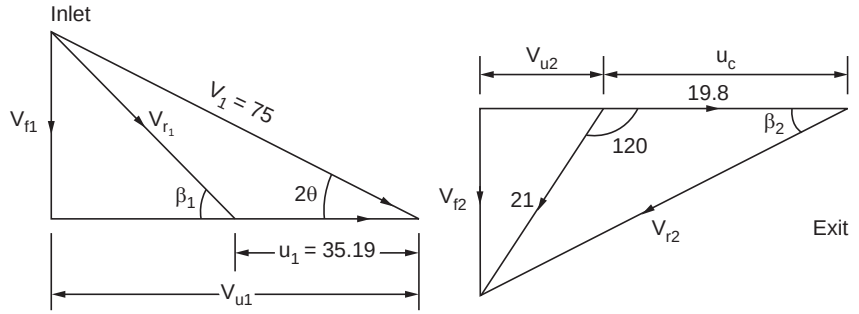


Figure P. 13.5

Blade velocity $u_1 = \pi dN = \pi \times 1.6 \times 7 = 35.19$ m/s

$$u_2 = \frac{9}{16} \times 35.19 = 19.8 \text{ m/s}$$

$$\tan \beta_1 = \frac{V_1 \sin \alpha_1}{V_1 \cos \alpha_1 - u} = \frac{75 \times \sin 20}{75 \times \cos 20 - 35.19}$$

Solving $\beta_1 = \mathbf{36°}$

$$\tan \beta_2 = \frac{21 \sin 60}{19.8 + 21 \cos 60}$$

Solving $\beta_2 = \mathbf{30.97°}$

$\mathbf{T} = \dot{m} [V_{u_1} r_1 + V_{u_2} r_2]$ (in this case, V_{u_2} is in the opposite direction)

$\therefore$ $\Delta V_w = V_{u_1} + V_{u_2}$

$= 400 [0.8 \times 75 \cos 20 + 0.45 \times 21 \cos 60] = \mathbf{24443\ Nm}$

Power $= 24443 \times \omega = 24443 \times 2\pi \times 7 = 1075042\ W$

or Ω **1075 kW.**

Power in the jet $= \frac{75^2}{2} \times 400 = 1125000$ W or **1125 kW**

$$\eta = \frac{1075}{1125} = 0.955 \text{ or } \mathbf{95.5\%}$$

Radial force $= 400 (75 \sin 20 - 21 \sin 60) = \mathbf{2986\ N.}$

Problem 13.6 *A jet of water with a velocity of 30 m/s impinges on a series of vanes moving at 12 m/s at 30 to the direction of motion. The vane angle at outlet is 162° to the direction of motion.* ***Complete (i) the vane angle at inlet*** *for shockless entry and (ii)* ***the efficiency of power transmission.***

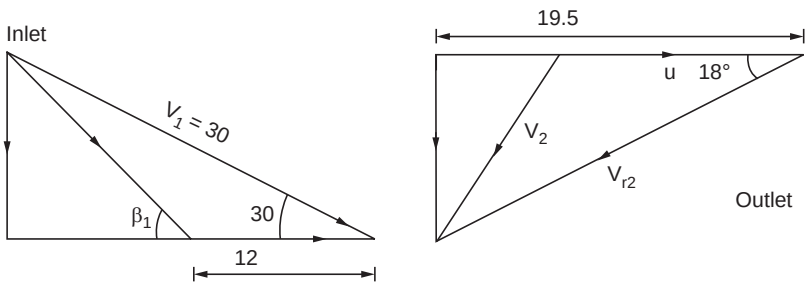


Figure P. 13.6

$$\tan \beta_1 = \frac{V_1 \sin \alpha_1}{V_1 \cos \alpha_1 - u} = \frac{30 \sin 30}{30 \cos 30 - 12} = 1.073$$

$\therefore$ $\beta_1 = \mathbf{47°}$

$$\sin \beta_1 = \frac{30 \sin 30}{V_{r_1}}$$

$$\therefore \quad V_{r_1} = \frac{30 \sin 30}{\sin \beta_1} = 20.5 \text{ m/s} = V_{r_2}$$

$V_{r_2} \cos \beta_2 > u_1$ $\therefore$ hence the shape of the triangle.

$$V_{u_1} = 30 \cos 30 = 25.98 \text{ m/s}$$

$$V_{u_2} = 20.5 \cos 18 - 12 = 7.5 \text{ m/s}$$

Assuming unit mass flow rate :

$$\mathbf{P} = u\,[V_{w_1} + V_{w_2}] = 12\,[25.98 + 7.5] = \mathbf{401.76\ W/kg/s}$$

Energy in the jet $= \dfrac{30^2}{2} = 450\ W.$

$$\therefore \quad \eta = \frac{401.76}{450} = \mathbf{0.893} \text{ or } 89.3\%$$

EXERCISE QUESTIONS

E 13.1 Derive the linear momentum equation using the control volume approach and determine the force exerted by the fluid flowing through a pipe bend.

E 13.2 Derive the expression for the force exerted by a water jet on a plate moving in the same direction of the jet with a velocity less than that of the jet.

E 13.3 A horizontal Y is shown in figure. Determine the x and y components of the force exerted in the pipe.

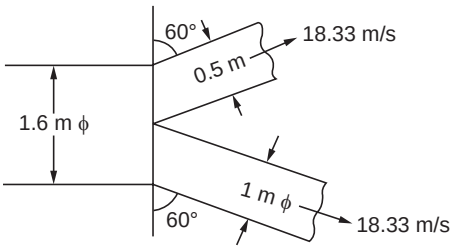


Figure E. 13.3

E 13.4 A nozzle of 5 cm diameter is fixed at the end of a pipe of 15 cm diameter with water flowing in the pipe at a velocity of 3 m/s. The jet discharges into the air. Determine the force exerted in the nozzle.

E 13.5 Water flows through a right angled reducer bend with inlet diameter of 60 cm and exit diameter of 40 cm. The entrance velocity is 6 m/s. If the bend lies on a horizontal plane, determine the magnitude and direction of the force on the bend.

E 13.6 A jet of 5 cm diameter enters a blade in the x direction with a velocity of 60 m/s. The blade angle at inlet is 0°. The outlet angle is 120° with x direction. If the blade moves with a velocity of 25 m/s along the x direction, determine the forces in the x and y directions. Also determine the energy transfer rate.

E 13.7 A jet of water 6 cm dia has a velocity of 30 m/s. If it impinges on a curved vane which turns the jet by 90° determine forces on the vane if the vane moves in the direction of the jet at a velocity of 14 m/s.

E 13.8 A series of vanes is acted upon by a 7.5 cm water jet having a velocity of 30 m/s. $\alpha_1 = \beta_1 = 0°$. If the force acting on the vane in the direction of the jet is 900 N determine the angle by which the jet is turned by the vane. The vane velocity is 15 m/s.

E 13.9 A 5 cm^2 area water jet impinges on a series of vanes as shown in figure. The absolute velocities and their directions are indicated on the figure. What is the power transmitted? Also determine the blade speed and blade inlet angle.

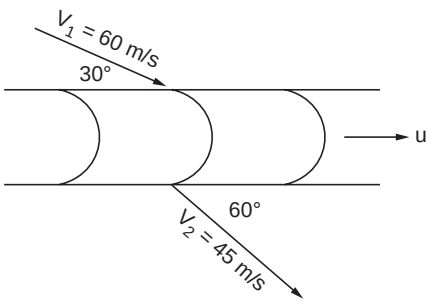


Figure E. 13.9

E 13.10 A water jet with a velocity of 60 m/s enters a series of curved vanes at an angle of 20° to the direction of blade movement. The peripheral speed of the disc on which the blades are mounted is 25 m/s. Calculate the vane inlet angle. If at the exit the component of absolute velocity along the direction of motion is zero, determine the outlet blade angle. Assume shockless enters and exit.

14 Hydraulic Turbines

14.0 INTRODUCTION

Most of the electrical generators are powered by turbines. Turbines are the primemovers of civilisation. Steam and Gas turbines share in the electrical power generation is about 75%. About 20% of power is generated by hydraulic turbines and hence thier importance. Rest of 5% only is by other means of generation. Hydraulic power depends on renewable source and hence is ever lasting. It is also non polluting in terms of non generation of carbon dioxide.

14.1 HYDRAULIC POWER PLANT

The main components of hydraulic power plant are (*i*) The storage system. (*ii*) Conveying system (*iii*) Hydraulic turbine with control system and (*iv*) Electrical generator

The storage system consists of a reservoir with a dam structure and the water flow control in terms of sluices and gates etc. The reservoir may be at a high level in the case of availability of such a location. In such cases the potential energy in the water will be large but the quantity of water available will be small. The conveying system may consist of tunnels, channels and steel pipes called penstocks. Tunnels and channels are used for surface conveyance. Penstocks are pressure pipes conveying the water from a higher level to a lower level under pressure. The penstock pipes end at the flow control system and are connected to nozzles at the end. The nozzles convert the potential energy to kinetic energy in free water jets. These jets by dynamic action turn the turbine wheels. In some cases the nozzles may be replaced by guide vanes which partially convert potential energy to kinetic energy and then direct the stream to the turbine wheel, where the remaining expansion takes place, causing a reaction on the turbine runner. Dams in river beds provide larger quantities of water but with a lower potential energy.

The reader is referred to books on power plants for details of the components and types of plants and their relative merits. In this chapter we shall concentrate on the details and operation of hydraulic turbines.

14.2 CLASSIFICATION OF TURBINES

The main classification depends upon the type of action of the water on the turbine. These are

(*i*) **Impulse turbine** (*ii*) **Reaction Turbine**. In the case of impulse turbine all the potential energy is converted to kinetic energy in the nozzles. The imulse provided by the jets is used to turn the turbine wheel. The pressure inside the turbine is atmospheric. This type is found suitable when the available potential energy is high and the flow available is comparatively low. Some people call this type as tangential flow units. Later discussion will show under what conditions this type is chosen for operation.

(*ii*) In reaction turbines the available potential energy is progressively converted in the turbines rotors and the reaction of the accelerating water causes the turning of the wheel. These are again divided into radial flow, mixed flow and axial flow machines. Radial flow machines are found suitable for moderate levels of potential energy and medium quantities of flow. The axial machines are suitable for low levels of potential energy and large flow rates. The potential energy available is generally denoted as "head available". With this terminology plants are designated as "high head", "medium head" and "low head" plants.

14.3 SIMILITUDE AND MODEL TESTING

Hydraulic turbines are mainly used for power generation and because of this these are large and heavy. The operating conditions in terms of available head and load fluctuation vary considerably. In spite of sophisticated design methodology, it is found the designs have to be validated by actual testing. In addition to the operation at the design conditions, the characteristics of operation under varying in put output conditions should be established. It is found almost impossible to test a full size unit under laboratory conditions. In case of variation of the operation from design conditions, large units cannot be modified or scrapped easily. The idea of similitude and model testing comes to the aid of the manufacturer.

In the case of these machines more than three variables affect the characteristics of the machine, (speed, flow rate, power, head available etc.). It is rather difficult to test each parameter's influence separately. It is also not easy to vary some of the parameters. Dimensional analysis comes to our aid, for solving this problem. In chapter 8 the important dimension less parameters in the case of turbomachines have been derived (problem 8.16).

The relevant parameters in the case of hydraulic machines have been identified in that chapter. These are

1. The head coefficient, gH/N^2D^2 (14.3.1)
2. The flow coefficient, Q/ND^3 (14.3.2)
3. The power coefficient, $P/\rho N^3D^5$ (14.3.3)
4. The specific sheed, $N\sqrt{p}/\rho^{1/2}(gH)^{5/4}$ (14.3.4)

Consistant sets of units should be used to obtain numerical values. All the four dimensionless numbers are used in model testing. The last parameter has particular value when it comes to choosing a particular type under given available inputs and outputs. It has been established partly by experimentation and partly by analysis that the specific speed to some extent indicates the possible type of machine to provide the maximum efficiency under the given conditions. Figure 14.1 illustrates this idea. The representation is qualitative only. Note that as head decreases for the same power and speed, the specific speed increases.

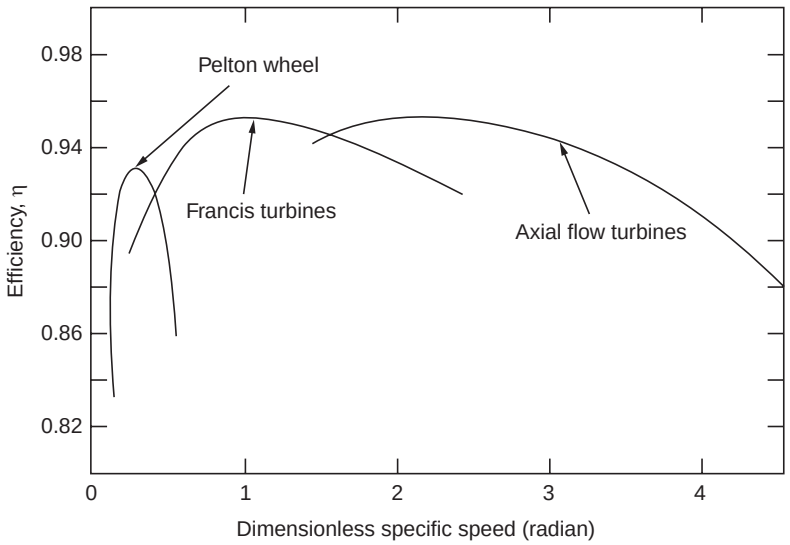


Figure 14.3.1 Variation of efficiency with specific speed.

Figure 14.3.2 gives another information provided by the specific speed. As flow rate increases for the same specific speed efficiency is found also to increase. For the same flow rate, there is an increase in efficiency with specific speed.

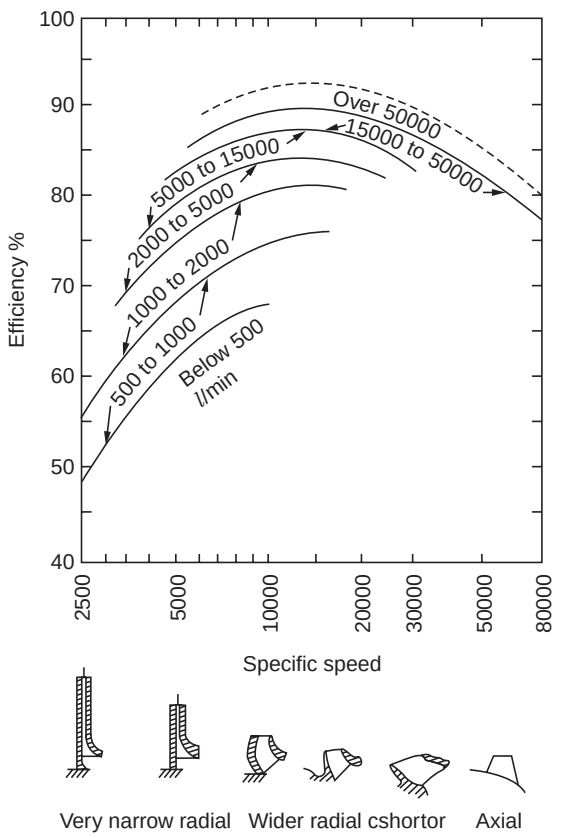


Figure 14.3.2 Variation of efficiency and flow passage with specific speed.

The type of flow passage also varies with specific speed as shown in the figure.

As the flow rate increases, the best shape is chosen for the maximum efficiency at that flow. The specific speed is obtained from the data available at the location where the plant is to be installed. Flow rate is estimated from hydrological data. Head is estimated from the topography.

Power is estimated by the product of head and flow rate. The speed is specified by the frequency of AC supply and the size. Lower the speed chosen, larger will be the size of the machine for the same power. These data lead to the calculation of the specific speed for the plant. The value of the specific speed gives a guidance about the choice of the type of machine. Worked examples will illustrate the idea more clearly.

The expressions given in the equation 14.3.1 to 4 are dimensionless and will give the same numerical value irrespective of the system of units adopted. In practice dimensional specific speed is popularly used

$$N_s = \frac{N\sqrt{p}}{H^{5/4}}$$

where N is in rps, P in W and H in m. This is also shown in Table 14.1.

Table 14.1 provides some guidance about the type of turbine suitable at various ranges of specific speeds.

Table 14.1 Best specific Speed Range for Different Type of Hydraulic Turbines

Dimensionles specific speed range	Dimensional specific speed in SI system	Type of turbine having the best efficiency at these values
0.015—0.053	8—29	Single jet Pelton turbine
0.047—0.072	26—40	Twin jet Pelton turbine
0.72—0.122	40—67	Multiple jet Pelton turbine
0.122—0.819	67—450	Radial flow turbine Francis type (H < 350m)
0.663—1.66	364—910	Axial flow Kaplan turbine. (H < 60m)

There is considerable variation in the specific speeds indicated by various authors. Speed N is used as rpm and power in kW by some authors.

Such use is dimensionly complex and will vary with values given in table 14.1. In the non dimensional form, speed should be in rps and power should be in W. Then only the value becomes dimensionless.

Consider the following

$$\frac{N\sqrt{p}}{\rho^{1/2}(gH)^{5/4}} \rightarrow \frac{1}{S}\frac{N^{1/2}\,m^{1/2}}{S^{1/2}}\cdot\frac{m^{3/2}}{kg^{1/2}}\cdot\frac{s^{10/4}}{m^{5/4}}\cdot m^{5/4}$$

Substituting for Newton $N^{1/2}$ as $kg^{1/2}\ m^{1/2}/s$ the expression will be dimensionless. This can be checked

$$N_S \rightarrow \frac{1}{s}\cdot\frac{kg^{1/2}\,m^{1/2}\,m^{1/2}}{s^{1/2}\,h^{s}}\cdot\frac{m^{1.5}}{kg^{1/2}}\cdot\frac{s^{2.5}}{m^{2.5}} \rightarrow M^{\circ}\,L^{\circ}\,T^{\circ}$$

$$\frac{1}{s}\cdot\frac{k^{1/2}m_1{}^{1/2}}{s}\cdot\frac{m^{1/2}}{s^{1/2}}\cdot\frac{m^{1.5}\,.\,s^{2.5}}{kg^{1/2}m^{2.5}} \rightarrow M^{\circ}L^{\circ}T^{\circ}$$

As a check for the dimensional value listed, the omitted quantities in this case are $\rho^{1/2}$ $g^{1.2} = 549$

$\therefore$ $0.015 \times 549 = 8.24$ as in the tabulation.

In the discussions the specific speed values in best efficiency is as given in table 14.1. But in the solved problems and examples, these conditions are generally not satisfied. Even with same actual installation data, the specific speed is found to vary from the listed best values for the type.

Significance of specific speed. Specific speed does not indicate the speed of the machine. It can be considered to indicate the flow area and shape of the runner. When the head is large, the velocity when potential energy is converted to kinetic energy will be high. The flow area required will be just the nozzle diameter. This cannot be arranged in a fully flowing type of turbine. Hence the best suited will be the impulse turbine. When the flow increases, still the area required will be unsuitable for a reaction turbine. So multi jet unit is chosen in such a case. As the head reduces and flow increases purely radial flow reaction turbines of smaller diameter can be chosen. As the head decreases still further and the flow increases, wider rotors with mixed flow are found suitable. The diameter can be reduced further and the speed increased up to the limit set by mechanical design. As the head drops further for the same power, the flow rate has to be higher. Hence axial flow units are found suitable in this situation. Keeping the power constant, the specific speed increases with N and decreases with head. The speed variation is not as high as the head variation. Hence specific speed value increases with the drop in available head. This can be easily seen from the values listed in table 14.1.

Example 14.1. *Determine the specific speed for the data available at a location as given below (Both dimensionless and dimensional). Head available : 900 m.*

Power estimated 40000 kW, Speed required : 417.5 rpm. Also indicate the suitable type of turbine

Dimensionless specific speed : units to be used :

$$N \to \text{rps},\ P \to W \text{ or } Nm/s,\ \rho \to \text{kg/m}^3,\ g \to \frac{\text{m}}{s^2},\ H \to \text{m}$$

$\therefore$ $$N_s = \frac{417.5}{60} \times \frac{(40{,}000000)^{1/2}}{1000^{1/2} \times 9.81^{5/4} \times 900^{5/4}} = \mathbf{0.0163.}$$

Hence single jet pelton turbine is suitable.

Non dimensional specific speed.

$$N_s = \frac{417.5}{60} \times \frac{40{,}000000^{1/2}}{900^{5/4}} = \mathbf{8.92.}$$

Agrees with the former value.

Single jet impulse turbine will be suitable.

Example 14.2. *At a location the head available was estimated as 200 m. The power potential was 50,000 kW. The sheed chosen is 600 rpm.* ***Determine the specific speed****. Indicate what type of turbine is suitable.*

Dimensionless. $N \to rps, P \to W, g \to \frac{m^3}{kg}, g \to \frac{m}{s^2}, H \to m$

$$N_s = \frac{600\sqrt{40{,}000000}}{60 \times (200 \times 9.81)^{5/4} \times 1000^{1/2}} = \mathbf{0.153.}$$

Hence **Francis type of turbine is suitable.**

Dimensional. $$N_s = \frac{600}{60} \times \frac{(40{,}000000)^{1/2}}{200^{5/4}} = \mathbf{84.09.}$$

Hence **agrees with the previous value.**

Example 14.3. *At a location, the head available was 50 m. The power estimated is 40,000 kW. The speed chosen is 600 rpm.* ***Determine the specific speed and indicate the suitable type*** *of turbine.*

Dimensionless $$N_s = \frac{600}{60} \cdot \frac{\sqrt{40{,}000{,}000}}{1000^{1/2}\,(9.81 \times 50)^{5/4}} = \mathbf{0.866}$$

Hence **axial flow Kaplan turbine** is suitable.

Dimensional $$N_s = \frac{600}{60} \times \frac{40{,}000{,}000^{1/2}}{50^{1.25}} = \mathbf{475.}$$

14.3.1 Model and Prototype

It is found not desirable to rely completely on design calculations before manufacturing a large turbine unit. It is necessary to obtain test results which will indicate the performance of the large unit. This is done by testing a "homologous" or similar model of smaller size and predicting from the results the performance of large unit. Similarity conditions are three fold namely geometric similarity, kinematic similarity and dynamic similarity. Equal ratios of geometric dimensions leads to geometric similarity.

Similar flow pattern leads to kinematic similarity. Similar dynamic conditions in terms of velocity, acceleration, forces etc. leads to dynamic similarity. A model satisfying these conditions is called "Homologous" model. In such case, it can be shown that specific speeds, head coefficients flow coefficient and power coefficient will be identical between the model and the **large machine called prototype.** It is also possible from these experiments to predict part load performance and operation at different head speed and flow conditions.

The **ratio between linear dimensions is called scale.** For example an one eight scale model means that the linear dimensions of the model is 1/8 of the linear dimensions of the larger machine or the prototype. For kinematic and dynamic similarity the flow directions and the blade angles should be equal.

Example 14.4. *At a location investigations yielded the following data for the installation of a hydro plant. Head available = 200 m, power available = 40,000 kW. The speed chosen was 500 rpm. A model study was proposed. In the laboratory head available was 20 m. It was proposed to construct a 1/6 scale model.* ***Determine the speed and dynamo meter capacity*** *to test the model. Also* ***determine the flow rate required*** *in terms of the prototype flow rate.*

The dimensional specific speed of the proposed turbine

$$\mathbf{N_s} = \frac{N\sqrt{P}}{H^{5/4}} = \frac{500}{60}\frac{\sqrt{40{,}000{,}000}}{200^{5/4}} = \mathbf{70.0747}$$

The specific speed of the model should be the same. As two unknowns are involved another parameter has to be used to solve the problem.

Choosing head coefficient, (as both heads are known)

$$\frac{H_m}{N_m^{\,2}\,D_m^{\,2}} = \frac{H_p}{N_p^{\,2}\,D_p^{\,2}} \quad \therefore \quad N_m^{\,2} = \frac{H_m}{H_p}\cdot N_p^{\,2}\left(\frac{D_p}{D_m}\right)^2$$

$$\therefore \quad \mathbf{N_m} = \left[\frac{20}{200}\times 500^2\,(6^2)\right]^{0.5} = \mathbf{948.7\ rpm}$$

Substituting in the specific speed expression,

$$70.0747 = \frac{984.7\sqrt{P_m}}{60\times 20^{5/4}}$$

Solving $P_m = 35062\ W = \mathbf{35.062\ kW}$

$\therefore$ The model is to have a capacity of 35.062 kW and run at 948.7 rpm.

The flow rate ratio can be obtained using flow coefficient

$$\frac{Q_m}{N_m\,D_m^{\,3}} = \frac{Q_p}{N_p\,D_p^{\,3}}$$

$$\therefore \quad \frac{Q_m}{Q_p} = \frac{N_m\,D_m^{\,3}}{N_p\,D_p^{\,3}} = \frac{948.7}{500}\times\frac{1}{6^3} = 0.08777$$

or $\mathbf{Q_m}$ **is** $\frac{1}{113}$ **of** $\mathbf{Q_p}$.

Example 14.5. *In example 14.4, the data in the proposed plant is given. These are 200 m, head, 40000 kW power and 500 rpm.*

A one sixth scale model is proposed. The test facility has a limited dynamometer capacity of 40 kW only whereas the speed and head have no limitations. Determine the speed and head required for the model.

The value of dimensional specific speed of the proposed plant is taken from example 14.4 as 70.0747.

In this case it is preferable to choose the power coefficient

$$\frac{P_m}{N_m^{\,3}\,D_m^{\,5}} = \frac{P_p}{N_p^{\,3}\,D_p^{\,5}} \quad \therefore \quad N_m^{\,3} = \frac{P_m}{P_p}\times N_p^{\,3}\times\left(\frac{D_p}{D_m}\right)^5$$

$$\therefore \quad \mathbf{N_m} = \left[\frac{40}{40{,}000}\times 500^3\times 6^5\right]^{1/3} = \mathbf{990.6\ rpm}$$

Using the specific speed value (for the model)

$$70.0747 = \frac{990.6}{60} \times \frac{\sqrt{40{,}000}}{H_m^{5/4}} \text{ or } H_m^{5/4} = \frac{990.6\sqrt{40{,}000}}{60 \times 70.0747}$$

Solving $\mathbf{H_m = 21.8\ m}$. Test head required is 21.8 m and test speed is 990.6 rpm.

The flow rate can be obtained using the flow coefficient

$$\frac{Q_m}{Q_p} = \frac{N_m D_m^3}{N_p D_p^3} = \frac{990.6}{500} \times \frac{1}{6^3} = 0.0916 \textbf{ or } \frac{\mathbf{1}}{\mathbf{109.9}} \text{ times the flow in prototype.}$$

Example 14.6. *Use the data for the proposed hydro plant given in example 14.4. The test facility has only a constant speed dynamometer running at 1000 rpm. In this case determine power of the model and the test head required.*

The specific speed of the proposed plant is 70.0747 and the models should have the same value of specific speed.

In this case the head coefficient is more convenient for solving the problem.

$$\mathbf{H_m} = H_p \frac{N_m^2}{N_p^2} \cdot \left(\frac{D_m}{D_p}\right)^2 = 200 \times \left(\frac{1000}{500}\right)^2 \left(\frac{1}{6}\right)^2 = \mathbf{22.22\ m}$$

Substituting in the specific speed expression,

$$70.0747 = \frac{1000\sqrt{P}}{60 \times 22.22^{5/4}}.$$

Solving P = 41146 W or **40.146 kW**

The flow ratio $$\frac{\mathbf{Q_m}}{\mathbf{Q_p}} = \frac{N_m}{N_p} \cdot \frac{D_m^3}{D_p^3} = \frac{1000}{500} \times \frac{1}{6^3} = 0.00926$$

or $\dfrac{\mathbf{1}}{\mathbf{108}}$ times the prototype flow.

14.3.2 Unit Quantities

The dimensionless constants can also be used to predict the performance of a given machine under different operating conditions. As the linear dimension will be the same, the same will not be taken into account in the calculation. Thus

Head coefficient will now be

$$\frac{H_1}{N_1^2 D^2} = \frac{H_2}{N_2^2 D^2} \text{ or } \frac{H_2}{H_1} = \frac{N_2^2}{N_1^2}$$

The head will vary as the square of the speed.

The flow coefficient will lead to

$$\frac{Q_1}{N_1 D^3} = \frac{Q_2}{N_2 D^3} \quad \text{or} \quad \frac{Q_2}{Q_1} = \frac{N_2}{N_1}$$

Flow will be proportional to N and using the previous relation

$$\frac{Q_2}{Q_1} = \sqrt{\frac{H_2}{H_1}} \quad \text{or} \quad \frac{\mathbf{Q}}{\sqrt{\mathbf{H}}} = \textbf{constant for a machine.}$$

The constant is called unit discharge.

Similarly $$\frac{N_2}{N_1} = \sqrt{\frac{H_2}{H_1}} \quad \text{or} \quad \frac{\mathbf{N}}{\sqrt{\mathbf{H}}} = \textbf{constant.}$$

This constant is called unit speed.

Using the power coefficient :

$$\frac{P_1}{N^3 D^5} = \frac{P_2}{N^3 D^5} \text{ or } \frac{P_2}{P_1} = \frac{N_2^3}{N_1^3} = \left(\frac{H_2}{H_1}\right)^{3/2}$$

or $$\frac{\mathbf{P}}{\mathbf{H^{3/2}}} = \textbf{constant.}$$ This constant is **called unit power.**

Hence when H is varied in a machine the other quantities can be predicted by the use of unit quantities.

Example 14.7. *A turbine is operating with a head of 400 m and speed of 500 rpm and flow rate of 5 m^3/s producing the power of 17.66 MW. The head available changed to 350 m. It no other corrective action was taken what would be the speed, flow and power ? Assume efficiency is maintained.*

1. $$\frac{H_1}{N_1^2} = \frac{H_2}{N_2^2}$$

$\therefore$ $$\frac{N_2}{N_1} = \left[\frac{350}{400}\right]^{0.5} = 0.93541 \text{ or } N_2 = 500 \times 0.93541 = \textbf{467.7 rpm}$$

2. $$\frac{Q_1}{N_1} = \frac{Q_2}{N_2} \quad \text{or} \quad \frac{Q_2}{Q_1} = \frac{N_2}{N_1} = \sqrt{\frac{H_2}{H_1}} = \left(\frac{350}{400}\right)^{0.5}$$

$\therefore$ $\mathbf{Q_2} = 5 \times 0.93531 = \mathbf{4.677\ m^3/s}$

3. $$\frac{P_2}{P_1} = \left(\frac{H_2}{H_1}\right)^{3/2} \quad \therefore \quad \mathbf{P_2} = 17.66 \times \left(\frac{350}{400}\right)^{0.5} = \textbf{14.45 MW}$$

14.4 TURBINE EFFICIENCIES

The head available for hydroelectric plant depends on the site conditions. Gross head is defined as the difference in level between the reservoir water level (called head race) and the level of water in the stream into which the water is let out (called tail race), both levels to be observed at the same time. During the conveyance of water there are losses involved. The difference between the gross head and head loss is called the net head or effective head. It can be measured

by the difference in pressure between the turbine entry and tailrace level. The following efficiencies are generally used.

1. Hydraulic efficiency : It is defined as the ratio of the power produced by the turbine runner and the power supplied by the water at the turbine inlet.

$$\eta_H = \frac{\text{Power produced by the runner}}{\rho Q g H} \tag{14.4.1}$$

where Q is the volume flow rate and H is the net or effective head. Power produced by the runner is calculated by the Euler turbine equation $P = Q\rho\ [u_1 V_{u1} - u_2 V_{u2}]$. This reflects the runner design effectiveness.

2. Volumetric efficiency : It is possible some water flows out through the clearance between the runner and casing without passing through the runner.

Volumetric efficiency is defined as the ratio between the volume of water flowing through the runner and the total volume of water supplied to the turbine. Indicating Q as the volume flow and ΔQ as the volume of water passing out without flowing through the runner.

$$\eta_v = \frac{Q - \Delta Q}{Q} \tag{14.4.2}$$

To some extent this depends on manufacturing tolerances.

3. Mechanical efficiency : The power produced by the runner is always greater than the power available at the turbine shaft. This is due to mechanical losses at the bearings, windage losses and other frictional losses.

$$\eta_m = \frac{\text{Power available at the turbine shaft}}{\text{Power produced by the runner}} \tag{14.4.3}$$

4. Overall efficiency : This is the ratio of power output at the shaft and power input by the water at the turbine inlet.

$$\eta_0 = \frac{\text{Power available at the turbine shaft}}{\rho QgH} \tag{14.4.4}$$

Also the overall efficiency is the product of the other three efficiencies defind

$$\eta_0 = N_H\ N_m N_v \tag{14.4.5}$$

14.5 EULER TURBINE EQUATION

The fluid velocity at the turbine entry and exit can have three components in the tangential, axial and radial directions of the rotor. This means that the fluid momentum can have three components at the entry and exit. This also means that the force exerted on the runner can have three components. **Out of these the tangential force only can cause the rotation of the runner and produce work.** The axial component produces a thrust in the axial direction, which is taken by suitable thrust bearings. The radial component produces a bending of the shaft which is taken by the journal bearings.

Thus it is necessary to consider the tangential component for the determination of work done and power produced. The work done or power produced by the tangential force equals the product of the mass flow, tangential force and the tangential velocity. As the tangential velocity varies with the radius, the work done also will be vary with the radius. It is not easy to sum up this work. The help of moment of momentum theorem is used for this purpose. It states that **the torque on the rotor equals the rate of change of moment of momentum of the fluid as it passes through the runner.**

Let u_1 be the tangental velocity at entry and u_2 be the tangential velocity at exit.

Let V_{u1} be the tangential component of the absolute velocity of the fluid at inlet and let V_{u2} be the tangential component of the absolute velocity of the fluid at exit. Let r_1 and r_2 be the radii at inlet and exit.

The tangential momentum of the fluid at inlet $= \dot{m}\ V_{u1}$

The tangential momentum of the fluid at exit $= \dot{m}\ V_{u2}$

The moment of momentum at inlet $= \dot{m}\ V_{u1}\, r_1$

The moment of momentum at exit $= \dot{m}\ V_{u2}\, r_2$

$\therefore$ Torque, $\tau = \dot{m}(V_{u1}\, r_1 - V_{u2}\, r_2)$ (14.5.1)

Depending on the direction of V_{u2} with reference to V_{u1}, the – sign will become + ve sign.

$$\text{Power} = \omega\tau \text{ and } \omega = \frac{2\pi N}{60} \tag{14.5.2}$$

where N is rpm.

$$\therefore \quad \text{Power} = \dot{m}\,\frac{2\pi N}{60}\,(V_{w1}\, r_1 - V_{w2}\, r_2) \tag{14.5.3}$$

But $\dfrac{2\pi N}{60}\, r_1 = u_1$ and $\dfrac{2\pi N}{60}\, r_2 = u_2$

$$\therefore \quad \text{Power} = \dot{m}(V_{u1}\, u_1 - V_{u2}\, u_2) \tag{14.5.4}$$

Equation (14.5.4) is known as Euler Turbine equation.

14.5.1 Components of Power Produced

The power produced can be expressed as due to three effects. These are the **dynamic, centrifugal and acceleration effects.** Consider the general velocity triangles at inlet and exit of turbine runner, shown in figure 14.5.1.

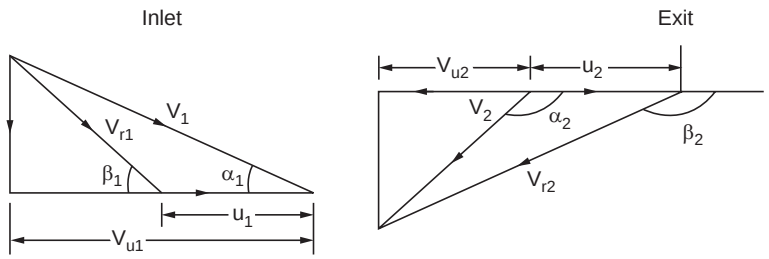


Figure 14.5.1 Velocity triangles

V_1, V_2 Absolute velocities at inlet and outlet.

V_{r1}, V_{r2} Relative velocities at inlet and outlet.

u_1, u_2 Tangential velocities at inlet and outlet.

V_{u1}, V_{u2} Tangential component of absolute velocities at inlet and outlet.

From inlet velocity triangle, ($V_{u1} = V_1 \cos \alpha_1$)

$$V_{r1}^2 = V_1^2 + u_1^2 - 2u_1 V_1 \cos \alpha_1$$

or
$$u_1 V_1 \cos \alpha_1 = V_{u1} u_1 = \frac{V_1^2 + u_1^2 - v_{r1}^2}{2} \quad \text{(A)}$$

From outlet velocity triangle ($V_{u2} = V_2 \cos \alpha_2$)

$$V_{r2}^2 = V_2^2 + u_2^2 - 2 u_2 V_2 \cos \alpha_2$$

or
$$u_2 V_2 \cos \alpha_2 = u_2 V_{u2} = (V_2^2 - u_2^2 + V_{r2}^2)/2 \quad \text{(B)}$$

Substituting in Euler equation,

Power per unit flow rate (here the V_{u2} is in the opposite to V_{u1})

$$\dot{m}(u_1 V_{u1} + u_2 V_{u2}) = \dot{m} \frac{1}{2} [(V_1^2 - V_2^2) + (u_1^2 - u_2^2) + (V_{r2}^2 - V_{r1}^2)]$$

$\frac{V_1^2 - V_2^2}{2}$ is the dynamic component of work done

$\frac{u_1^2 - u_2^2}{2}$ is the centrifugal component of work and this will be present only in the radial flow machines

$\frac{u_{r2}^2 - V_{r1}^2}{2}$ is the accelerating component and this will be present only in the reaction turbines.

The first term only will be present in Pelton or impulse turbine of tangential flow type.

In pure reaction turbines, the last two terms only will be present.

In impulse reaction turbines of radial flow type, all the terms will be present. (Francis turbines is of this type).

In impulse reaction turbines, the degree of reaction is defined by the ratio of energy converted in the rotor and total energy converted.

$$R = \frac{(u_1^2 - u_2^2) + (V_{r2}^2 - V_{r1}^2)}{(V_1^2 - V_2^2) + (u_1^2 - u_2^2) + (V_{r2}^2 - V_{r1}^2)} \quad (14.5.5)$$

The degree of reaction is considered in detail in the case of steam turbines where speed reduction is necessary. Hydraulic turbines are generally operate of lower speeds and hence degree of reaction is not generally considered in the discussion of hydraulic turbines.

14.6 PELTON TURBINE

This is the only type used in high head power plants. This type of turbine was developed and patented by L.A. Pelton in 1889 and all the type of turbines are called by his name to honour him.

A sectional view of a horizontal axis Pelton turbine is shown in figure 14.6.1. The main components are (1) The runner with the (vanes) buckets fixed on the periphery of the same. (2) The nozzle assembly with control spear and deflector (3) Brake nozzle and (4) The casing.

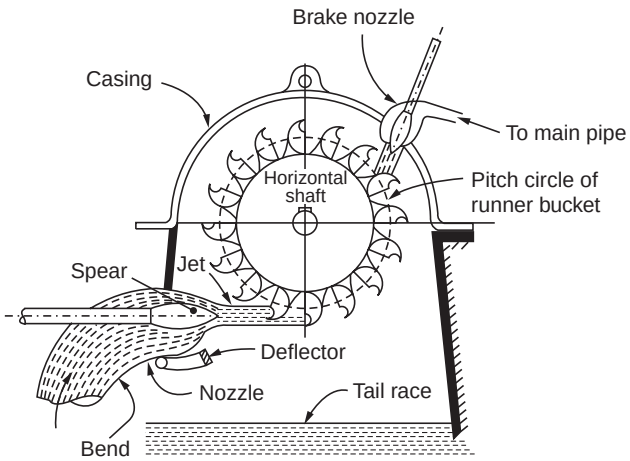


Figure 14.6.1 Pelton turbine

The rotor or runner consists of a circular disc, fixed on suitable shaft, made of cast or forged steel. Buckets are fixed on the periphery of the disc. The spacing of the buckets is decided by the runner diameter and jet diameter and is generally more than 15 in number. These buckets in small sizes may be cast integral with the runner. In larger sizes it is bolted to the runner disc.

The buckets are also made of special materials and the surfaces are well polished. A view of a bucket is shown in figure 14.6.2 with relative dimensions indicated in the figure. Originally spherical buckets were used and pelton modified the buckets to the present shape. It is formed in the shape of two half ellipsoids with a splilter connecting the two. A cut is made in the lip to facilitate all the water in the jet to usefully impinge on the buckets. This avoids interference of the incoming bucket on the jet impinging on the previous bucket. Equations are available to calculate the number of buckets on a wheel. The number of buckets, Z,

$$Z = (D/2d) + 15$$

where D is the runner diameter and d is the jet diameter.

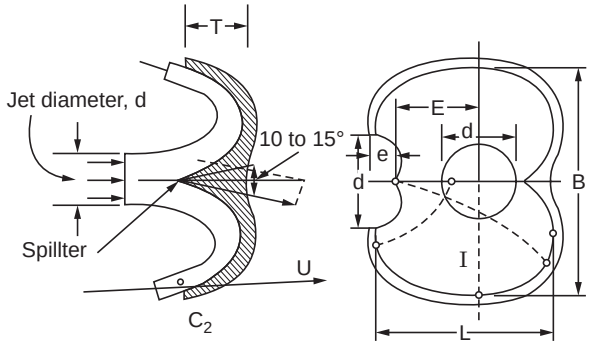


Figure 14.6.2 Pelton turbine bucket

Bucket and wheel dimensions

D/d	*B/d*	*L/d*	*T/d*	Notch width
14 – 16	2.8 – 4	2.5 – 2.8	0.95	1.1 *d* + 5 mm

The nozzle and controlling spear and deflector assembly

The head is generally constant and the jet velocity is thus constant. A fixed ratio between the jet velocity and runner peripheral velocity is to be maintained for best efficiency. The nozzle is designed to satisfy the need. But the load on the turbine will often fluctuate and some times sudden changes in load can take place due to electrical circuit tripping. The velocity of the jet should not be changed to meet the load fluctuation due to frequency requirements. The quantity of water flow only should be changed to meet the load fluctuation. A governor moves to and fro a suitably shaped spear placed inside the nozzle assembly in order to change the flow rate at the same time maintaining a compact circular jet.

When load drops suddenly, the water flow should not be stopped suddenly. Such a sudden action will cause a high pressure wave in the penstock pipes that may cause damage to the system. To avoid this a deflector as shown in figure 14.6.3 is used to suddenly play out and deflect the jet so that the jet bypasses the buckets. Meanwhile the spear will move at the safe rate and close the nozzle and stop the flow. The deflector will than move to the initial position. Even when the flow is cut off, it will take a long time for the runner to come to rest due to the high inertia. To avoid this a braking jet is used which directs a jet in the opposite direction and stops the rotation. The spear assembly with the deflector is shown in figure 14.6.3. Some other methods like auxiliary waste nozzle and tilting nozzle are also used for speed regulation. The first wastes water and the second is mechanically complex. In side **the casing** the pressure is atmospheric and hence no need to design the casing for pressure. It mainly serves the purpose of providing a cover and deflecting the water downwards. The casing is cast in two halves for case of assembly. The casing also supports the bearing and as such should be sturdy enough to take up the load.

When the condition is such that the specific speed indicates more than one jet, a vertical shaft system will be adopted. In this case the shaft is vertical and a horizontal nozzle ring with several nozzle is used. The jets in this case should not interfere with each other.

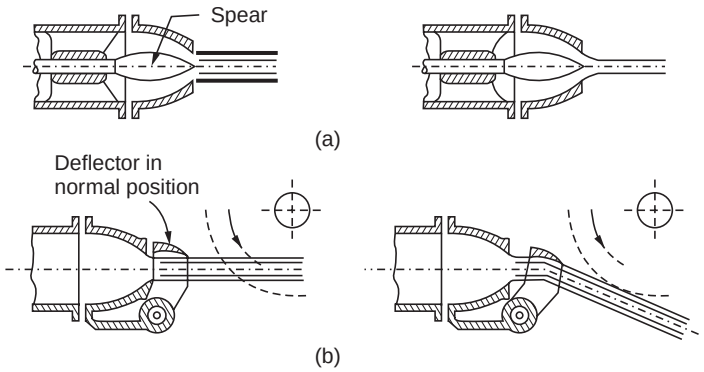


Figure 14.6.3 Nozzle assembly

Generally the turbine directly drives the generator. The speed of the turbine is governed by the frequency of AC. Power used in the region. The product of the pairs of poles used in the generator and the speed in rps gives the number of cycles per second. Steam turbines operate at 3000 rpm or 50 rps in the areas where the AC frequency is 50 cycles per second. Hydraulic turbines handle heavier fluid and hence cannot run at such speeds. In many cases the speed in the **range to 500 rpm**. As the water flows out on both sides equally axial thrust is minimal and heavy thrust bearing is not required.

14.6.1 Power Development

The bucket splits the jet into equal parts and changes the direction of the jet by about 165°. The velocity diagram for Pelton turbine is shown in figure 14.6.4.

The diagram shown is for the conditions $V_{r2} \cos \beta > u$, and $V_2 \cos \alpha_2$ is in the opposite direction to V_{u1} and hence ΔV_{u1} is additive.

In this case the jet direction is parallel to the blade velocity or the tangential velocity of the runner.

Hence $$V_{u1} = V_1 \tag{A}$$

and $$V_{r1} = V_1 - u \tag{B}$$

In the ideal case $V_{r2} = V_{r1}$. But due to friction $V_{r2} = k\, V_{r1}$ and $u_2 = u_1$.

$$F = \dot{m}\,(V_{u1} \pm V_{u2}) \tag{14.6.1}$$

$$\tau = \dot{m}\,(V_{u1} \pm V_{u2})\, r \tag{14.6.2}$$

$$P = \dot{m}\,(V_{u1} \pm V_{u2})\, u \tag{14.6.3}$$

where $\dot{m}$ is given by ρAV at entry.

Hydraulic efficiency

$$\eta_h = \frac{\dot{m}\,(V_{u1} \pm V_{u2})\, u}{\dot{m}\, V_1^2 / 2} = \frac{2u\,(V_{u1} \pm V_{u2})\, u}{V_1^2} \tag{14.6.4}$$

Once the effective head of turbine entry is known V_1 is fixed given by $V_1 = C_v \sqrt{2gH}$. For various values of u, the power developed and the hydraulic efficiency will be different. In fact the out let triangle will be different from the one shown it $u > V_{r2} \cos \beta$. In this case V_{u2} will be in the same direction as V_{u1} and hence the equation (14.6.3) will read as

$$P = \dot{m}\ (V_{u1} - V_{u2})\ u$$

It is desirable to arrive at the optimum value of u for a given value of V_1. Equation 14.6.4 can be modified by using the following relations.

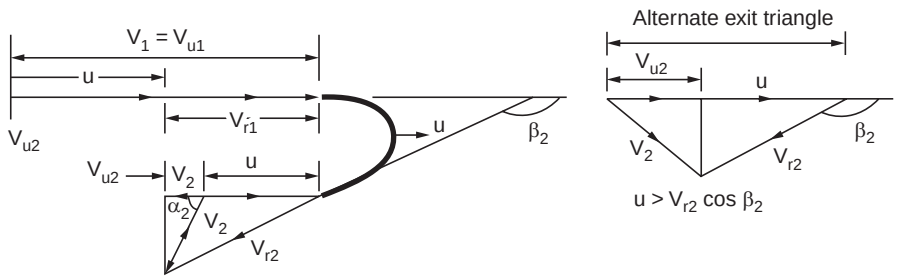


Figure 14.6.4 Velocity triangles Pelton turbine

$$V_{u1} = V_1,\ V_{u2} = V_{r2} \cos \beta_2 - u = kV_{r1} \cos \beta_2 - u = k(V_1 - u) \cos \beta_2 - u$$

$$\therefore \quad V_{u1} + V_{u2} = V_1 + k\ V_1 \cos \beta_2 - u \cos \beta_2 - u$$

$$= V_1\ (1 + k \cos \beta_2) - u(1 + k \cos \beta_2)$$

$$= (1 + k \cos \beta_2)\ (V_1 + u)$$

Substituting in equation (14.6.4)

$$\eta_H = \frac{2u}{V_1^2} \times (1 + k \cos \beta_2)\ (V_1 + u) \qquad (14.6.4a)$$

$$= 2(1 + k \cos \beta_2) \left[\frac{u}{V_1} - \frac{u^2}{V_1^2} \right]$$

$\dfrac{u}{V_1}$ is called speed ratio and denoted as ϕ.

$$\therefore \quad \eta_H = 2(1 + k \cos \beta_2)\ [\phi - \phi^2] \qquad (14.6.5)$$

To arrive at the optimum value of ϕ, this expression is differentiated with respect to ϕ and equated to zero.

$$\frac{d\eta_H}{d\phi} = 2(1 + k \cos \beta_2)\ (1 - 2\ \phi)$$

$$\therefore \quad \phi = \mathbf{\frac{u}{V_1} = \frac{1}{2}} \quad \textbf{or} \quad \mathbf{u = 0.5\ V_1} \qquad (14.6.6)$$

In practice the value is some what lower at $\mathbf{u = 0.46\ V_1}$

Substituting equation (14.6.6) in (14.6.4*a*) we get

$$\eta_H = 2(1 + k \cos \beta_2)\ [0.5 - 0.5^2]$$

$$= \frac{1 + k \cos \beta_2}{2}$$

It may be seen that in the case $k = 1$ and $\beta = 180°$,

$$\eta_H = 1 \quad \text{or} \quad 100 \text{ percent.}$$

But the actual efficiency in well designed units lies between 85 and 90%.

Example 14.8. *The head available at a plant location is 500 m.* ***For various values of ϕ determine the work done 1 kg.*** *Assume $\beta_2 = 165°$ and $C_v = 0.97$, $V_{r2} = V_{r1}$.*

$$V_j = 0.97 \sqrt{2 \times 9.81 \times 500} = 96 \text{ m/s}$$

1. $\phi = 0.2$, $u = 19.2$, $V_{r1} = 76.8$ m/s $= V_{r2}$

   $V_{w2} = (76.8 \times \cos 15 - 19.2) = 54.98$

   $W = (96 + 54.98) \times 19.2 =$ **2898.8 Nm/kg/s**

2. $\phi = 0.3$, $u = 28.8$, $V_{r1} = 67.2 = V_{\pi 2}$

   $V_{w2} = (67.2 \times \cos 15 - 28.8) = 36.11$ m/s

   $W = (96 + 36.11) \times 28.8 =$ **3804.8 Nm/kg/s**

3. $\phi = 0.4$, $u = 38.4$, $V_{r1} = V_{r2} = 57.6$ m/s

   $V_{w2} = (57.6 \cos 15 - 38.4) = 17.24$

   $W = (96 + 17.24) \times 38.4 =$ **4348.3 Nm/kg/s**

4. $\phi = 0.45$, $u = 43.2$, $V_{r1} = V_{r2} = 52.8$

   $V_{w2} = (52.8 \cos 15 - 43.2) = 7.8$

   $W = (96 + 7.8) \times 43.2 =$ **4484 Nm/kg/s**

5. $\phi = 0.5$, $u = 48$, $V_{r1} = V_{r2} = 48$

   $V_{w2} = (48 \cos 15 - 48) = - 1.64$

   $W = (96 - 1.64) \times 48 =$ **4529** Nm/kg/s

6. $\phi = 0.6$, $u = 57.6$, $V_{r1} = V_{r2} = 38.4$

   $V_{w2} = (38.4 \cos 15 - 57.6) = - 20.5$

   $W = (96 - 20.5) \times 57.6 =$ **4348.3** Nm/kg/s

7. $\phi = 0.7$, $u = 67.2$, $V_{r1} = V_{r2} = 28.8$

   $V_{w2} = (28.8 \cos 15 - 67.2) = - 39.4$

   $W = (96 - 39.4) \times 67.2 =$ **3804.8** Nm/kg/s

8. $\phi = 0.8$, $u = 76.8$, $V_{r1} = V_{r2} = 19.2$

   $V_{w2} = (19.2 \cos 15 - 76.8) = - 58.3$

   $W = (96 - 58.3) \times 76.8 =$ **2898.8 Nm/kg/s**

The result is shown plotted in Figure 14.6.5.

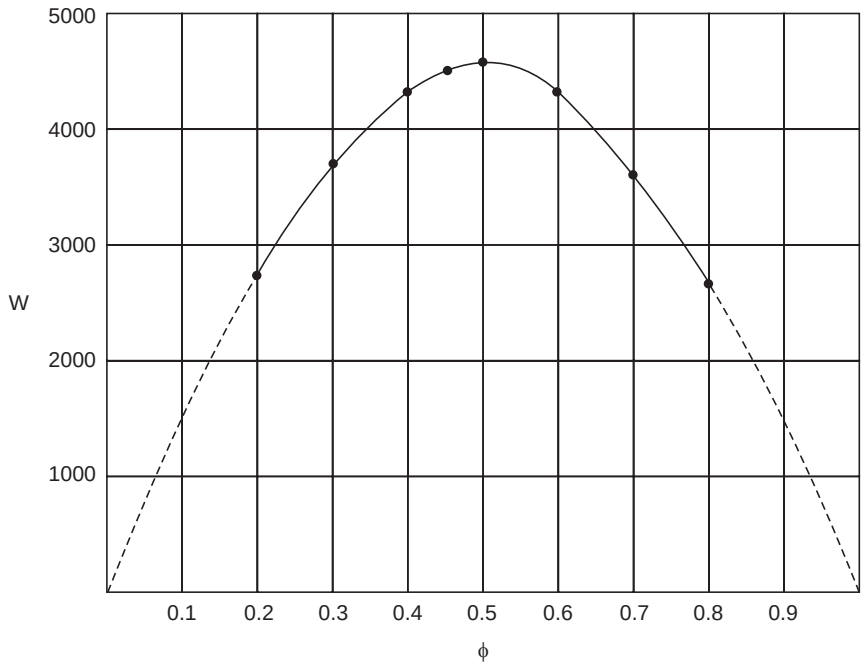


Figure 14.6.5 Variation of power with variation of ϕ for constant jet velocity

The shape of the velocity diagram at exit up to $\phi = 0.45$ is given by Fig. 14.6.6 (*a*) and beyond $\phi = 10.4$ by Fig 14.6.6 (*b*)

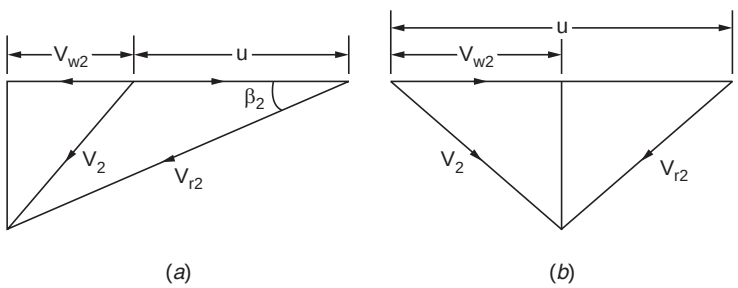


Figure 14.6.6 Exit velocity diagrams for pelton turbine

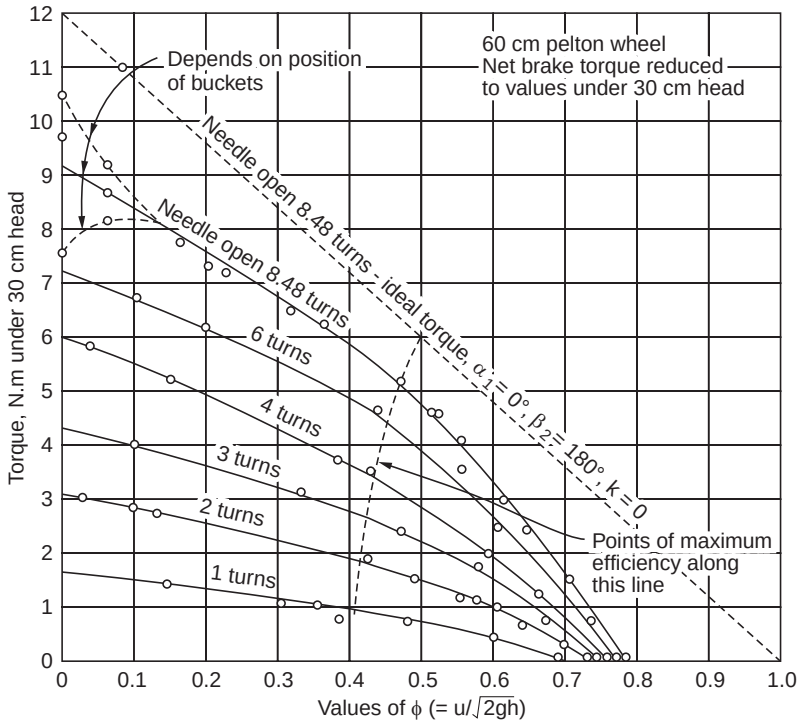


Figure 14.6.7 Relation between torque and speed ratio at constant head for various nozzle openings.

14.6.2 Torque and Power and Efficiency Variation with Speed Ratio

It is useful to study the variation torque with speed. Instead of speed the dimensionless speed ratio ϕ can be used for generality. In the ideal case the torque will be maximum at $u = 0$ or $\phi = 0$ and zero at $\phi = 1$, or $u = V_1$. **The actual variation of torque with speed ratio is shown in figure 14.6.7.** It is noted that the maximum efficiency lies in all cases between $\phi = 0.4$ and 0.5. Also torque is found to be zero at values less than $\phi = 1$. This is done to friction and exit loss ($V_2^2/2$) variation with various values of u.

The power variation for constant value of V_1 with ϕ is shown in figure 14.6.8.

The power can be calculated from the torque curves. In the ideal case power is zero both at $\phi = 0$ and $\phi = 1$. In the actual case power is zero even at ϕ is between 0.7 and 0.8. As the torque versus ϕ is not a straight line, the actual power curve is not a parabola.

The efficiency variation with speed ratio is similar to power versus speed ratio curve as the input $V_1^2/2$ is the same irrespective of u/V_1. Efficiency is some what higher in larger sizes as compared to small sizes homologous units. But this does not increase in the same proportion as the size. At higher heads any unit will operate at a slightly higher efficiency.

It is interesting to observe the variation of efficiency with load at a constant speed (Figure 14.6.9). Most units operate at a constant speed but at varying loads. The curve is rather flat and hence impulse turbine can be operated at lower loads with reduced losses.

It can be shown that the specific speed of impulse turbine is dependent on the jet diameter, (d) wheel diameter (D) ratio or called jet ratio in short.

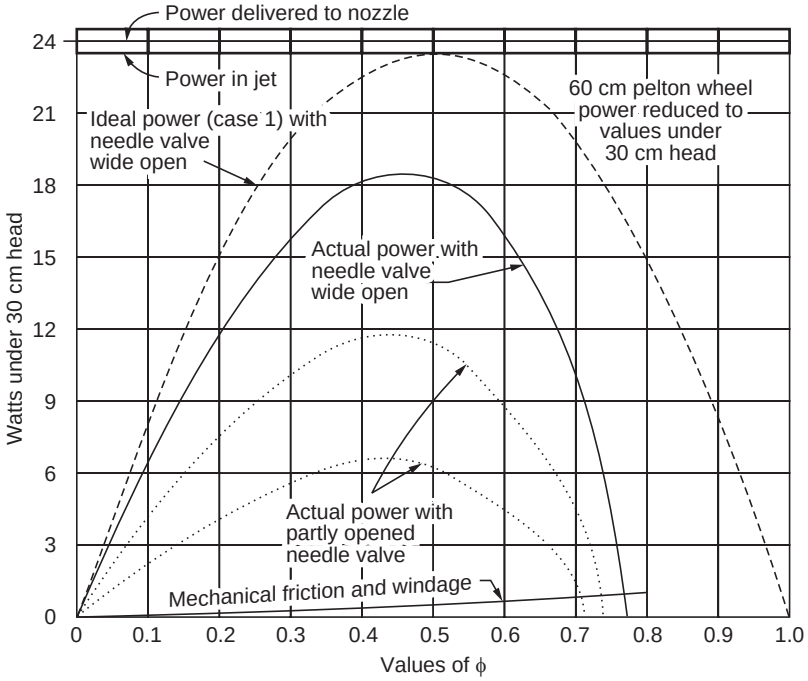


Figure 14.6.8 Speed ratio V_s power developed by pelton turbine

$$N_s \propto \frac{N\sqrt{P}}{H^{5/4}}$$

$$N \propto \frac{u}{D} \propto \frac{\phi}{D} V_1 \propto \frac{\phi\sqrt{H}}{D}$$

$$P \propto Q\,H \propto d_2\,V_1\,H \propto d^2\sqrt{H}\;H \propto d^2\,H^{3/2}$$

$$\sqrt{P} \propto d\,H^{3/4}$$

$$\therefore \qquad N_s \propto \frac{\phi\,H^{1/2}\,d\,.\,H^{3/4}}{H^{5/4}} \propto \phi\,\frac{d}{D} = \text{constant}\;\phi\;\frac{d}{D}$$

where d is the jet diameter. ϕ does not vary much and the constant made up of efficiency and C_v also does not vary much. Hence specific speed of an impulse turbine is mainly dependent on the jet diameter wheel diameter ratio. Inversely at the specific speed at which efficiency is maximum, there is a specific value of jet speed wheel speed ratio. For single nozzle unit, the best value of diamensional specific speed is about 17 and at that condition the wheel diameter is about 12 times the jet diameter. This is shown in figure 17.6.10.

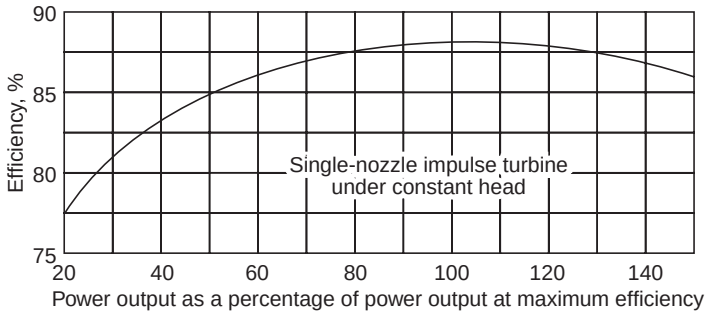


Figure 14.6.9 Variation of efficiency with load of constant speed for an impulse turbine.

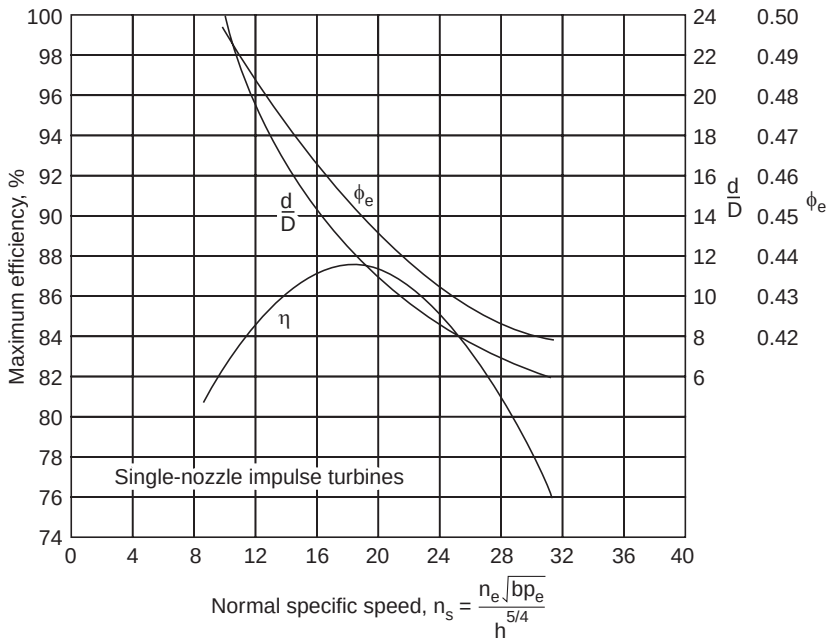


Figure 14.6.10 Variation of efficiency at the $\frac{D}{d}$ ratio for single jet inpulse turbine

14.7 REACTION TURBINES

The functioning of reaction turbines differs from impulse turbines in two aspects.

1. In the impulse turbine the potential energy available is completely converted to kinetic energy by the nozzles before the water enters the runner. The pressure in the runner is constant at atmospheric level.

In the case of reaction turbine the potential energy is partly converted to kinetic energy in the stater guide blades. The remaining potential energy is gradually converted to kinetic energy and absorbed by the runner. The pressure inside the runner varies along the flow.

2. In the impulse turbine only a few buckets are engaged by the jet at a time.

In the reaction turbine as it is fully flowing all blades or vanes are engaged by water at all the time. The other differences are that reaction turbines are well suited for low and medium heads (300 m to below) while impulse turbines are well suited for high heads above this value.

Also due to the drop in pressure in the vane passages in the reaction turbine the relative velocity at outlet is higher compared to the value at inlet. In the case of impulse turbine there is no drop in pressure in the bucket passage and the relative velocity either decreases due to surface friction or remains constant. In the case of reaction turbine the flow area between two blades changes gradually to accomodate the change in static pressure. In the case of impulse turbine the speed ratio for best efficiency is fixed as about 0.46. As there is no such limitation, reaction turbines can be run at higher speeds.

14.7.1 Francis Turbines

Francis turbine is a radial inward flow turbine and is the most popularly used one in the medium head range of 60 to 300 m. Francis turbine was first developed as a purly radial flow turbine by James B. Francis, an American engineer in 1849. But the design has gradually changed into a mixed flow turbine of today.

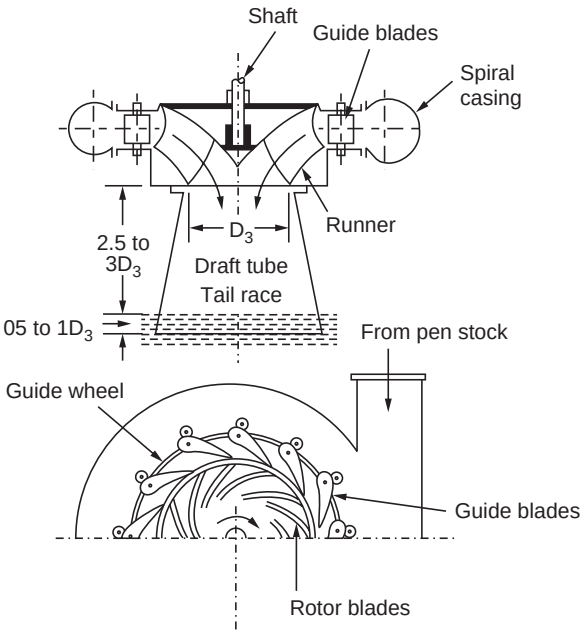


Figure 14.7.1 Typical sectional and front view of a modern Francis turbine.

A sectional view of a typical Francis turbine of today is shown in figure 14.7.1.

The main components are (*i*) The spiral casing (*ii*) Guide vanes (*iii*) Runner (*iv*) Draft tube and (*v*) Governor mechanism. Most of the machines are of vertical shaft arrangement while some smaller units are of horizontal shaft type.

14.7.1.1 Spiral Casing

The spiral casing surrounds the runner completely. Its area of cross section decreases gradually around the circumference. This leads to uniform distribution of water all along the circumference of the runner. Water from the penstock pipes enters the spiral casing and is distributed uniformly to the guide blades placed on the periphery of a circle. The casing should be strong enough to withstand the high pressure.

14.7.1.2 Guide Blades

Water enters the runner through the guide blades along the circumference. The number of guide blades are generally fewer than the number of blades in the runner. These should also be not simple multiples of the runner blades. The guide blades in addition to guiding the water at the proper direction serves two important functions. The water entering the guide blades are imparted a tangential velocity by the drop in pressure in the passage of the water through the blades. **The blade passages act as a nozzle in this aspect**.

The guide blades rest on pivoted on a ring and can be rotated by the rotation of the ring, whose movement is controlled by the governor. In this way the area of blade passage is changed to vary the flow rate of water according to the load so that the speed can be maintained constant. The variation of area between guide blades is illustrated in Figure 14.7.2. The control mechanism will be discussed in a later section.

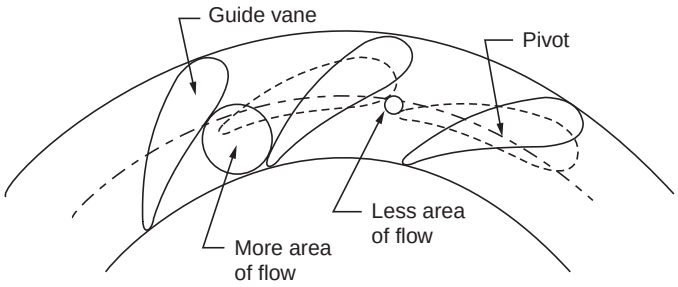


Figure 14.7.2. Guide vanee and giude wheel

14.7.1.3 The Runner

The runner is circular disc and has the blades fixed on one side. In high speed runners in which the blades are longer a circular band may be used around the blades to keep them in position.

The shape of the runner depends on the specific speed of the unit. These are classified as (*a*) slow runner (*b*) medium speed runner (*c*) high speed runner and (*d*) very high speed runner.

The shape of the runner and the corresponding velocity triangles are shown in figure 14.7.3. The development of mixed flow runners was necessitated by the limited power capacity of the purely radial flow runner. A larger exit flow area is made possible by the change of shape from radial to axial flow shape. This reduces the outlet velocity and thus increases efficiency. As seen in the figure the velocity triangles are of different shape for different runners. It is seen from the velocity triangles that the blade inlet angle β_1 changes from acute to obtuse as the speed increases. The guide vane outlet angle α_1 also increases from about 15° to higher values as speed increases.

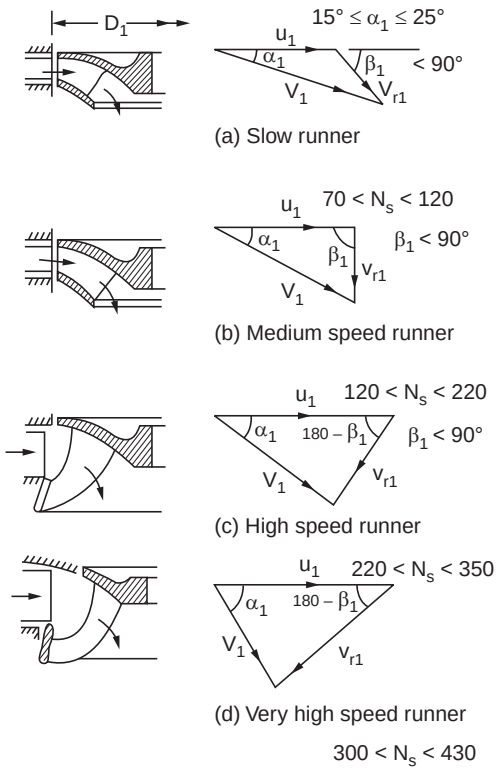


Figure 14.7.3 Variation of runner shapes and inlet velocity triangles with specific speed

In all cases, the outlet angle of the blades are so designed that there is no whirl component of velocity at exit ($V_{u2} = 0$) or absolute velocity at exit is minimum.

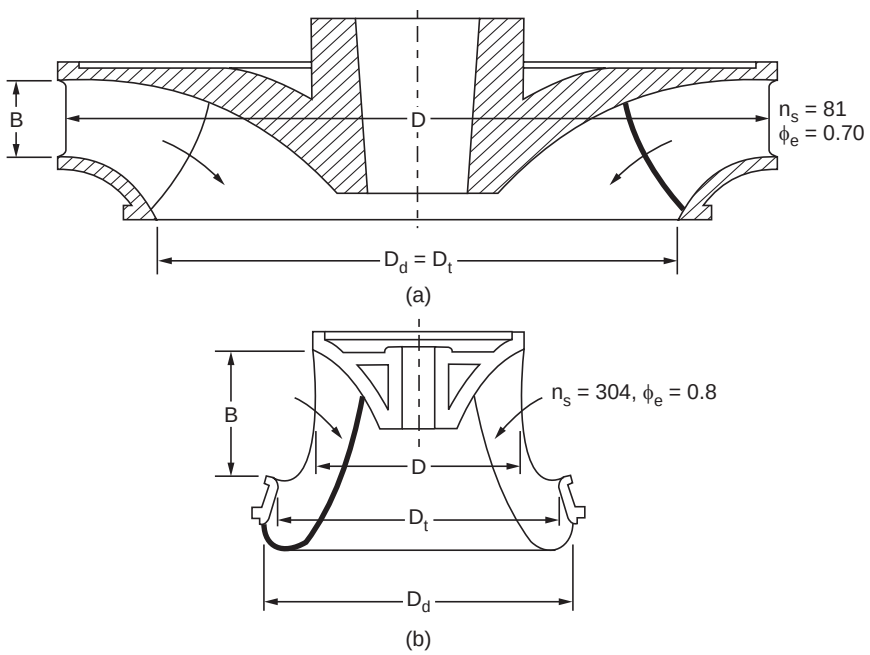


Figure 14.7.4 Slow speed and highspeed runner shapes.

The runner blades are of doubly curved and are complex in shape. These may be made separately using suitable dies and then welded to the rotor. The height of the runner along the axial direction (may be called width also) depends upon the flow rate which depends on the head and power which are related to specific speed. As specific speed increases the width also increase accordingly. Two such shapes are shown in figure 7.1.4.

The runners change the direction and magnitute of the fluid velocity and in this process absorb the momentum from the fluid.

14.7.1.4 Draft Tube

The turbines have to be installed a few meters above the flood water level to avoid innundation. In the case of impulse turbines this does not lead to significant loss of head. In the case of reaction turbines, the loss due to the installation at a higher level from the tailrace will be significant. This loss is reduced by connecting a fully flowing diverging tube from the turbine outlet to be immersed in the tailrace at the tube outlet. This reduces the pressure loss as the pressure at the turbine outlet will be below atmospheric due to the arrangement. **The loss in effective head is reduced by this arrangement. Also because of the diverging section of the tube the kinetic energy is converted to pressure energy which adds to the effective head.** The draft tube thus helps (1) to regain the lost static head due to higher level installation of the turbine and (2) helps to recover part of the kinetic energy that otherwise may be lost at the turbine outlet. A draft tube arrangement is shown in Figure 14.7.1 (as also in figure 14.7.5). Different shapes of draft tubes is shown in figure 14.7.6.

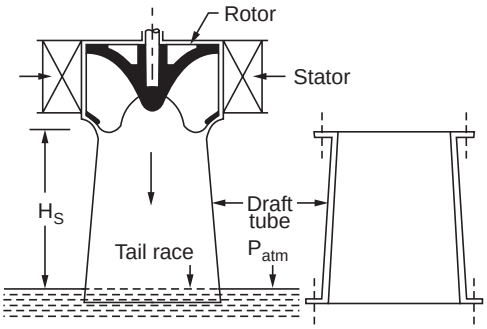


Figure 14.7.5 Draft tube

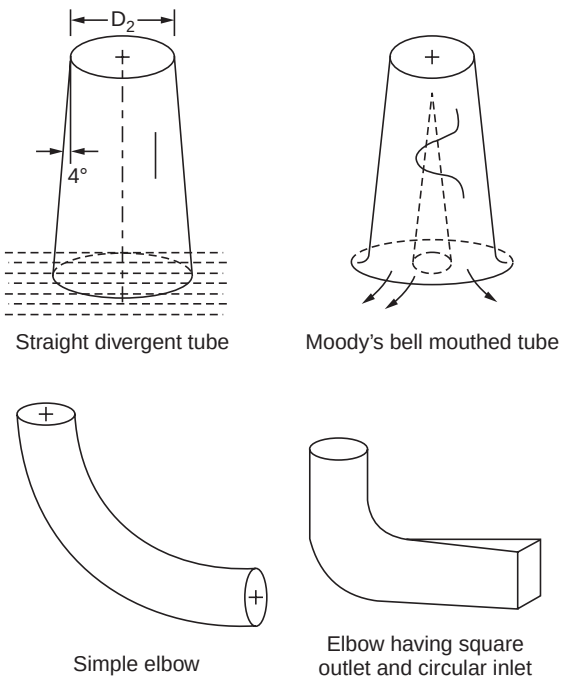


Figure 14.7.6 Various shapes of draft tubes

The head recovered by the draft tube will equal the sum of the height of the turbine exit above the tail water level and the difference between the kinetic head at the inlet and outlet of the tube less frictional loss in head.

$$H_d = H + (V_1^2 - V_2^2)/2g - h_f$$

where H_d is the gain in head, H is the height of turbine outlet above tail water level and h_f is the frictional loss of head.

Different types of draft tubes are used as the location demands. These are (*i*) Straight diverging tube (*ii*) Bell mouthed tube and (*iii*) Elbow shaped tubes of circular exit or rectangular exit.

Elbow types are used when the height of the turbine outlet from tailrace is small. Bell mouthed type gives better recovery. The divergence angle in the tubes should be less than 10° to reduce separation loss.

The height of the draft tube will be decided on the basis of cavitation. This is discussed in a later section.

The efficiency of the draft tube in terms of recovery of the kinetic energy is defined us

$$\eta = \frac{V_1^2 - V_2^2}{V_1^2}$$

where V_1 is the velocity at tube inlet and V_2 is the velocity at tube outlet.

14.7.1.5 Energy Transfer and Efficiency

A typical velocity diagrams at inlet and outlet are shown in Figure 14.7.7.

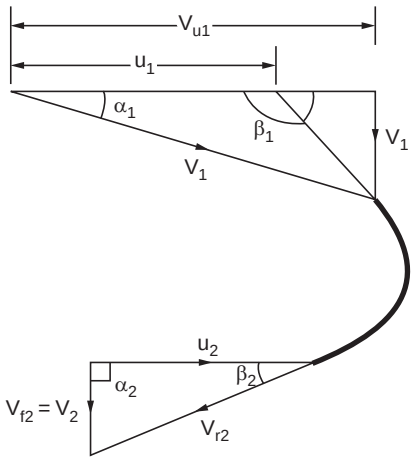


Figure 14.7.7 Velocity diagram for Francis runner

Generally as flow rate is specified and the flow areas are known, it is directly possible to calculate V_{f1} and V_{f2}. Hence these may be used as the basis in calculations. By varying the widths at inlet and outlet suitably the flow velocity may be kept constant also.

From Euler equation, power

$$P = \dot{m}(V_{u1}\, u_1 - V_{u2}\, u_2)$$

where $\dot{m}$ is the mass rate of flow equal to Q ρ where Q is the volume flow rate. As Q is more easily calculated from the areas and velocities, Q ρ is used by many authors in placed $\dot{m}$.

In all the turbines to minimise energy loss in the outlet the absolute velocity at outlet is minimised. This is possible only if $V_2 = V_{f2}$ and then $V_{u2} = 0$.

$\therefore$ $$P = \dot{m}\, V_{u1}\, u_1$$

For unit flow rate, the energy transfered from fluid to rotor is given by

$$E_1 = V_{u1}\, u_1$$

The energy available in the flow per kg is

$$E_a = g\, H$$

where H is the effective head available.

Hence the hydraulic efficiency is given by

$$\eta_H = \frac{V_{u1}\, u_1}{g\, H}$$

It friction and expansion losses are neglected

$$W = g\, H - \frac{{V_2}^2}{2g}$$

$\therefore$ It may be written in this case

$$\eta = \frac{gH - \dfrac{{V_2}^2}{2g}}{gH} = 1 - \frac{{V_2}^2}{2gH}.$$

The values of other efficiencies are as in the impulse turbine *i.e.* volumetric efficiency and mechanical efficiency and over all efficiency.

$$V_{f1} = Q / \pi\,(D_1 - zt)\, b_1 \,\underline{\Omega}\, Q / \pi\, D_1 b_1 \text{ (neglecting blade thickness)}$$

$$V_{u1} = u_1 + V_{f1} / \tan \beta_1 = u_1 + V_{f1} \cot \beta_1$$

$$= \frac{\pi\, D_1\, N}{60} + V_{f1} \cot \beta_1$$

$$u_1 = \frac{\pi\, D_1\, N}{60}$$

$\therefore$ $V_{u1}\, u_1$ can be obtained from Q_1 , D_1 , b_1 and N_1

For other shapes of triangles, the + sign will change to – sign as β_2 will become obtuse.

Runner efficiency or Blade efficiency

This efficiency is calculated not considering the loss in the guide blades.

From velocity triangle :

$$V_{u1} = V_{f1} \cot \alpha_1$$

$$u_1 = V_f [\cot \alpha_1 + \cot \beta_1]$$

$$\therefore \quad u_1 V_{u1} = V_{f1}^2 \cot \alpha_1 [\cot \alpha_1 + \cot \beta_1]$$

Energy supplied to the runner is

$$u_1 V_{u1} + \frac{V_2^2}{2} = u_1 V_{u1} + \frac{V_{f2}^2}{2} = u_1 V_{u1} + \frac{V_{f1}^2}{2}$$

(Assume $V_{f2} = V_{f1}$)

$$\therefore \quad \eta_b = \frac{V_{f1}^2 \cot \alpha_1 [\cot \alpha_1 + \cot \beta_1]}{\frac{V_{f1}^2}{2} + V_{f1}^2 \cot \alpha_1 [\cot \alpha_1 + \cot \beta_1]}$$

Multiply by 2 and add and subtract V_{f1}^2 in the numerator to get

$$\eta_b = 1 - \frac{1}{1 + 2\cot \alpha_1 (\cot \alpha_1 + \cot \beta_1)}$$

In case $\beta_1 = 90°$

$$\eta_b = 1 - \frac{1}{1 + \frac{2}{\tan^2 \alpha_1}} = \frac{2}{2 + \tan^2 \alpha_1}$$

In this case $\mathbf{V_{u1} = u_1}$

The characteristics of Francis turbine is shown in Figure 14.7.8.

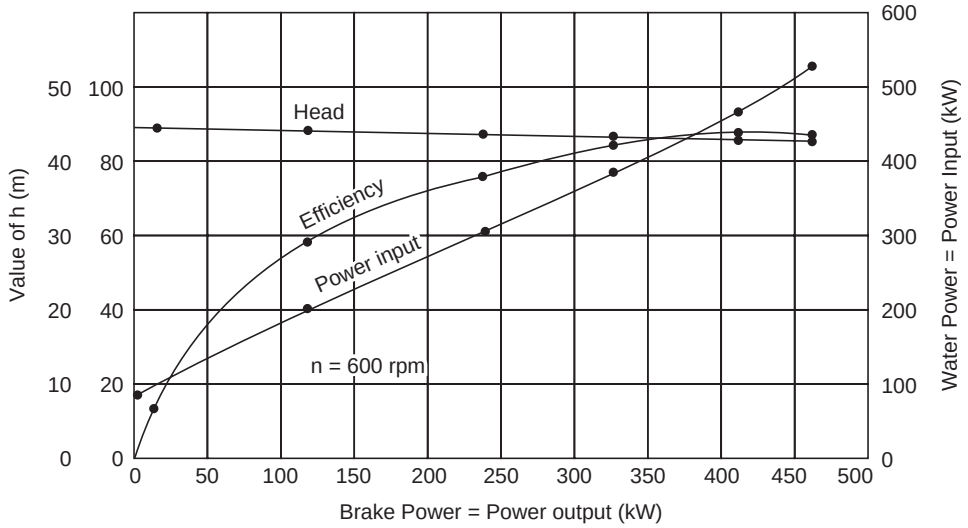


Figure 14.7.8

The efficiency curve is not as flat as that of impulse turbine. At part loads the efficiency is relatively low. There is a drop in efficiency after 100% load.

The characteristics of Francis turbine at various speeds but at constant head is shown in figure 14.7.9.

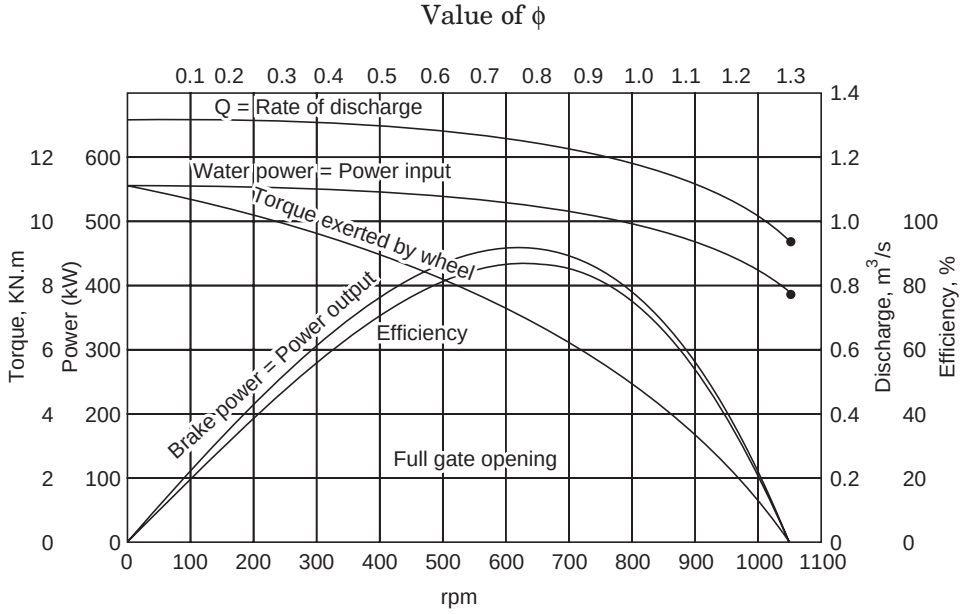


Figure 14.7.9 Francis turbine characteristics at variable speed and constant head

14.8 AXIAL FLOW TURBINES

The popular axial flow turbines are the Kaplan turbine and propeller turbine. In propeller turbine the blades are fixed. In the Kaplan turbines the blades are mounted in the boss in bearings and the blades are rotated according to the flow conditions by a servomechanism maintaining constant speed. In this way a constant efficiency is achieved in these turbines. The system is costly and where constant load conditions prevail, the simpler propeller turbines are installed.

In the discussions on Francis turbines, it was pointed out that as specific speed increases (more due to increased flow) the shape of the runner changes so that the flow tends towards axial direction. This trend when continued, the runner becomes purely axial flow type.

There are many locations where large flows are available at low head. In such a case the specific speed increases to a higher value. In such situations axial flow turbines are gainfully employed. A sectional view of a kaplan turbines in shown in figure 14.8.1. These turbines are suited for head in the range 5 – 80 m and specific speeds in the range 350 to 900. The water from supply pipes enters the spiral casing as in the case of Francis turbine. Guide blades direct the water into the chamber above the blades at the proper direction. The speed governor in this case acts on the guide blades and rotates them as per load requirements. The flow rate is changed without any change in head. The water directed by the guide blades enters the runner which has much fewer blades (3 to 10) than the Francis turbine. The blades are also rotated by the governor to change the inlet blade angle as per the flow direction from the guide blades, so

that entry is without shock. As the head is low, many times the draft tube may have to be elbow type. The important dimensions are the diameter and the boss diameter which will vary with the chosen speed. At lower specific speeds the boss diameter may be higher.

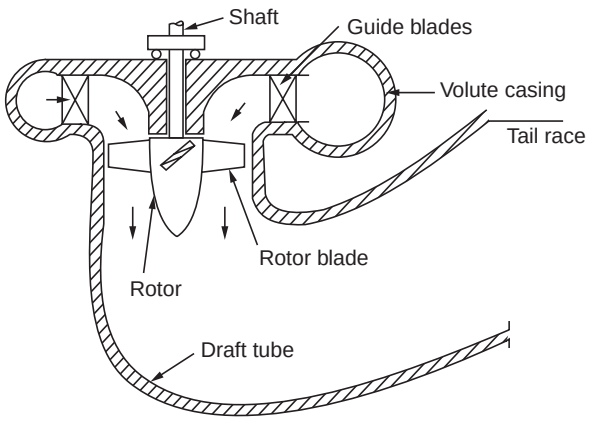


Figure 14.8.1 Sectional view of kaplan furbine

The number of blades depends on the head available and varies from 3 to 10 for heads from 5 to 70 m. As the peripheral speed varies along the radius (proportional to the radius) the blade inlet angle should also vary with the radius. Hence twisted type or Airfoil blade section has to be used. The speed ratio is calculated on the basis of the tip speed as $\phi = u/\sqrt{2gH}$ and varies from 1.5 to 2.4. The flow ratio lies in the range 0.35 to 0.75.

Typical velocity diagrams at the tip and at the hub are shown in Figure 14.8.2. The diagram is in the axial and tangential plane instead of radial and tangential plane as in the other turbines.

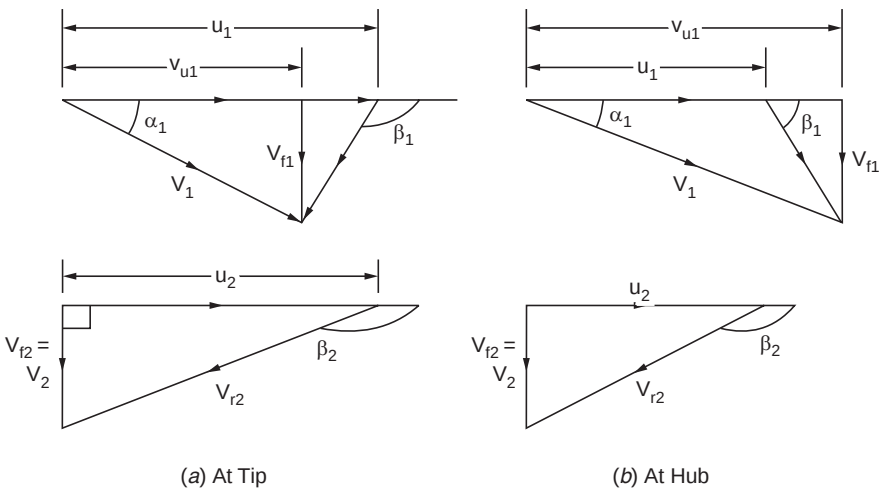


Figure 14.8.2 Typical velocity diagrams for Kaplan turbine

Work done = $u_1 V_{u1}$ (Taken at the mean diameter)

$$\eta_H = \frac{u_1 V_{u1}}{g H}$$

All other relations defined for other turbines hold for this type also. The flow velocity remains constant with radius. As the hydraulic efficiency is constant all along the length of the blades, $u_1 V_{u1}$ = Constant along the length of the blades or V_{u1} decreases with redius.

Kaplan turbine has a flat characteristics for variation of efficiency with load. Thus the part load efficiency is higher in this case. In the case of propeller turbine the part load efficiency suffers as the blade angle at these loads are such that entry is with shock.

The load efficiency characteristic of the four types of turbines is shown in figure 14.8.3.

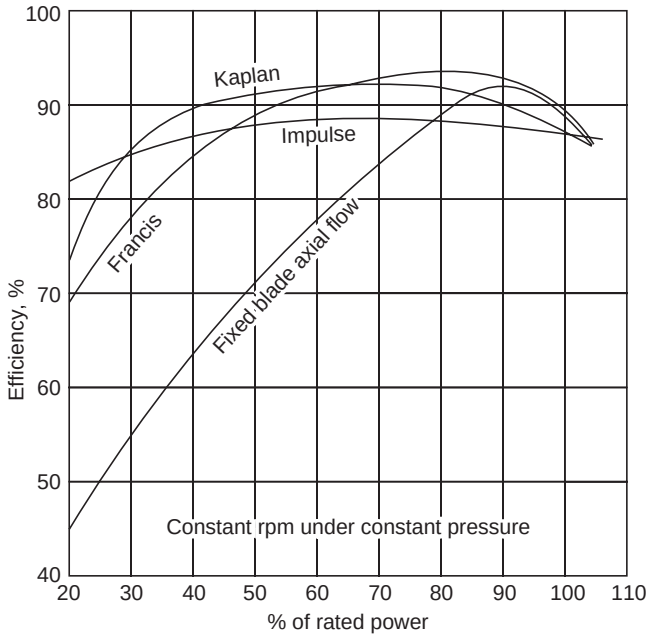


Figure 14.8.3 Load efficiency characteristics of hydraulic turbines

14.9 CAVITATION IN HYDRAULIC MACHINES

If at any point in the flow the pressure in the liquid is reduced to its vapour pressure, the liquid will then will boil at that point and bubbles of vapour will form. As the fluid flows into a region of higher pressure the bubbles of vapour will suddenly condense or collapse. This action produces very high dynamic pressure upon the adjacent solid walls and since the action is continuous and has a high frequency the material in that zone will be damaged. Turbine runners and pump impellers are often severely damaged by such action. The process is called cavitation and the damage is called cavitation damage. In order to avoid cavitation, the absolute pressure at all points should be above the vapour pressure.

Cavitation can occur in the case of reaction turbines at the turbine exit or draft tube inlet where the pressure may be below atmospheric level. In the case of pumps such damage may occur at the suction side of the pump, where the absolute pressure is generally below atmospheric level.

In addition to the damage to the runner cavitation results in undesirable vibration noise and loss of efficiency. The flow will be disturbed from the design conditions. In reaction turbines the most likely place for cavitation damage is the back sides of the runner blades near their trailing edge. The critical factor in the installation of reaction turbines is the vertical distance from the runner to the tailrace level. For high specific speed propeller units it may be desirable to place the runner at a level lower than the tailrace level.

To compare cavitation characteristics a cavitation parameter known as Thoma cavitation coefficient, σ, is used. It is defined as

$$\sigma = \frac{h_a - h_r - z}{h} \tag{14.8.1}$$

where h_a is the atmospheric head h_r is the vapour pressure head, z is the height of the runner outlet above tail race and h is the total operating head. The minimum value of σ at which cavitation occurs is defined as critical cavitation factor σ_e. Knowing σ_c the maximum value of z can be obtained as

$$z = h_a - h_v - \sigma_e h \tag{14.8.2}$$

σ_c is found to be a function of specific speed. In the range of specific speeds for Francis turbine σ_c varies from 0.1 to 0.64 and in the range of specific speeds for Kaplan turbine σ_c varies from 0.4 to 1.5. The minimum pressure at the turbine outlet, h_0 can be obtained as

$$h_0 = h_a - z - \sigma_c H \tag{14.8.3}$$

There are a number of correlations available for the value of σ_c in terms of specific speed, obtained from experiments by Moody and Zowski. The constants in the equations depends on the system used to calculate specific speed.

For Francis runners $\sigma_c = 0.006 + 0.55\ (N_s/444.6)^{1.8}$ (14.8.4)

For Kaplan runners $\sigma_c = 0.1 + 0.3\ [N_s/444.6]^{2.5}$ (14.8.5)

Other empirical corrlations are

Francis runner $$\sigma_c = 0.625 \left[\frac{N_s}{380.78}\right]^2 \tag{14.8.6}$$

For Kaplan runner $$\sigma_c = 0.308 + \frac{1}{6.82}\left(\frac{N_s}{380.78}\right)^2 \tag{14.8.7}$$

Example 14.9. *The total head on a Francis turbine is 20 m. The machine is at an elevation where the atmosphic pressure is 8.6 m. The pressure corresponding to the water temperature of 15° C is 0.17 m. It critical cavitation factor is 0.3, determine the level of the turbine outlet above the tail race.*

$$\mathbf{z} = P_a - P_v - \sigma_c h = 8.6 - 0.17 - 0.3 \times 20 = \mathbf{2.43\ m.}$$

The turbine outlet can be set at 2.43 m above the tailrace level.

14.9 GOVERNING OF HYDRAULIC TURBINES

Hydraulic turbines drive electrical generators in power plants. The frequency of generation has to be strictly maintained at a constant value. This means that the turbines should run at constant speed irrespective of the load or power output. It is also possible that due to electrical tripping the turbine has to be stopped suddenly.

The governing system takes care of maintaining the turbine speed constant irrespective of the load and also cutting off the water supply completely when electrical circuits trip.

When the load decreases the speed will tend to rise if the water supply is not reduced. Similarly when suddenly load comes on the unit the speed will decrease. The governor should step in and restore the speed to the specified value without any loss of time.

The **governor should be sensitive** which means that it should be able to act rapidly even when the change in speed is small. At the same time **it should not hunt**, which means that there should be no ups and downs in the speed and stable condition should be maintained after the restoration of the speed to the rated value. It should not suddenly cut down the flow completely to avoid damage to penstock pipes.

In hydraulic power plants the available head does not vary suddenly and is almost constant over a period of time. So **governing can be achieved only by changing the quantity of water that flows into the turbine runner**. As already discussed the water flow in pelton turbines is controlled by the spear needle placed in the nozzle assembly. The movement of the spear is actuated by the governor to control the speed. In reaction turbines the guide vanes are moved such that the flow area is changed as per the load requirements.

Hydraulic system is used to move the spear in the nozzle or to change the positions of the guide blades because the force required is rather high.

The components of governing system are

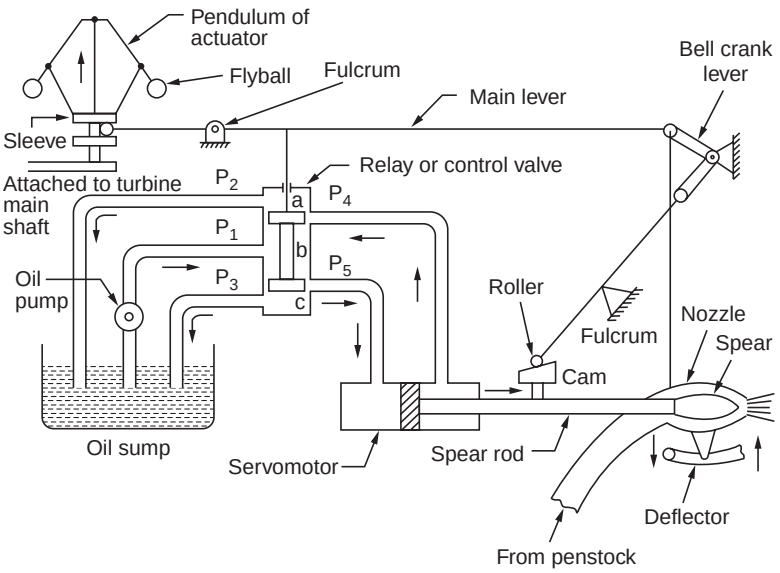


Figure 14.9.1 Governing system for Pelton turbine

(*i*) The speed sensing element which actuates the system (*ii*) Hydraulic power pack with suitable pump and valves. (*iii*) Distributing valve also called relay valve (*iv*) Power cylinder which provides the force required.

In the older systems a centrifugal governor was used as the sensing element. In the modern system electronic means of frequency detection is used to actuate the system.

The older type of system used in the case of pelton turbine is shown in figure 14.9.1.

The mechanical centrifugal governor is driven by the turbine shaft. The weights carry a sleeve which can move up and down the drive spindle. When the load decreases the turbine speeds up and the governor weights fly apart moving the sleeve up. The reverse happens when load increase on the turbine. The sleeve carries a lever which moves the control value in the relay cylinder. Oil under pressure is maintained at the central position of the realy cylinder. The top and bottom are connected on one side to the power cylinder and to the sump on the other side. Under steady load conditions the value rod closes both inlets to the power cylinder and the spear remains at a constant position. When the turbine speeds up, the valve rod moves down connecting the oil supply to the left side of the power cylinder. The piston in the power cylinder mover to reduce the flow. At the some time the right side of the power cylinder is connected to the sump so that the oil in the right side can flow out. The opposite movement takes place when the turbine speed reduces.

As sudden cut off is not desirable, a deflector is actuated by suitable mechanism to deflect the flow when sudden and rapid increase in speed takes place.

In the case of reaction turbines, the power cylinder and the sensing system are the same. The guide vanes are mounted on a ring and so mounted that these rotate when the ring rotates. The rotation of the ring is actuated by the power cylinder when the load changes. This part of the system is shown in figure 14.9.2.

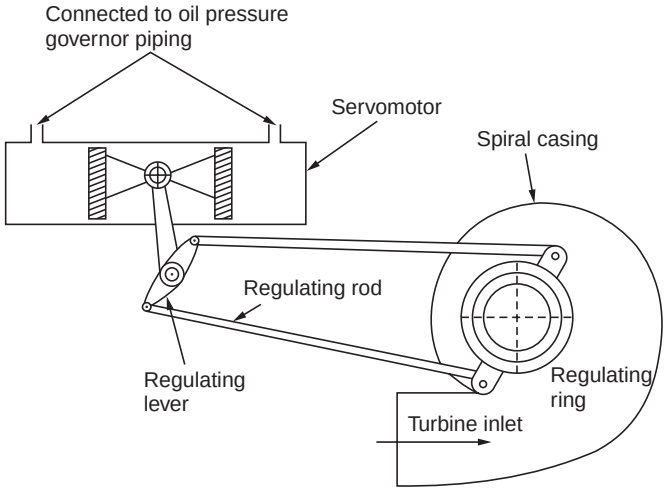


Figure 14.9.2 Reaction turbine governor linkage

WORKED EXAMPLES

Problem 14.1 *A lawn sprinkler is shown in figure. The sectional area at outlet is 1 cm^2. The flow rate is 1ls on each side.*

Calculate the angular speed of rotation and the torque required to hold it stationary. Neglect friction.

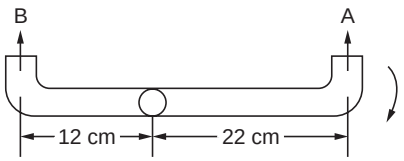


Figure P. 14.1

The flow velocity is $10^{-3}/10^{-4}$ = **10 m/s.**

The jets A and B exert forces in the opposite direction. As arm A is longer the sprinkler will rotate in the clockwise direction. Let it rotate at an angular velocity ω.

The absolute velocity of

Jet A = (10 – 0.22 ω).

The absolute velocity of jet B = 10 + 0.12 ω

As no external torque is applied and as there is no friction, the resultant torque is zero.

$\therefore$ $(10 - 0.22\ \omega)\ 0.22 = (10 + 0.12\ \omega)\ 0.12$

Solving ω **= 15.923** radions/second

$$\omega = \frac{2\pi N}{60}, \text{ Substituting and solving for } N$$

N = 152 rpm

Torque when stationary : (as mass) flow is equal = 1 kg/s

$$= 1(V_a r_a - V_B r_b) = 1(10 \times 0.22 - 10 \times 0.12)$$

= 1 Nm.

Problem 14.2 *A 20 cm pipe 600 m long with friction factor of 0.02 carries water from a reservoir to a turbine with a difference in head of 90 m. The friction loss in the nozzle is 0.05 V_s^2/ 2g.* ***Determine the diameter of the jet which will result in maximum power.***

The energy equation is

$$90 - \frac{600 \times 0.02 \times V_p^2}{2g \times 0.2} - \frac{0.05\, V_j^2}{2g} = \frac{V_j^2}{2g}$$

$$90 \times 2 \times 9.81 - \frac{600 \times 0.02}{0.2}\ V_p^2 = 1.05\ V_j^2$$

$$V_p = V_j D_j^2 / 0.2^2,\ V_p^2 = V_j^2 D_j^4 / 0.2^4 = 625\ V_j^2 D_j^4$$

$$1765.8 - \frac{600 \times 625 \times 0.02}{0.2}\ .\ V_j^2 D_j^4 = 1.05\ V_j^2$$

or $V_j^2 (1.05 + 37500\, D_j^4) = 1765.8$

Solving by trial by assuming D_j, power in the jet is determined as $\dot{m}\ V_j^2 / 2$

D_j, m	V_j, m/s	$\dot{m} = (\pi D_j^2/4)\, V_j \times 1000$	Power in jet = $\frac{\dot{m} V_1^2}{2}$ / 1000 kW
0.04	40.92	51.40 kg/s	43.05 kW
0.06	33.90	95.85	50.07
0.08	26.39	132.60	46.19
0.10	19.18	150.64	27.7

Maximum power is around D_j = 0.06 m or 60 mm.

Problem 14.3 *At a location selected for installation of a hydro electric plant, the head available was estimated as 115 m and water flow rate was estimated as 15 m^3/s. For convenience of maintenance it is desired to select two units for the plant. Select turbines.*

This is a problem for which there could be a number of solutions. At first sight, the head will suggest two Francis turbines.

$$\text{Power} = 15 \times 10^3 \times 9.81 \times 115 = 15230 \times 10^3 \text{ W}.$$

In order to calculate the specific speed, the working speed of the turbine is required. Let us try 250 rpm (for 50 cycle operation, 12 pairs of pole generator)

$$N_s = \frac{250}{60} \frac{\sqrt{15230 \times 10^3 / 2}}{115^{1.25}} = 30.53$$

This is not a suitable range for Francis turbine. A higher speed of operation say 500 rpm will give $N_s \underline{\Omega}$ 60. Which is for a narrow rotor, which may not be suitable.

Let us try impulse turbine : operating at 125 rpm.

N_s = 15.26, A single nozzle unit can be selected.

Diameter : Assuming speed ratio of 0.46

$$u = 0.46 \sqrt{2 \times 9.81 \times 115} = 21.85 \text{ m/s}$$

$$\therefore \quad D = \frac{21.85 \times 60}{\pi \times 125} = 3.34 \text{ m}$$

Doubling the speed will reduce the diameter but N_s also will be doubled and twin nozzle unit may have to be chosen.

Problem 14.4 *A turbine operates at 500 rpm at a head of 550 m. A jet of 20 cm is used. Determine the specific speed of the machine. Assume C_v = 0.97 and efficiency is 88%. If ϕ = 0.46* ***determine the pitch diameter of the runner.***

$$\text{Power} = \eta\ \dot{m} g H = \eta \cdot \frac{\pi D_j^2}{4} \times V_j\, g\, H \times \rho$$

$$V_j = 0.97\sqrt{2g\,H}$$

$$\therefore \quad \text{Power} = 0.85 \times \frac{\pi \times 0.2^2}{4} \times 1000 \times 0.97\sqrt{2 \times 9.81 \times 550} \times 9.81 \times 550/1000$$

$$= 14517.9 \text{ kW} = 14.518 \times 10^6 \text{ W}$$

$$N_s = \frac{500}{60}\frac{\sqrt{145178 \times 10^6}}{550^{5/4}} = 11.92$$

Dimensionless specific speed is = 0.022

A single jet pelton turbine is suitable

$$V_j = 0.97\sqrt{2 \times 9.81 \times 550}\,,\ u = 0.46\ V_j$$

$$u = 0.46 \times 0.97\sqrt{2 \times 9.81 \times 550} = 46.35 \text{ m/s}$$

$$u = \frac{\pi DN}{60},\ \boldsymbol{D} = \frac{60\,u}{\pi N} = \frac{60 \times 46.35}{\pi \times 500} \textbf{ = 1.77 m}$$

Problem 14.5 *At a location for a hydroelectric plant, the head available (net) was 335 m. The power availability with an overall efficiency of 86% was 15500 kW. The unit is proposed to run at 500 rpm. Assume* $C_v = 0.98$, $\phi = 0.46$, *Blade velocity coefficient is 0.9. If the bucket outlet angle proposed is 165°* ***check for the validity of the assumed efficiency***

First flow rate is calculated

$$Q = \frac{15500000}{0.86 \times 1000 \times 9.81 \times 335} = 5.484 \text{ m}^3/\text{s}$$

Jet velocity is next calculated

$$V_j = 0.98\sqrt{2 \times 9.81 \times 335} = 79.45 \text{ m/s}$$

Blade velocity $u = 0.46 \times 79.45 = 36.55$ m/s

Runner diameter $D = \dfrac{36.55 \times 60}{\pi \times 500} = 1.4$ m

Jet diameter assuming single jet,

$$d = \left(\frac{Q \times 4}{\pi V_j}\right)^{0.5} = \left(\frac{5.484 \times 4}{\pi \times 79.45}\right)^{0.5} = 0.296 \text{ m}$$

$$\frac{D}{d} = \frac{1.4}{0.296} = 4.72, \text{ not suitable should be at least 10.}$$

Assume 4 jets, then

$$d = \left(\frac{5.484 \times 4}{4 \times \pi \times 79.45}\right)^{0.5} = 0.1482 \text{ m},\ \frac{D}{d} = 9.5$$

may be suggested

$$\text{Per jet, } N_s = \frac{500}{60}\frac{\sqrt{15500000/4}}{335^{5/4}} = 11.44$$

Dimensionless $N_s = 0.0208$ $\therefore$ acceptable

Such units are in operation in Himachal Pradesh.

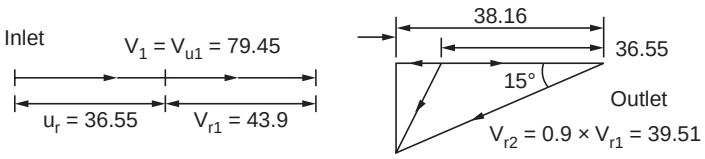


Figure P. 14.5

The velocity diagrams are given above

$$V_{u1} = 79.45,\ V_{u2} = 38.16 - 36.55 = 1.61 \text{ m/s}$$

$\therefore$ $$\text{W/kg} = 36.55(79.45 + 1.61) = 2962.9 \text{ Nm/kg}$$

$$\eta_H = 2962.9\ (9.81 \times 335) = \mathbf{0.9 \text{ or } 90\%}$$

Assumed value is lower as it should be because, overall efficiency < hydraulic efficiency.

Problem 14.6. *A Pleton turbine running at 720 rmp uses 300 kg of water per second. If the head available is 425 m* ***determine the hydraulic efficiency****. The bucket deflect the jet by 165°. Also* ***find the diameter of the runner and jet.*** *Assume $C_v = 0.97$ and $\phi = 0.46$, Blade velocity coefficient is 0.9.*

The velocity diagram is shown in figure

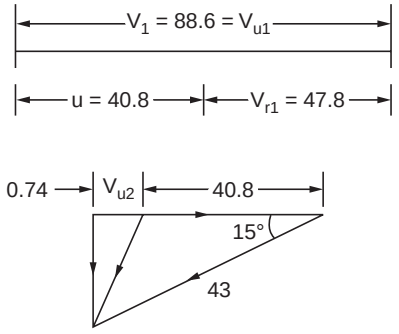


Figure P. 14.6

$$V_j = C_v\sqrt{2g\,H} = 0.97\sqrt{2 \times 9.81 \times 425} = 88.6 \text{ m/s}$$

$$u = 0.46 \times 88.6 = 40.8 \text{ m/s}$$

$$V_{r1} = 88.6 - 40.8 = 47.8 \text{ m/s}$$

$$V_{r2} = 0.9 \times 47.8 = 43 \text{ m/s}$$

$$V_{u1} = 88.6 \text{ m/s}$$

$$V_{u2} = 43 \cos 15 - 40.8 = 0.74 \text{ m/s}$$

$$\textbf{Power} = 300 \times 40.8\ (88.6 + 0.74)/1000 = \textbf{1093.5 kW}$$

$$\textbf{Hydraulic efficiency} = \frac{1093.5 \times 10^3 \times 2}{300 \times 88.6^2} = 0.9286 = \textbf{92.86\%}$$

$$D = \frac{u \times 60}{\pi N} = \frac{40.8 \times 60}{\pi \times 720} = \textbf{1.082 m}$$

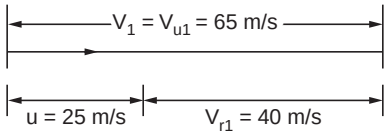
$$d = \left(\frac{4Q}{\pi V_1}\right)^{0.5} = \left(\frac{4 \times 0.3}{\pi \times 88.6}\right)^{0.5} = \textbf{0.06565 m}$$

$$\frac{D}{d} = \frac{1.082}{0.06565} = 16.5$$

$$\text{Overall efficiency} = \frac{1093.5 \times 10^3}{300 \times 9.81 \times 425} = 0.8743 \text{ or } 87.43\%$$

$$N_s = \frac{720}{60} \cdot \frac{\sqrt{1093.5 \times 10^3}}{425^{5/4}} = 3.79$$

Problem 14.7 *The jet velocity in a pelton turbine is 65 m/s. The peripheral velocity of the runner is 25 m/s. The jet is defleted by 160° by the bucket.* ***Determine the power developed and hydraulic efficiency*** *of the turbine for a flow rate of 0.9 m^3/s. The blade friction coefficient is 0.9.*



$$V_1 = V_{u1} = 65 \text{ m/s}$$

$$u = 25 \text{ m/s}$$

$$V_{r1} = 65 - 25 = 40 \text{ m/s}$$

$$V_{r2} = 0.9 \times V_{r1} = 36 \text{ m/s}$$

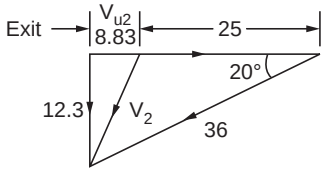


Figure P. 14.7

As 36 cos 20 = 33.82 < 25 the shape of the exit triangle is as in figure.

$$V_{u2} = 36 \cos 20 - 25$$

$$= 33.83 - 25 = 8.83 \text{ m/s}$$

In the opposite direction of V_{u1} hence addition

$$P = 900 \times 25\ (65 + 8.83) = \textbf{1.661 × 10}^{\textbf{6}}\ \textbf{W}$$

$$\eta_H = \frac{1.661 \times 10^6 \times 2}{900 \times 65^2} = \textbf{87.37\%}$$

$$\text{Exit loss} = m \frac{V_2^2}{2}$$

$$V_2^2 = 36^2 + 25^2 - 2 \times 36 \times 25 \times \cos 20 = 229.55$$

$\therefore$ **Exit loss of power** = $\dfrac{900 \times 229.35}{2}$ = **103.3 × 10³ W**

Problem 14.8. *A Pelton turbine is to produce 15 MW under a head of 480 m when running at 500 rpm. If D/d = 10,* ***determine the number of jets required.***

Assume $\eta_0 = 85\%$, $C_v = 0.97$, $\phi = 0.46$

$$Q = \frac{15 \times 10^6}{0.85 \times 1000 \times 9.81 \times 480} = 3.75 \text{ m}^3/\text{s}$$

$$V_j = 0.97\sqrt{2 \times 9.81 \times 480} = 94.13 \text{ m/s}$$

$$u = 0.46 \times 94.13 = 43.3 \text{ m/s}$$

$$D = \frac{43.3 \times 60}{\pi \times 500} = 1.65 \text{ m.}$$

$\therefore$ $d = 0.165$ m

Volume flow in a jet = $\dfrac{\pi \times 0.165^2}{4} \times 94.13 = 2.01 \text{ m}^3/\text{s}$

Total required = 3.75. $\therefore$ **Two jets will be sufficient.**

The new diameter of the jets

$$\boldsymbol{d'} = \left(\frac{4 \times 3.75}{2 \times \pi \times 94.13}\right)^{0.5} = \mathbf{0.1593 \text{ m}}$$

$$N_s = \frac{500}{60} \cdot \frac{\sqrt{15 \times 10^6}}{480^{5/4}} = 14.37$$

Problem 14.9. *The head available at a location was 1500 m. It is proposed to use a generator to run at 750 rpm. The power available is estimated at 20,000 kW. Investigate whether a single jet unit will be suitable. Estimate the number of jets and their diameter.* ***Determine the mean diameter of the runner and the number of buckets.***

The specific speed is calculated to determine the number of jets.

$$N_s = \frac{750}{60} \frac{\sqrt{20{,}000{,}000}}{1500^{5/4}} = 5.99$$

So a single jet unit will be suitable.

In order to determine the jet diameter, flow rate is to be calculated. The value of overall efficiency is necessary for the determination. It is assumed as 0.87

$$20{,}000{,}000 = 0.87 \times Q \times 1000 \times 9.81 \times 1500$$

$\therefore$ $Q = 1.56225 \text{ m}^3/\text{s}$

To determine the jet velocity, the value of C_v is required. It is assumed as 0.97

$$V = 0.97\sqrt{2 \times 9.81 \times 1500} = 166.4 \text{ m/s}$$

$$\therefore \quad 1.56225 = \frac{\pi d^2}{4} \times 166.4. \text{ Solving, } d = 0.1093 \text{ m}$$

In order to determine the runner diameter, the blade velocity is to be calculated. The value of ϕ is assumed as 0.46.

$$\therefore \quad u = 166.4 \times 0.46 \text{ m/s}$$

$$\frac{\pi DN}{60} = u,$$

$$\therefore \quad \frac{166.4 \times 0.46 \times 60}{\pi \times 750} = D \quad \therefore \quad \boldsymbol{D = 1.95 \text{ m}}$$

The **number of buckets** $= Z \cdot \frac{D}{2d} + 15 = \frac{1.95}{2 \times 0.1093} + 15$

$$= 8.9 + 15 \simeq \textbf{24 numbers.}$$

Problem 14.10. *At a location selected to install a hydro electric plant, the head is estimated as 550 m. The flow rate was determined as 20 m^3/s. The plant is located at a distance of 2 m from the entry to the penstock pipes along the pipes. Two pipes of 2 m diameter are proposed with a friction factor of 0.029. Additional losses amount to about 1/4th of frictional loss. Assuming an overall efficiency of 87%,* ***determine how many single jet unit running at 300 rpm*** *will be required.*

The specific speed is to be determined first.

Net head = Head available – loss in head.

Frictional loss $= f L V_p^2 / 2g D.$

$V_p \times A_p \times$ number of pipes $= Q = 20 \text{ m}^3/\text{s}$

$$\therefore \quad V_p = 20 \Big/ \left(\frac{\pi 2^2}{4} \times 2 \right) = 3.183 \text{ m/s}$$

$$L = 2000 \text{ m}, D = 2 \text{ m}, f = 0.029$$

$$\therefore \quad h_f = 0.029 \times 2000 \times 3.183^2 / 2 \times 9.81 \times 2 = 14.98 \text{ m.}$$

Total loss of head $= \frac{5}{4} \times 14.8 = 18.72 \text{ m}$

$$\therefore \quad \text{Net head} = 550 - 18.72 = 531.28 \text{ m}$$

$$\text{Power} = \eta \times Q \times \rho \times g \times H = 0.87 \times 20 \times 1000 \times 9.81 \times 531.28$$

$$= 90.6863 \times 10^6 \text{ W}$$

$$N_s = \frac{300}{60} \cdot \frac{\sqrt{90.6863 \times 10^6}}{531.28^{5/4}} = 18.667$$

Dimensionless specific speed = 0.034

This is within the range for a single jet unit. Discussion of other consideration follow.

Discussions about suitability of single jet unit.

$$V_j = C_v \sqrt{2g\,H} = 0.98 \times \sqrt{2 \times 9.81 \times 531.28}$$

$$= 100.05 \text{ m/s}$$

$$\frac{\pi d^2}{4} \times V_j = Q \quad \therefore \quad d = \left(\frac{4Q}{\pi V_j} \right)^{0.5}$$

Jet diameter, d $= \left(\dfrac{4 \times 20}{\pi \times 100.05} \right)^{0.5}$ **= 0.5 m** (farily high)

$$\frac{\pi DN}{60} = 0.46 \times 100.05, \quad \therefore \quad D = \frac{0.46 \times 100.05 \times 60}{300\,\pi} = 2.93 \text{ m}$$

$$\text{Jet speed ratio} = \frac{2.95}{0.5} = 6 \text{ too low.}$$

Consider a twin jet unit in which case, d = 0.35 m and Jet speed ratio : 8.3. low side.

If three jets are suggested, then, d = 0.29

Jet speed ratio is about 10. Suitable

In this case $$N_s = \frac{300}{60} \cdot \frac{\sqrt{90.6863 \times 10^6 / 3}}{531.28^{5/4}}$$

$$= 10.77, \text{ Dimensionless value} = 0.018$$

Hence a three jet unit can be suggested.

Alternate will be three single jet units.

Problem 14.11. *The following data refers to a Pelton turbine. It drives a 15 MW generator. The effective head is 310 m. The generator and turbine efficiencies are 95% and 86% respectively. The speed ratio is 0.46. Jet ratio is 12. Nozzle velocity coefficient is 0.98.* ***Determine the jet and runner diameters, the speed and specific speed of the runner****.*

From the power and efficiencies the flow rate is determined

$$\eta_T \eta_g Q \rho g H = 15 \times 10^6$$

$$\therefore \quad Q = \frac{15 \times 10^6}{0.95 \times 0.86 \times 1000 \times 9.81 \times 310} = 6.0372 \text{ m}^3/\text{s}$$

The velocity of the jet is determined from the head and C_v

$$V_j = 0.98\sqrt{2 \times 9.81 \times 310} = 76.43 \text{ m/s}$$

Runner tangential velocity is

$$u = 0.46 \times 76.43 = 35.16 \text{ m/s}$$

Jet diameter is found from flow rate and jet velocity.

$$\frac{\pi d^2}{4} \times V_j = Q,\ d = \left[\frac{6.0372 \times 4}{\pi \times 76.43}\right]^{0.5} = \mathbf{0.3171\ m}$$

Jet speed ratio is $\frac{D}{d} = 12, \quad \therefore \quad D = 12 \times 0.3171 = 3.8$ m

The turbine rotor speed is determined from the tangential velocity

$$\frac{\pi DN}{60} = u,$$

$$\mathbf{N} = \frac{u \times 60}{\pi D} = 35.16 \times 60 / \pi \times 3.8 = \mathbf{176.71\ rpm}$$

$$\mathbf{N_s} = \frac{176.61}{60} \cdot \frac{\sqrt{15 \times 10^6}}{310^{5/4}} = \mathbf{8.764}$$

Problem 14.12. *Show that for the following constants, the dimensionless specific speed is* $0.2096\frac{d}{D}$, $\phi = 0.48$, $\eta_o = 90\%$, $C_v = 0.98$

$$N_s = \frac{N\sqrt{P}}{\rho^{1/2}\,(gH)^{5/4}}, \text{ where } N \text{ is rps.}$$

$$Q = \frac{\pi d^2}{4} \times V_j = \frac{\pi d^2}{4} \times 0.98\,[2g\,H]^{1/2}$$
$$= 3.4093\,d^2\,H^{1/2}$$
$$P = \eta_o \times \rho\, g\, Q\, H = 0.9 \times 1000 \times 9.81 \times 3.4093\,d^2\,H^{1/2} \times H$$
$$= 30100.73\,d^2\,H^{1.5}$$

$$N = \frac{u}{\pi d},\ u = 0.48\,V_j = 0.48 \times 0.98\sqrt{2g}\ \ H^{1/2} = 2.0836\,H^{1/2}$$

$\therefore \quad N = 0.6632\,H^{1/2}\,D^{-1}$

$\therefore \quad \mathbf{N_s} = \frac{0.6632\,H^{1/2}}{D}\,[30100.73\,d^2H^{1.5}]^{1/2} / 1000^{1/2}\,g^{1.25}\,H^{1.25}$

$$= 115.068\frac{dH^{1.25}}{D} / 549\,H^{1.25} = \mathbf{0.2096}\frac{\mathbf{d}}{\mathbf{D}}$$

Problem 14.13. *The outer diameter of a Francis runner is 1.4 m. The flow velocity at inlet is 9.5 m/s. The absolute velocity at the exit is 7 m/s. The speed of operation is 430 rpm. The power developed is 12.25 MW, with a flow rate of 12* m^3*/s. Total head is 115 m. For shockless entry determine the angle of the inlet guide vane. Also find the absolute velocity at entrance, the*

runner blade angle at inlet and the loss of head in the unit. Assume zero whirl at exit. Also fluid the specific speed.

The runner speed $u_1 = \dfrac{\pi D N}{60} = \dfrac{\pi \times 430 \times 1.4}{60} = 31.52$ m/s

As $V_{u2} = 0$,

Power developed $= \dot{m}\ V_{u1}\ u_1$

$12.25 \times 10^6 = 12 \times 10^3 \times V_{u1} \times 31.52$

Solving $\mathbf{V_{u1} = 32.39\ m/s}$

$V_{u1} > u_1$

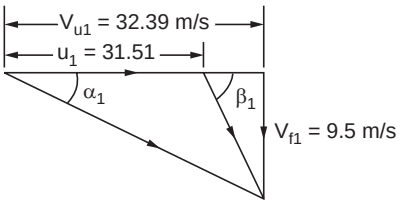


Figure P. 14.13

∴ The shape of the inlet

Velocity triangle is as given. Guide blade angle α_r

$$\tan \alpha_1 = \frac{9.5}{32.39} \quad \therefore \quad \mathbf{\alpha_1 = 16.35°}$$

$$\mathbf{V_1} = (V_{f1}^2 + V_{u1}^2)^{0.5} = [9.5^2 + 32.39^2]^{0.5} = \mathbf{33.75\ m/s}$$

Blade inlet angle β_1

$$\tan \beta_1 = 9.5 / (32.39 - 31.52)$$

$$\therefore \quad \mathbf{\beta_1 = 84.77°}$$

Total head = 115 m. **head equal for Euler work =** $\dot{m}\ V_{u1}\ u_1/g$

$$= \frac{32.39 \times 31.52}{9.81} = \mathbf{104.07\ m}$$

Head loss in the absolute velocity at exit

$$= \frac{7^2}{2 \times 9.81} = \mathbf{2.5\ m}$$

∴ Loss of head = 115 – 104.07 – 2.5 = **8.43 m**

$$\mathbf{N_s} = \frac{430}{60} \cdot \frac{\sqrt{12.25 \times 10^6}}{115^{1.25}} = \mathbf{223.12}$$

As the inner diameter is not known blade angle at outlet cannot be determined.

Problem 14.14. *A Francis turbine developing 16120 kW under an a head of 260 m runs at 600 rpm. The runner outside diameter is 1500 mm and the width is 135 mm. The flow rate is 7 m^3/s . The exit velocity at the draft tube outlet is 16 m/s. Assuming zero whirl velocity at exit and neglecting blade thickness determine the overall and hydraulic efficiency and rotor blade angle at inlet. Also find the guide vane outlet angle :*

$$\text{Overall efficiency} = \frac{\text{Power developed}}{\text{Hydraulic power}} = \frac{16120 \times 10^3}{7 \times 1000 \times 9.81 \times 260}$$

$$\eta_o = \mathbf{0.9029\ or\ 90.29\%}$$

Assuming no friction and other losses,

Hydraulic efficiency $= \left(H - \frac{V_2^2}{2g} \right) / H$

where V_2 is the exit velocity into the tailrace

$$\eta_H = (260 - (16^2/2 \times 9.81) / 260$$
$$= \mathbf{0.9498 \quad or \quad 94.98\%}$$

As V_{u2} is assumed to be zero,

$$V_{u1} = \eta_H (gH)/u_1$$

$$u_1 = \pi DN / 60 = \frac{\pi \times 1.5 \times 600}{60} = \mathbf{47.12\ m/s}$$

$\therefore$ $V_{u1} = 0.9498 \times 9.81 \times 260 / 847.12 = \mathbf{51.4\ m/s}$

$V_{u1} > u$

$\therefore$ The shape of the velocity triangle is as given. β is the angle taken with the direction of blade velocity.

$$V_{f1} = \frac{Q}{\pi D_1 b_1} = \frac{7}{\pi \times 1.5 \times 0.135} = \mathbf{11\ m/s}$$

$$\tan \alpha_1 = 11 / 51.4$$

$\therefore$ $\alpha_1 = \mathbf{12.08°}$

$$\tan \beta_1 = 11/(51.4 - 47.12)$$

$\therefore$ $\beta_1 = \mathbf{68.74°}$

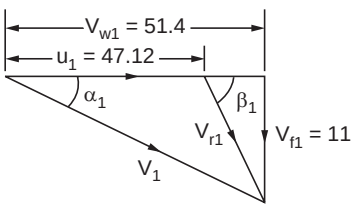


Figure P. 14.14

The specific speed of the unit

$$= \frac{600}{60} \frac{\sqrt{16120000}}{260^{1.25}} = \mathbf{38.46}$$

It is on the lower side.

Problem 14.15. *A small Francis turbine develops 2555 kW working under a head of 25 m. The overall efficiency is 0.9. The diameter and width at inlet are 1310 mm and 380 mm. At the outlet these are 1100 mm and 730 mm. The runner blade angle at inlet is 135° along the direction of the blade velocity. The whirl is zero at exit. Determine the runner speed, whirl velocity at inlet, the guide blade outlet angle and the flow velocity at outlet. Assume* $\eta_v = 0.98$, $\eta_m = 0.97$.

The flow rate $\mathbf{Q} = P / \eta_o gH$

$$= 2555 \times 10^3 / 0.9 \times 9.81 \times 25 = \mathbf{11.58\ m^3/s}$$

Hydraulic efficiency = Overall efficiency/(Mechanical efficiency × Volumetric efficiency)

$\therefore$ $\boldsymbol{\eta_H} = 0.9 / 0.98 \times 0.97 = \mathbf{0.9468}$

$$= u_1 V_{u1} / gH$$

$\therefore$ $\mathbf{u_1 V_{u1}} = 0.9468 \times 9.81 \times 25 = \mathbf{232.2\ m}$

The flow velocity at inlet

$$V_{f1} = 11.55/\pi \times 1.31 \times 0.38$$
$$= \mathbf{7.385\ m/s}$$

$$\tan (180 - 135) = V_{f1} / (u_1 - V_{u1})$$

$$\therefore \quad u_1 - V_{u1} = 7.385 \times \tan (180 - 135)$$
$$= 7.385$$

$$u_1 (u_1 - 7.385) = 232.2$$

$$u_1^2 - 7.385\ u_1 - 232.2 = 0$$

$$\therefore \quad \mathbf{u_1 = 19.37\ m/s}, \quad \mathbf{V_{u1} = 11.99\ m/s}$$

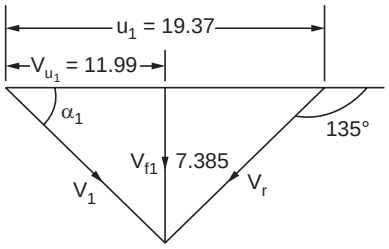


Figure P. 14.15

$$u_1 = \frac{\pi DN}{60}, \mathbf{N} = 4 \times 60/\pi D = 19.37 \times 60 / \pi \times 1.31 = \mathbf{282.4\ rpm}$$

$$\tan \alpha_1 = 7.385 / 11.9 \quad \therefore \quad \boldsymbol{\alpha_1 = 31.63°}$$

$$\mathbf{V_{f2}} = 11.58/\pi \times 1.1 \times 0.73 = \mathbf{4.59\ m/s} = \mathbf{V_2}$$

Blade velocity at outlet

$$\mathbf{u_2} = \frac{\pi D_2 N}{60} = \frac{\pi \times 1.1 \times 282.4}{60} = \mathbf{16.26\ m/s}$$

The exit triangle is right angled

$$\tan (180 - \beta_2) = \frac{4.59}{16.26}$$

$$\therefore \quad 180 - \beta_2 = 15.76°, \boldsymbol{\beta_2 = 164.24°}$$

Specific speed $= \dfrac{N\sqrt{P}}{H^{5/4}}$

$$\mathbf{N_s} = \frac{282.4\sqrt{2555000}}{60 \times 25^{1.25}} = \mathbf{134.58.}$$

Problem 14.16. *A Francis turbine works under a head of 120 m. The outer diameter and width are 2 m and 0.16 m. The inner diameter and width are 1.2 m and 0.27 m. The flow velocity at inlet is 8.1 m/s. The whirl velocity at outlet is zero. The outlet blade angle is 16°. Assume η_H = 90%. Determine, power, speed and blade angle at inlet and guide blade angle.*

The outlet velocity diagram is a right angled triangle as shown

$$V_{f2} = V_{f1} \times D_1 b_1 / D_2 b_2$$
$$= 8.1 \times 2 \times 0.16 / 1.2 \times 0.27 = 8\ \text{m/s}$$

$$\therefore \quad u_2 = 8/\tan 16 = \mathbf{27.9\ m/s}$$

$$\frac{\pi D_2 N}{60} = u_2, \frac{\pi \times 1.2 \times N}{60} = 27.9.$$

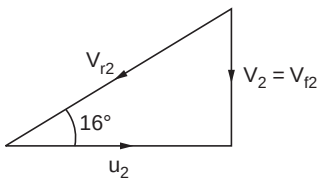


Figure P. 14.16

Solving N = 444 rpm

$$\mathbf{u_1} = \frac{\pi D_1 N}{60} = \frac{\pi \times 2 \times 266.4}{60} = 46.5 \text{ m/s}$$

$$\eta_H = \frac{u_1 V_{u1}}{g H}, \; V_{u1} = \frac{0.9 \times 9.81 \times 120}{46.5} = \mathbf{22.8 \text{ m/s}}$$

$u_1 > V_{u1}$.

The shape of the inlet triangle is shown.

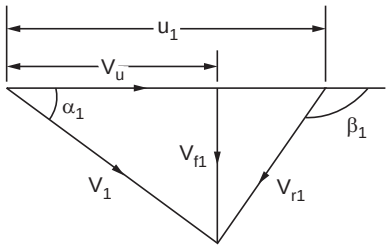


Figure P. 14.16

$$\tan \alpha_1 = \frac{V_{f1}}{V_{u1}} = \frac{8.1}{22.8}$$

$\therefore$ $\mathbf{\alpha_1 = 19.55°}$

$$\tan (180 - \beta_1) = \frac{V_{f1}}{u_1 - V_{u1}} = \frac{8.1}{46.5 - 22.8}$$

$\therefore$ $\mathbf{\beta_1 = 161°}$

Flow rate = $\pi D_1 b_1 V_{f1} = \pi \times 2 \times 0.16 \times 8.1 = 8.143 \text{ m}^3/\text{s}$

Power = $0.9 \times 120 \times 9.81 \times 8.143 \times 10^3 / 10^3 =$ **8627 kW**

$$\mathbf{N_s} = \frac{444}{60} \frac{\sqrt{8627 \times 10^2}}{120^{5/4}} = \mathbf{54.72}$$

Problem.14.17 *An inward flow reaction turbine of the Francis type operates with a flow rate of 1.67 m^3/s runs at 416 rpm. The available head is 81 m. The blade inlet angle is 120 with the direction of wheel velocity. The flow ratio is 0.2. Hydraulic efficiency is 92%. Determine runner diameter, the power developed and the speed ratio*

Power developed = $0.92 \times 81 \times 9.81 \times 1.67 \times 10^3/10^3$

$P =$ **1220.8 KW**

$u_1 V_{u1} = 1220.8 \times 10^3/1.67 \times 10^3 =$ **731.04 $\mathbf{m^2/s^2}$**

$$\text{Flow ratio} = 0.2 = \frac{V_{f1}}{\sqrt{2gH}}, \quad \therefore \quad \mathbf{V_{f1}} = 0.2\sqrt{2 \times 9.81 \times 81} = \mathbf{7.973 \text{ m/s}}$$

The shape of the velocity triangle is as shown. $\beta_1 > 90°$

$$V_{u1} = u_1 - \frac{7.973}{\tan 60} = u_1 - 4.6$$

$$u_1 V_{u1} = 731.04 = u_1 (u_1 - 4.6)$$

or $u_1^2 - 4.6\, u_1 - 731.04 = 0$. Solving, $\boldsymbol{u_1}$ **= 29.44 m/s**

$\therefore$ $\boldsymbol{V_{u1}} = 29.44 - 4.6 =$ **24.84 m/s**

Speed ratio $\phi = \frac{u_1}{V_1}$, $\boldsymbol{V_1} = (24.84^2 + 7.973^2)^{1/2} =$ **26.09 m/s**

$$\therefore \qquad \phi = \frac{29.44}{26.09} = \mathbf{1.128}$$

$$\frac{\pi D_1 N}{60} = 29.44$$

$$\therefore \qquad \boldsymbol{D = 1.35\ m}$$

$$N_s = \frac{416}{60} \cdot \frac{\sqrt{1220 \times 10^3}}{81^{1.25}}$$

$$= \mathbf{31.53}$$

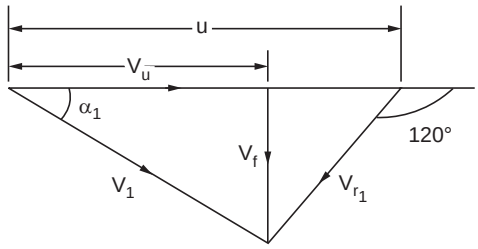


Figure P. 14.17

Problem 14.18 *Determine the diameters and blade angles of a Francis turbine running at 500 rpm under a head of 120 m and delivering 3 MW. Assume flow ratio as 0.14 and* $D_2 = 0.5\ D_1$ *and* $b_1 = 0.1\ D_1$*. The hydraulic efficiency is 90% and the overall efficiency is 84%.*

$$V_{f1} = 0.14\sqrt{2g\,H} = 0.14\sqrt{2 \times 9.81 \times 120} = \mathbf{6.79\ m/s}$$

From the overall efficiency and power delivered

$$Q = \frac{3 \times 10^6}{10^3 \times 9.81 \times 120 \times 0.84} = \mathbf{3.034\ m^3/s}$$

$$Q = \pi D_1 b_1 V_{f1} = \pi D_1 \times 0.1\, D_1 \times 6.79$$

Solving $\qquad \mathbf{D_1 = 1.193\ m},\ \mathbf{D_2 = 0.5965\ m}.$

$$\mathbf{b_1 = 0.1193\ m},\ \mathbf{b_2 = 0.2386\ m}$$

$$u_1 = \frac{\pi D_1 N}{60} = \frac{\pi \times 1.193 \times 500}{60} = \mathbf{31.23\ m/s}$$

$$\therefore \qquad u_2 = \mathbf{15.615\ m/s}$$

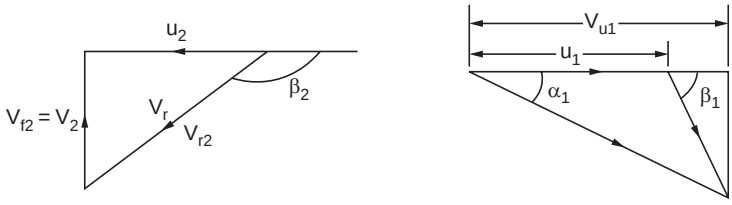


Figure P. 14.18

The outlet triangle is as shown as $V_{u2} = 0$, assuming $V_{f2} = V_{f1}$

$$\tan(180 - \beta_2) = \frac{6.79}{15.615}$$

$\therefore$ Solving $\qquad \boldsymbol{\beta_2 = 156.5°}$ (23.5°)

To solve inlet angles, V_{u_1} is required

$$0.9 = \frac{u_1 V_{u1}}{g\,H}$$

$$\therefore \quad V_{u1} = \frac{0.9 \times 9.81 \times 120}{31.23} = \mathbf{33.93\ m/s}$$

$V_{u1} > u_1$. $\therefore$ The triangle is as shown

$$\tan \alpha_1 = \frac{V_{f1}}{V_{u1}} = \frac{6.79}{33.93} \quad \therefore \quad \boldsymbol{\alpha_1 = 11.32°}$$

$$\tan \beta_2 = \frac{6.79}{33.93 - 31.23} \quad \therefore \quad \boldsymbol{\beta_1 = 68.3°}$$

Problem 14.19 *In an inward flow reaction turbine the working head is 10 m. The guide vane outlet angle is 20°. The blade inlet angle is 120°.* ***Determine the hydraulic efficiency*** *assuming zero whirl at exit and constant flow velocity. Assume no losses other than at exit.*

The velocity diagram is as shown in figure. As no velocity value

$$V_u = V_1 \cos 20 = 0.9397\ V_1 \qquad (\mathbf{1})$$

$$V_f = V_1 \sin 20 = 0.3420\ V_1 \qquad (\mathbf{2})$$

is avaivable, the method adoped is as below.

$$u = V_u + \frac{V_f}{\tan 60} = 0.9397\ V_1 + \frac{0.342\ V_1}{1.732} = 1.1372\ V_1 \qquad (\mathbf{3})$$

Work done = headlosses (all expressed as head)

$$\frac{u \,.\, V_u}{g} = H - \frac{V_f^2}{2g}$$

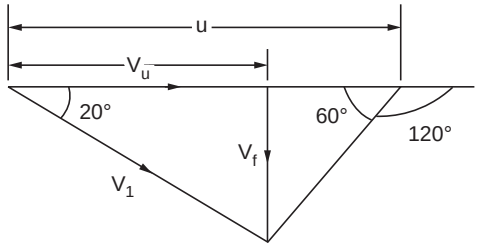


Figure P. 14.18

$$\frac{1.1372 \times 0.9397}{9.81} \,.\, V_1^2 = 10 - \frac{0.342^2\ V_1^2}{2 \times 9.81}$$

$$0.10893\ V_1^2 + 0.00596\ V_1^2 = 10$$

$$\therefore \quad V_1 = \left[\frac{10}{0.1093 + 0.00596}\right]^{0.5} = 9.33 \text{ m/s}$$

$$\therefore \quad u = 1.1372 \times 9.33 = 10.61 \text{ m/s}$$

$$V_{u1} = 0.9397 \times 9.33 = 8.767 \text{ m/s}$$

$$\eta_H = \frac{10.61 \times 8.767}{9.81 \times 10} = 0.9482 \text{ or } \mathbf{94.82\%}$$

Problem 14.20 *A Francis turbine delivers 16 MW with an overall efficiency of 85 percent and a hydraulic efficiency of 91 percent, when running at 350 rpm under a head of 100 m. Assume ID = 0.6 OD and width as 0.10 D. The flow ratio is 0.2 and blade blockage is 8 percent of flow area at inlet. Assume constant flow velocity and zero whirl at exit.* ***Determine the runner diameter, and blade angles.***

$$\text{Overall efficiency} = \frac{\text{Power delivered}}{\rho Q g H} \quad \therefore \quad Q = \frac{\text{Power delivered}}{\rho \eta_0 g H}$$

$$= \frac{16 \times 10^6}{1000 \times 0.85 \times 9.81 \times 100} = 19.1881 \text{ m}^3/\text{s}$$

$$Q = \pi D\, b\, V_f (1 - 0.08), \text{ Flow ratio} = \frac{V_f}{\sqrt{2g H}}$$

$\therefore$ $V_f = \text{flow ratio} \times \sqrt{2g H} = 0.2\sqrt{2 \times 9.81 \times 100} = 8.86$ m/s

$\therefore$ $19.1881 = \pi D \times 0.1 D \times 8.86 \times (1 - 0.08)$

$\therefore$ $D^2 = \dfrac{19.1881}{\pi \times 0.1 \times 8.86 \times 0.92}$

$\therefore$ $D = \mathbf{2.74\ m,\ b = 0.274\ m}$

$ID = \mathbf{1.64\ m}$

$$u_1 = \frac{\pi D_1 N}{60} = \frac{\pi \times 2.74 \times 350}{60} = 50.21 \text{ m/s},\ u_2 = 30.13 \text{ m/s}$$

$$\eta_H = \frac{u_1 V_{u1}}{g H},\ 0.91 = \frac{50.21 \times V_{u1}}{9.81 \times 100}$$

$\therefore$ $V_{u1} = 17.78$ m/s

$V_{u1} < u_1$

$\therefore$ The velocity diagram is as in figure

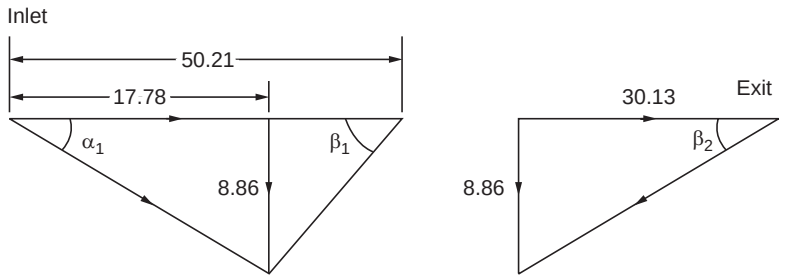


Figure P. 14.20

$$\tan \alpha_1 = \frac{8.86}{17.78}$$

$\therefore$ **Guide blade outlet angle is 26.5°**

$$\tan \beta_1 = \frac{8.86}{50.12 - 17.78}$$

$\therefore$ $\boldsymbol{\beta_1 = 15.3°}$ **(as in figure) or 164.7° (along + ve u)**

$$\tan \beta_2 = \frac{8.86}{30.13}$$

Outlet angle $\boldsymbol{\beta_2}$ = 16.4° **(as in figure) or 163.6° (with + ve u direction)**

Problem 14.21 *An inward low reaction turbine has a flow velocity of 4 m/s while the pheripheral velocity is 35 m/s. The whirl velocity is 26 m/s. There is no whirl at exit. If the hydraulic efficiency is 91%* ***determine the head available. Also find the inlet blade angle and the guide vane outlet angle.***

The velocity diagram is as shown

$$\tan\beta_1 = \frac{V_f}{u - V_u} = \frac{4}{35 - 26} = 0.444$$

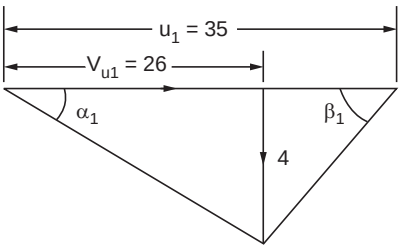


Figure P. 14.21

$\therefore$ Blade angle at inlet, $\beta_1 = 23.96°$ or **156.04°.**

$$\tan\alpha_1 = \frac{V_f}{Vu_1} = \frac{4}{26} = 0.1538,$$

$\therefore$ **Guide vane outlet angle = 8.75°**

As exit whirl is zero

$$\eta_H = \frac{u_1 V_{u1}}{g H}$$

$$\therefore \quad \mathbf{H} = \frac{u_1 V_{u1}}{g \times \eta_H} = \frac{35 \times 26}{9.81 \times 0.91} = \mathbf{101.94\ m}$$

Problem 14.22 *The diameter and blade angles of a Francis turbine with a specific speed of 95 are to be determined. The power delivered is 45 MW under a head of 180 m. Assume overall efficiency of 85% and hydraulic efficiency of 90%. Also* $b_1 = 0.1\ D_1$ *and blade thickness occupies 5% of flow area. The constant flow velocity is 15 m/s.*

To determine the speed

$$N_s = \frac{N}{60} \cdot \frac{\sqrt{P}}{H^{5/4}}$$

$$\therefore \quad \mathbf{N} = 95 \times 60 \times 180^{5/4} / \sqrt{45 \times 10^6}$$

$$= \mathbf{560.22.}\ \text{Say 560 rpm}$$

not suitable for 50 cycles.

500 rpm can be adopted (with 6 pairs of poles), power capacity cannot be changed.

$$\mathbf{N_s} = \frac{500}{60} \cdot \frac{\sqrt{45 \times 10^6}}{180^{1.25}} = \mathbf{84.8.}\ \text{May be adopted}$$

$$\mathbf{Q} = \frac{5 \times 10^6}{0.85 \times 1000 \times 9.81 \times 180} = \mathbf{29.98\ m^3/s}$$

$$Q = \pi D_1 b_1 V_f \times 0.95 = \pi \times D_1 \times 0.1 D_1 \times 15 \times 0.95 = 29.98$$

Solving, $\mathbf{D_1 = 2.59\ m,\ D_2 = 1.295\ m}$

$\mathbf{b_1 = 0.259\ m,\ b_2 = 0.518\ m}$

$$\mathbf{u_1} = \frac{\pi D_1 N}{60} = \frac{\pi \times 2.59 \times 500}{60} = \mathbf{67.8\ m/s}$$

$\mathbf{u_2 = 33.9\ m/s}$

$$0.9 = \frac{u_1 V_{u1}}{g H}$$

$$\therefore \quad \boldsymbol{V_{u1}} = \frac{0.9 \times 9.81 \times 180}{67.8} = \mathbf{23.44\ m/s}$$

Velocity triangles are as shown

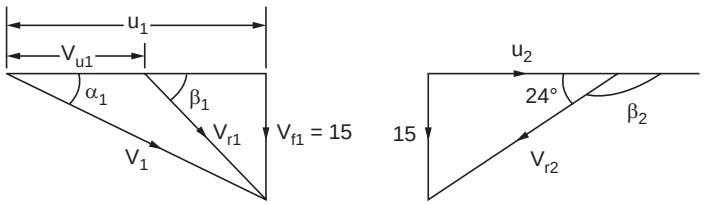


Figure P. 14.22

$$\tan \alpha_1 = 15 / 67.8 \qquad \therefore \quad \boldsymbol{\alpha_1 = 12.48°}$$

$$\tan \beta_1 = 15/ (67.8 - 23.44) \qquad \therefore \quad \boldsymbol{\beta_1 = 18.7°}$$

$$\tan (180 - \beta_2) = 15/33.9 \qquad \therefore \quad \boldsymbol{\beta_2 = 156°\ (24°)}$$

Problem 14.23. *In a Francis turbine the guide blade angle is 17° and the entry to the runner is in the radial direction. The speed of operation is 400 rpm. The flow velocity remains constant at 10 m/s. The inner diameter is 0.6 of outer diameter. The width at inlet is 0.12 times the diameter. Neglecting losses, determine the head, the diameter and power. Also fluid the angle at blade outlet. The flow area is blocked by vane thickness by 6%.*

This is a special case and the inlet velocity triangle is a right angled triangle.

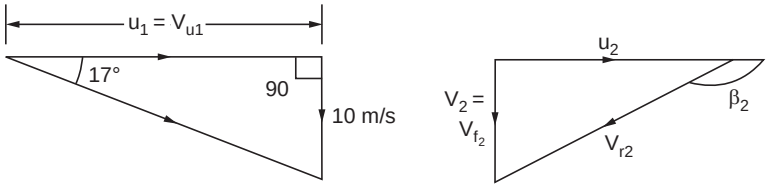


Figure P. 14.23

The inlet velocity diagram is as shown

$$\mathbf{u_1} = \frac{V_{f1}}{\tan \alpha_1} = \frac{10}{\tan 17°} = \mathbf{32.7\ m/s}$$

$$\mathbf{u_2} = u_1 \times 0.6 = \mathbf{19.63\ m/s}$$

$$\mathbf{D_1} = \frac{60 \times u_1}{\pi \times N} = \frac{60 \times 32.7}{\pi \times 400} = \mathbf{1.56\ m},\ \mathbf{b_1 = 0.1874\ m}$$

$$\mathbf{D_2 = 0.936\ m}$$

Neglecting losses head supplied

$$\mathbf{H} = \frac{u_1 V_{u1}}{g} + \frac{V_2^2}{2g} = \frac{32.7^2}{9.81} + \frac{10^2}{2 \times 9.81} = \mathbf{114.1\ m}$$

$$\mathbf{Q} = \pi D_1 b_1 V_{f1} \times 0.94 = \pi \times 1.56 \times 0.1874 \times 10 \times 0.94 = \mathbf{8.633\ m^3/s}$$

$$\therefore \quad \text{Power } \mathbf{P} = \frac{8.633 \times 10^3 \times 32.7^2}{10^3} = \mathbf{9221\ kW}$$

From the velocity triangle at outlet

$$\tan (180 - \beta_2) = \frac{10}{19.63}$$

$$\therefore \quad \beta_2 = \mathbf{153°}$$

Problem 14.24. *Show in the case* ***of a 90° inlet Francis*** *turbine, the* ***hydraulic efficiency*** $= \dfrac{\mathbf{2}}{\mathbf{2 + \tan^2 \alpha_1}}$.

The velocity diagram is as shown

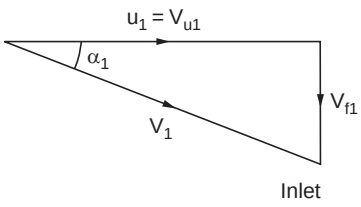


Figure P. 14.24

$$\tan \alpha_1 = \frac{V_{f1}}{u_1} = \frac{V_{f1}}{V_{u1}}$$

$$\therefore \quad u_1 V_{u1} = u_1^2 = \frac{V_{f1}^2}{\tan^2 \alpha_1}$$

Neglecting losses and assuming $V_{f2} = V_{f1}$

$$\text{Work input} = u_1 V_{u1} + \frac{V_{f1}^2}{2}$$

$$\therefore \quad \eta_H = \frac{V_{f1}^2 / \tan^2 \alpha_1}{\dfrac{V_{f1}^2}{\tan^2 \alpha_1} + \dfrac{V_{f1}^2}{2}}$$

Multiplying by 2 and also $\tan^2 \alpha_1$ both the numerator and denominator

$$\eta_H = \frac{2V_{f1}^2}{2V_{f1}^2 + V_{f1}^2 \tan^2 \alpha_1} = \frac{2}{2 + \tan^2 \alpha_1}.$$

Problem 14.25. *The following details are available about a Francis turbine. Diameters are 2.25 m and 1.5 m. Widths are 0.25 m and 0.375 m. The guide blade outlet angle is 18° runner blade angle is 85°. Both angles with the blade velocity direction. Frictional loss is 15% of the pressure head available between the inlet and outlet of the runner is 60 m. Calculate the speed and output of the turbine. Also fluid the blade outlet angle. Mechanical efficiency is 92%. Blade thickness blocks the flow area by 8%.*

In this case as

$$D_1 b_1 = D_2 b_2,$$

$$\therefore \quad V_{f2} = V_2 = V_{f1}$$

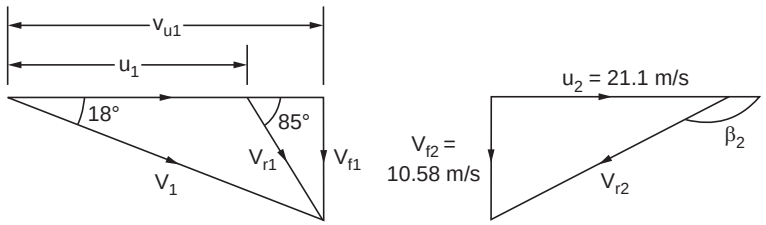


Figure P. 14.25

To determine the work output

u_1, V_{u1} are to be calculated

$$\boldsymbol{V_{u1}} = V_1 \cos 18° = 0.951\ V_1 \qquad \text{(A)}$$

$$\boldsymbol{V_{f1}} = V_1 \sin 18° = \mathbf{0.309\ V_1}$$

$$\boldsymbol{u_1} = V_{u1} - V_{f1} / \tan 85 = 0.951\ V_1 - 0.309\ V_1 / \tan 85$$

$$= \mathbf{0.924\ V_1}$$

$$\therefore \quad \boldsymbol{u_1 V_{u1}} = 0.951 \times 0.924\ V_1^2 = \mathbf{0.8787\ V_1^2}, \text{ as head} = \frac{\mathbf{0.8787\ V_1^2}}{\mathbf{g}}$$

Considering the runner inlet and outlet

$$h_1 + \frac{V_1^2}{2g} = h_2 + \frac{V_2^2}{2g} + W + h_L$$

$$\mathbf{h_1 - h_2 = 60\ m,\ h_L = 0.15 \times 60 = 9\ m,}\ V_2^2 = V_f^2 = 0.309^2\ V_1^2$$

$$V_2^2 = 0.0955\ V_1^2$$

$$\therefore \quad h_1 - h_2 - h_L = -\frac{V_1^2}{2g} + \frac{0.0955\ V_1^2}{2g} + \frac{0.8787\ V_1^2}{g}$$

$$51 = \frac{V_1^2}{g}\left[-\frac{1}{2} + \frac{0.0955}{2} + 0.8787\right] = 0.42645\ \frac{V_1^2}{g}$$

Solving, $\boldsymbol{V_1 = 34.25\ m/s}$ $\quad \therefore \quad \mathbf{u_1 = 31.65\ m/s,\ V_{u1} = 32.57\ m/s}$

$$V_{f1} = V_{f2} = \boldsymbol{V_2} = \mathbf{10.58\ m/s}$$

$$\frac{\pi D_1 N}{60} = 34.25,\ D_1 = 2.25 \text{ m}, \quad \therefore \quad \boldsymbol{N = 290.7 \text{ rpm}}$$

Flow rate $= \pi D_1 b_1 \times (1 - 0.08)\ V_f$

$= \pi \times 2.25 \times 0.25 \times 0.92 \times 10.58 \text{ m}^3/\text{s}$

$\boldsymbol{Q = 17.2 \text{ m}^3/\text{s}}$

$\therefore$ **Power developed** $= 17.2 \times 10^3 \times 32.57 \times 31.65/1000$

$= \mathbf{17730 \text{ kW}}$

Power delivered $= P \times \eta_{\text{mech.}} = \mathbf{16312 \text{ kW}}$

$\boldsymbol{u_2} = u_1 \times 1.5 / 2.25 = 34.25 \times 1.5 / 2.25 = \mathbf{21.1 \text{ m/s}}$

From Exit triangle, $\boldsymbol{V_2 = V_{f2} = 10.58 \text{ m/s}}$

$\tan (180 - \beta_2) = (10.58 / 21.1)$

Solving $\boldsymbol{\beta_2 = 153.4°}$ (as in figure)

Problem 14.26. *In a Francis turbine installation the runner inlet is at a mean height of 2 m from tailrace while the outlet is 1.7 m from the tailrace. A draft tube is connected at the outlet. The runner diameter is 1.5 m and runs at 375 rpm. The pressure at runner inlet is 35 m above atmosphere, while the pressure at exit is 2.2 m below the atmosphere. The flow velocity at inlet is 9 m/s. At output it is 7 m/s. Available head is 62 m. Hydraulic efficiency is 90%.* ***Determine the losses before the runner, in the runner and at exit.***

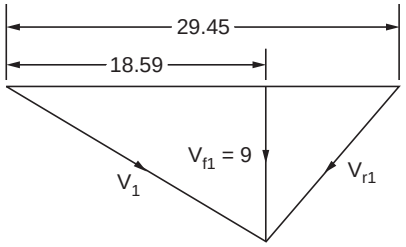


Figure P. 14.26

Runner outlet velocity

$$\boldsymbol{u_1} = \frac{\pi D N}{60} = \frac{\pi \times 1.5 \times 375}{60} = \mathbf{29.45 \text{ m/s}}$$

To find V_{u1},

$$0.9 = \frac{29.45 \times V_{u1}}{9.81 \times 62}$$

$\therefore \quad \boldsymbol{V_{u1} = 18.59 \text{ m/s}}$

Head loss upto the exit of guide blades or entry to runner. Denoting these locations as 1 and 2.

$$\mathbf{h_1 + \frac{V_1^2}{2g} + Z_o = h_2 + \frac{V_2^2}{2g} + Z_2 + h_{L1}}$$

LHS = 62 m, $h_2 = 35$ m, $Z_2 = 2$ m

$V_1^2 = (18.59^2 + 9^2) = 426.6 \text{ m}^2/\text{s}^2$

Substituting, $\therefore \quad \dfrac{V_1^2}{2g} = 21.74 \text{ m}$

$$62 = 35 + 2 + \frac{426.6}{2 \times 9.81} + h_{L1}$$

$$\mathbf{h_{L1}} = 62 - 35 - 2 - 21.74 = \mathbf{3.26\ m}$$

Considering the runner inlet and outlet : Denoting as 1 and 2

$$\mathbf{h_1 + \frac{V_1^2}{2g} + Z_1 = h_2 + Z_2 + W + \frac{V_2^2}{2g} + h_{L2}}$$

$$35 + 21.74 + 2 = -2.2 + 1.7 + \frac{29.45 \times 18.59}{9.81} + h_{L2} + \frac{7^2}{2 \times 9.81}$$

$$= -2.2 + 1.7 + 55.81 + 2.5 + h_{L2}$$

$\therefore$ $\mathbf{h_{L2} = 0.93\ m.}$

Considering the draft tube

Static head available = 1.7 m

Kinetic head available = 2.5 m

Total = 4.2 m

But actual head at turbine exit = 2.2 m

$\therefore$ Loss, h_{L3} = 2 m

Total loss = 3.26 + 0.93 + 2 = 6.19 m in **6.2 m**

10% of the total head. as hydraulic efficiency is 90%.

Problem 14.27. *A Kaplan turbine plant develops 3000 kW under a head of 10 m. While running at 62.5 rpm. The discharge is 350 m^3/s. The tip diameter of the runner is 7.5 m and the hub to tip ratio is 0.43.* ***Calculate the specific speed, turbine efficiency, the speed ratio and flow ratio.***

Speed ratio is based on tip speed.

Hub diameter **= 0.43 × 7.5 = 3.225 m**

Turbine efficiency $= P / \rho Q H g$

$$= \frac{30000 \times 10^3}{1000 \times 350 \times 10 \times 9.81} = 0.8737 \quad \text{or } \mathbf{87.37\%}$$

$$\textbf{Specific speed} = \frac{60}{60} \cdot \frac{\sqrt{30{,}000 \times 10^3}}{10^{1.25}} = \mathbf{308}$$

$$\text{Runner tip speed} = \frac{\pi \times 7.5 \times 60}{60} = 23.56 \text{ m/s}$$

$\therefore$ **Speed ratio** $= 23.56/\sqrt{2 \times 9.81 \times 10} = \mathbf{1.68}$

$$\text{Flow velocity} = \frac{350 \times 4}{\pi (7.5^2 - 3.225^2)} = 9.72 \text{ m/s}$$

$\therefore$ **Flow ratio** $= 9.72 / \sqrt{2 \times 9.81 \times 10} = \mathbf{0.69.}$

Problem 14.28 *A Kaplan turbine delivering 40 MW works under a head of 35 m and runs at 167 rpm. The hub diameter is 2.5 m and runner tip diameter is 5 m. The overall efficiency is 87%.* ***Determine the blade angles at the hub and tip*** *and also at a diameter of 3.75 m.* ***Also find the speed ratio*** *and flow ratio based on tip velocity. Assume* $\eta_H = 90\%$.

$$\text{Flow rate } \mathbf{Q} = \frac{40 \times 10^6}{10^3 \times 9.81 \times 35 \times 0.87} = \mathbf{133.9\ m^3/s}$$

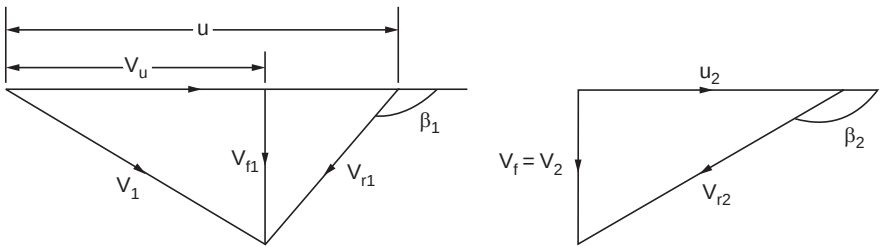


Figure P. 14.28

Assuming no obstruction by blades,

$$\boldsymbol{V_f} = \frac{133.9 \times 4}{\pi (5^2 - 2.5^2)} = \mathbf{9.09\ m/s}$$

$$\text{Blade tip velocity} = \frac{\pi \times 5 \times 167}{60} = \mathbf{43.72\ m/s}$$

$$\text{Hub velocity} = 43.72/2 = \mathbf{21.86\ m/s}$$

$$\text{Velocity at 3.75 m, } = \frac{\pi \times 3.75 \times 167}{60} = \mathbf{32.79\ m/s}$$

$$\boldsymbol{u_1\ V_{u1}} = 0.9 \times 9.81 \times 35 = \mathbf{309\ m^2/s^2}$$

$$= \text{Constant}$$

$\therefore$ $\boldsymbol{V_{u1}}$ at tip = 309 / 43.72 = **7.07 m/s**

$\boldsymbol{V_{u1}}$ at hub = 309 / 21.86 = **14.14 m/s**

$\boldsymbol{V_{u1}}$ at middle = 309 / 32.79 = **9.42 m/s**

In all cases $u > V_u$ $\therefore$ Shape of triangle is as given

$$\mathbf{\tan \beta_1 = \frac{V_f}{u - V_u}}$$

At tip $\tan (180 - \beta_1) = \dfrac{9.09}{43.72 - 7.07}$ $\therefore$ $\mathbf{\beta_1 = (180 - 13.92)° = 166.08°}$

At 3.75 *m* Dia, $\tan (180 - \beta_1) = \dfrac{9.09}{32.79 - 9.42}$ $\therefore$ $\mathbf{\beta = (180 - 21.25) = 158.75°}$

At hub $\tan(180 - \beta_1) = \dfrac{9.09}{21.86 - 14.14}$ $\therefore$ $\boldsymbol{\beta_1 = (180 - 49.66) = 130.34°}$

The trend is that (as measured with u direction) β_1 decreases with radius.

Outlet triangles are similar.

At tip: $\tan(180 - \beta_2) = 9.09 / 43.72$ $\therefore$ $\beta_2 = (180 - 11.75)°$

At 3.75 m Dia: $\tan(180 - \beta_2) = 9.09 / 32.79$ $\therefore$ $\boldsymbol{\beta_2 = (180 - 15.5)°}$

At hub: $\tan(180 - \beta_2) = 9.09 / 21.86$ $\therefore$ $\boldsymbol{\beta_2 = (180 - 22.6)°}$

The trend may be noted.

At the tip

$$\textbf{Speed ratio} = \frac{u}{\sqrt{2gH}} = \frac{43.72}{\sqrt{2 \times 9.81 \times 35}} = \mathbf{1.67}$$

$$\textbf{Flow ratio} = \frac{9.09}{\sqrt{2 \times 9.81 \times 35}} = \mathbf{0.35}$$

$$\textbf{Specific speed} = \frac{167\sqrt{40 \times 10^6}}{60 \times 35^{1.25}} = \mathbf{206.8}$$

Problem 14.29 *A Kaplan turbine delivers 30 MW and runs at 175 rpm. Overall efficiency is 85% and hydraulic efficiency is 91%. The tip diameter 5 m and the hub diameter is 2 m.* ***Determine the head and the blade angles at the mid radius****. The flow rate is 140 m^3/s.*

$\rho\, Q\, g\, H\, \eta_o$ = power delivered

$\therefore$ $\boldsymbol{H} = 30 \times 10^6 / 1000 \times 9.81 \times 140 \times 0.85 = \mathbf{25.7\ m}$

Power developed = Power available from fluid $\times\ \eta_H$

At midradius $= \dfrac{30}{0.85} \times 10^6 \times 0.93 = \mathbf{32.82\ kW}$

$$\boldsymbol{u} = \pi \times \frac{D \times N}{60} = \frac{\pi \times 3.5 \times 175}{60} = \mathbf{32.07\ m/s}$$

$$\dot{m}\, u_1\, V_{u1} = 32.82 \times 10^6 = 140 \times 10^3 \times 32.07 \times V_{u1}$$

$\therefore$ $\boldsymbol{V_{u1}} = \mathbf{7.14\ m/s}$

(note $u_1\, V_1$ = constant at all radii)

$$\boldsymbol{V_f} = 4 \times 140 / \pi\,(5^2 - 2^2) = \mathbf{8.5\ m/s}$$

$V_u < u$,

$\therefore$ The velocity diagram is as given

$$\tan(180 - \beta_1) = \frac{V_f}{u - V_u} = \frac{8.5}{32.07 - 7.14}$$

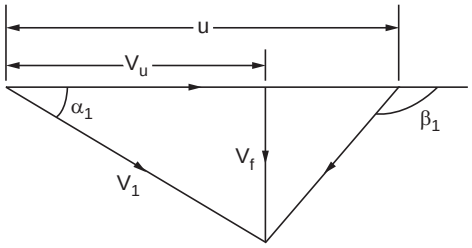


Figure P. 14.29

Chapter 14

$\therefore$ **$180 - \beta = 18.82°$,**

18.82° with – ve u direction and 161.18° with + ve u direction

Outlet triangle is right angled as $V_{u2} = 0$,

$$\tan (180 - \beta_2) = \frac{8.5}{32.07}, \ \boldsymbol{\beta_2 = 14.8°}$$

with –ve u direction 165.2° with + ve u direction

$$\tan \alpha_1 = \frac{8.5}{7.14} \quad \therefore \quad \boldsymbol{\alpha_1 = 50°}$$

Problem14.30. *A Kaplan turbine works under a head of 26.5 m, the flow rate of water being 170 m^3/s. The overall efficiency is 90%. Determine the power and specific speed. The turbine speed is 150 rpm.*

Power developed $= 0.9 \times 170 \times 10^3 \times 9.81 \times 26.5$ W

$= \mathbf{39.77 \times 10^6 \ W \text{ or } 39.77 \ MW}$

Dimensionless specific speed

$$= \frac{N\sqrt{P}}{\rho^{1/2} (gH)^{5/4}} = \frac{150}{60} \cdot \frac{\sqrt{39.77 \times 10^6}}{1000^{1/2} \times 9.81^{1.25} \times 26.5^{1.25}} = \mathbf{0.4776 \ rad}$$

Diamensional specific speed

$$= \frac{150}{60} \cdot \frac{\sqrt{39.77 \times 10^6}}{26.5^{1.25}} = \mathbf{262.22.}$$

Problem 14.31. *At a location it is proposed to install a Kaplan turbine with an estimated power of 30 MW at an overall efficiency of 0.89. The head available is 42 m.* ***Determine the speed it hub tip ratio is 0.5 and*** *the flow ratio and speed ratio are 0.5 and 1.8.*

The flow rate $= \mathbf{Q} = \dfrac{\text{Power}}{\eta_o \rho g H} = \dfrac{30 \times 10^6}{0.89 \times 1000 \times 9.81 \times 42} = \mathbf{81.81 \ m^3/s}$

Flow ratio $= V_f \sqrt{2gh} = 0.5.$

$\therefore \quad V_f = 0.5\sqrt{2 \times 9.81 \times 42} = \mathbf{14.35 \ m/s.}$

$$Q = \frac{\pi}{4} (D^2 - d^2) V_f$$

$\therefore \quad 81.81 = \dfrac{\pi}{4} D^2 (1 - 0.5^2) \ 14.35$

Solving $\boldsymbol{D = 3.11 \ m}$

$\boldsymbol{u} = 1.8\sqrt{2 \times 9.81 \times 42} = \mathbf{51.67 \ m/s}$

$$u = \frac{\pi D N}{60}, N = \frac{u \times 60}{\pi D} = \frac{51.67 \times 60}{\pi \times 3.11} = \mathbf{317.3\ rpm}$$

This may not suit 50 cycle operation. The nearest synchronous speed is 333.3 with a 9 pair of poles.

Problem 14.32. *A Kaplan turbine delivers 10 MW under a head of 25 m. The hub and tip diameters are 1.2 m and 3 m. Hydraulic and overall efficiencies are 0.90 and 0.85. If both velocity triangles are right angled triangles,* ***determine the speed, guide blade outlet angle and blade outlet angle.***

The inlet velocity diagram is shown in the figure.

Flow rate is calculated from power, head and overall efficiency

$$Q = \frac{10 \times 10^6}{10^3 \times 25 \times 9.81 \times 0.85} = \mathbf{47.97\ m^3/s}$$

$$V_f = \frac{47.97 \times 4}{\pi (3^2 - 1.2^2)} = \mathbf{8.08\ m/s}$$

Power developed

$$u\, V_{u1}\, \dot{m} = 10 \times 10^6 / 0.90$$

$$\dot{m} = 47.97 \times 10^3 \text{ kg/s and } u = V_{u1}$$

$$\therefore \quad 47.97 \times 10^3 \times u_1^2 = \frac{10 \times 10^6}{0.9}$$

Solving $u_1 = \mathbf{15.22\ m/s}$

$$\therefore \quad \tan \alpha_1 = V_{f1} / u_1 = \frac{8.08}{15.22}$$

$$\therefore \quad \boldsymbol{\alpha_1 = 28^\circ}$$

At the outlet $u_2 = u_1,\ V_{f2} = V_{f1}$

$$\therefore \quad \tan \beta = \frac{8.08}{15.22}$$

$$\therefore \quad \beta = \mathbf{28^\circ}$$

$$\frac{\pi D N}{60} = 15.22$$

$$\therefore \quad N = \frac{15.22 \times 60}{\pi \times 3} = \mathbf{96.9\ rpm}$$

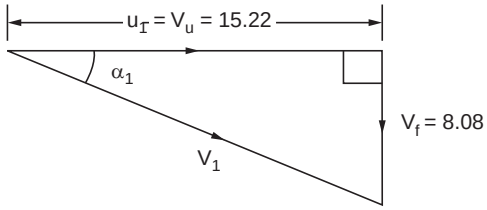


Figure P. 14.32

Problem 14.33. *In a low head hydro plant, the total head is 7 m. A draft tube is used to recover a part of the kinetic head. If the velocity at the turbine outlet or the draft tube inlet is 7 m/s and that at the outlet is 5 m/s,7 m/s, 5 m/s, determine the hydraulic efficiency if the draft tube efficiency is 100% and if the draft tube efficiency to recover kinetic energy is 80%. What will be the efficiency if all the exit velocity from the turbine is lost. Assume that there are no other losses.*

Total head = 7 m

Kinetic head at draft tube inlet $= \dfrac{7^2}{2 \times 9.81} = 2.5$ m

Kinetic head at the draft tube outlet $= \dfrac{5^2}{2 \times 9.81} = 1.27$ m

When 100% of kinetic head is recovered, head recovered

$= 2.5 - 1.27 = 1.23$ m

Case 1: The maximum gain : 1.23 m

$\therefore$ loss $= 2.5 - 1.23 = 1.27$

$\therefore$ $\eta_{\mathbf{H}} = (7 - 1.27) / 7 =$ **0.8186 = 81.86%**

Case 2: If 80% is recovered : gain $= 0.8\ (2.5 - 1.27) = 0.984$ m

Head lost $= (2.5 - 0.984) = 1.516$

$\therefore$ $\eta_{\mathbf{H}} = (7 - 1.516) / 7 = 0.7834$ or **78.34%**

Case 3: If no recovery is achieved

$\boldsymbol{\eta_{\mathbf{H}}} = (7 - 2.5) / 7 = 0.6429$ or **64.29%**

Problem 14.34. *In a draft tube arrangement for a propeller turbine the flow rate is 150 m^3/s. Inlet area of the draft tube is 15 m^2 while the outside area is 22.5 m^2. The turbine runner outlet or the draft tube inlet is 0.5 m below the tailrace level. If the kinetic head recovered by the draft tube is 80% determine the pressure head at turbine outlet.*

Considering the tailrace level as datum and denoting inlet and outlet by suffix 1 and 2.

$$h_1 + \frac{V_1^2}{2g} + Z_1 = h_2 + \frac{V_2^2}{2g} + Z_2 + \text{losses} \qquad \text{(A)}$$

$\boldsymbol{V_1} = 150 / 15 =$ **10 m/s**, $\boldsymbol{V_2} = 150 / 22.5 =$ **6.67 m/s**

$Z_1 =$ **– 0.5 m, $\mathbf{Z_2} = 0$**. Also $\mathbf{h_2}$ = atmospheric *pr* = **10 m** of water

$$\text{Losses} = \frac{0.2\,(V_1^2 - V_2^2)}{2g}$$

Rearranging the equation (A) and substituting the values, pressure at turbine exit is

$$h_1 = 10 + 0.5 + \left(\frac{6.67^2 - 10^2}{2 \times 9.81}\right) + 0.2\left(\frac{10^2 - 6.67^2}{2 \times 9.81}\right)$$

$= 10 + 0.5 - 2.83 + 0.57 =$ **8.24 m absolute** or **1.76 m vacuum**.

Problem 14.35. *The inlet of a draft tube of a reaction turbine is 2.5 m above the tail race level. The outlet area is 3 times the inlet area. Velocity at inlet is 8 m/s. Kinetic head recovery is 80%. Considering atmospheric head as 10 m water column, determine the pressure at the draft tube inlet.*

Considering tail race level as daum and denoting inlet and outlet by suffixes 1 and 2

$$h_1 + \frac{V_1^2}{2g} + Z_1 = h_2 + \frac{V_2^2}{2g} + Z_2 + \text{Losses} \qquad \text{(A)}$$

$$V_1 = \mathbf{8\ m/s},\ V_2 = \mathbf{8 \times \frac{1}{3} = 2.67\ m/s}$$

$$Z_1 = \mathbf{2.5\ m},\ Z_2 = \mathbf{0},\ h_2 = \mathbf{10\ m}$$

$$\text{Losses} = 0.2\left(\frac{V_1^2 - V_2^2}{2g}\right)$$

Substituting the values,

$$\mathbf{h_1} = 10 - 2.5 + \left(\frac{2.67^2 - 8^2}{2 \times 9.81}\right) + 0.2\left(\frac{8^2 - 2.67^2}{2 \times 9.81}\right)$$

$$= \mathbf{5.18\ m\ absolute\ or\ 4.82\ m\ vacuum}$$

Head lost by friction and outlet velocity are

$$\mathbf{h_2} = 0.2\left(\frac{8^2 - 2.67^2}{2 \times 9.81}\right) + \frac{2.67^2}{2 \times 9.81} = 0.58 + 0.36 = \mathbf{0.94\ m.}$$

REVIEW QUESTIONS

1. Explain how hydraulic turbines are classified.
2. What are the types of turbines suitable under the following conditions : (*a*) high head and low discharge (*b*) medium head and medium discharge and (*c*) low head and large discharge.
3. What is the advantage gained by diverting the water jet on both sides by the splitter in the buckets of Pelton wheel.
4. List the range of dimensional and non dimensional specific speeds for the various types of hydraulic turbines.
5. For a given power and speed which factor controls the value of specific speed.
6. What are the main advantages of model testing ?
7. Explain why model testing becomes almost mandatory in the case of hydraulic turbines.
8. List the conditions to be satisfied by a model so that it can be considered similar to the prototype.
9. List the dimensionless coefficients used in model testing of hydraulic turbines.
10. Explain how unit quantities are useful in predicting the performance of a given machine under various input output conditions.
11. Explain why a notch is made in lips of Pelton turbine buckets.
12. Explain why penstock pipes are of larger diameter compared to the jet diameters.
13. What is the advantage and limitations in doubling the diameter of a penstock pipe.
14. What is the function of the casing in the pelton turbines ?

15. Explain why the casing is of spiral shape with uniform change in area, in the case of reaction turbines.
16. Explain why inward radial flow turbines are popular compared to outward flow units.
17. Explain the reason for higher part load efficiency in the case of kaplan turbine as compared to Francis or propeller units.
18. List the functions of guide vanes in the case of Francis and kaplan turbines.
19. Which is the factor to be controlled in governing of hydraulic turbines ? Why this factor ?
20. Explain the function of draft tube in the case of reaction turbines.
21. What is cavitation ? How does it affect the performance of hydraulic machines?
22. Define Thoma's cavitation number.
23. Explain why it may be necessary in certain cases to install the turbine below the tail water level.
24. Explains how the specific speed can be interpreted as a runner shape factor.
25. What are the main differences between impulse and reaction turbines.
26. List the various efficiencies used to express the performance of hydraulic turbines.
27. Explain the factor in the design of blade angles at outlet which maximises the energy utilisation.
28. Define speed ratio and flow ratio.
29. List the range of speed ratio for the various types of turbines.
30. Explain how aerodynamic blading improves the efficiency of kaplan turbine.

OBJECTIVE QUESTIONS

I. Fill in the blanks with suitable words.

1. The specific speed is lowest in the case of _____ turbines.
2. The specific speed is highest in the case of _____ turbines.
3. Axial thrust is minimal in the case of _____ turbine.
4. The turning angle of the jet in the case of pelton turbine is about _____.
5. The casing is under pressure in the case of _____ and _____ turbines.
6. Spiral casing is used in the case of _____ and _____ turbines.
7. A jet deflector is used in pelton turbine when the load _____.
8. Governing of hydraulic turbines is by varying the _____ of water.
9. In Kaplan turbine the rotor blades can be _____ to suit the load.
10. Sudden cutting off the water flow will cause _____ in the penstock.

Answers

(1) Pelton (2) Kaplan/propeller (3) Pelton (4) 165° (5) Francis, Kaplan (6) Francis, Kaplan (7) suddenly dropped (8) flow rate (9) rotated (10) water hammer

Choose the correct answer

1. Dimensionless specific speed for pelton turbine is about
   (*a*) 0.02 (*b*) 0.2 (*c*) 2 (*d*) 20.
2. Dimensionless specific speed for Francis turbine is about
   (*a*) 0.6 (*b*) 6 (*c*) 60 (*d*) 600.

3. Dimensionless specific speed for kaplan turbine is about
   (*a*) 0.9 (*b*) 9 (*c*) 90 (*d*) 900.
4. Dimensional specific speed for hydraulic turbines is
   (*a*) $\frac{P\sqrt{N}}{H^{3/4}}$ (*b*) $\frac{N\sqrt{P}}{H^{1.25}}$ (*c*) $\frac{N\sqrt{H}}{P^{3/4}}$ (*d*) $\frac{P\sqrt{H}}{N^{3/4}}$.
5. Overall efficiency of Hydraulic turbines is (the suffixes have the useral meaning)
   (*a*) $\eta_v / \eta_m \cdot \eta_H$ (*b*) $\eta_H / \eta_u \cdot \eta_v$ (*c*) $\eta_m \eta_H \eta_v$ (*d*) $\eta_H \cdot \eta_v / \eta_m$.
6. A low specific speed Francis turbine is
   (*a*) Axial flow turbine (*b*) tangential flow turbine
   (*c*) Mixed flow turbine (*d*) radial flow turbine.
7. A high specific speed Francis turbine is
   (*a*) Radial flow turbine (*b*) Tangential flow turbine
   (*c*) Mixed flow turbine (*d*) Axial flow turbine.
8. Example of a pure reaction turbine is
   (*a*) Francis turbine (*b*) Propeller turbine
   (*c*) Kaplan turbine (*d*) Lawn sprinkler.
9. Pelton turbine is a
   (*a*) Reaction turbine (*b*) Impulse turbine
   (*c*) Radial flow turbine (*d*) Axial flow turbine.
10. Flow ratio is
    (*a*) Flow velocity / blade velocity (*b*) Flow velocity / Relative velocity
    (*c*) Flow velocity / $\sqrt{2g\,H}$ (*d*) Flow velocity /whirl velocity.

Answers

(1) *a* (2) *a* (3) *a* (4) *b* (5) *c* (6) *d* (7) *c* (8) *d* (9) *b* (10) *c*.

EXERCISE PROBLEMS

E 14.1. At a location the power potential was estimated as 4 MW. The head available was 35 m. The speed chosen is 300 rpm. Determine the dimensional and non dimensional specific speeds. If the speed ratio is 0.8 and the flow ratio is 0.35 **determine the diameter of the runner.**

E 14.2. The head available at a location is 1000 m. The unit chosen is to run at 300 rpm. The power potential is 120 MW. **Estimate how many units will be required** if these are of single jet type with dimensionless specific speed of 0.021.

E 14.3. A model of scale 1/4 is to be designed. The prototype is to develop 8 MW and run at 600 rpm. The head available is 40 m. If the dynamometer facility is limited to 25 kW, **determine the head required and the speed of the model. If the** overall efficiency is 85% in both cases, determine the flow rates.

E 14.4. Water is brought by four penstocks of 0.9 m diameter over a distance of 1.6 km to the turbine. At the nozzle inlet the head available is 500 m. The diameter of the four nozzles is 0.2 m each, $C_v = 0.98$, $f = 0.032$.

Determine the power available in each jet and the head lost by friction. (14.96 MW, 66.8 m).

E 14.5. A Pelton turbine delivers 14.25 MW when running at 600 rpm under a head of 900 m. Assuming an overall efficiency of 89.2%. **Determine the jet diameter and the wheel diameter.** Assume C_v = 0.98 and speed ratio as 0.46. (**134 mm, 1910 mm**).

E 14.6. Pelton turbine deliver 15 MW under a head of 1000 m when running at 300 rpm. The blade speed ratio is 0.48 and C_v = 0.98. The jet is turned through 165° by the bucket. The relative velocity is reduced by 15% by blade friction. **Determine the hydraulic efficiency. Also calculate the wheel diameter and nozzle diameter.** The mechanical and volumetric efficiencies are 0.96 and 0.98. Also find the number of buckets.

E 14.7. A Pelton turbine delivers 13 MW at 430 rpm when the head available was 800 m. The overall efficiency is 85% **Determine the diameter of the wheel and diameter of the nozzle.** Assume C_v = 0.98 and blade speed ratio as 0.46. (**2 m, 0.2 m**).

E 14.8. **Determine the number and diameter of jets for** a Pelton turbine producing 20 MW under a head of 450 m running at a speed of 475 rpm. The jet diameter is not to exceed one twelfth of the wheel diameter. Also find the diameter of the wheel and water flow rate. Assume overall efficiency is 86% and C_v = 0.97 and ϕ = 0.46.

E 14.9. A Pelton wheel has a bucket speed of 20 m/s and jet speed of 42 m/s. The flow rate is 450 kg/s. The jet is deflected by 165°. The relative velocity is reduced by 12% by friction. **Determine the power generated**. What is the efficiency of the unit ? (**36.5 kW, 93.47%**)

E 14.10. At a site the head available is 315 m and the maximum power potential is 35 MW. If the dimensionless specific speed is to be in the range 0.015 to 0.02 determine the number of units required and the power of each unit.

If the overall efficiency is expected to be 85%, determine the flow rate. If C_v = 0.97 and ϕ = 0.46, **determine the speed if the wheel diameter is 10 times the jet diameter.**

E 14.11. A Pelton wheel running at 500 rpm develops 5 MW under a net head of 300 m. Assume C_v = 0.97, speed ratio = 0.46, η_o = 80%, jet diameter to wheel diameter is 1:10. **Determine the wheel diameter and the number of jets required.** (**1.3 m, 4**)

E 14.12. In an impulse turbine of the Pelton type, the jet is turned by the bucket by 165°. The head available at the nozzle is 750 m and blade speed ratio is 0.46. C_v = 0.98. Relative velocity is reduced by 12% due to friction. Determine the hydraulic efficiency. If the flow available is 20 m^3/s. What is the power potential. Assuming 5 units of equal power, **determine the jet diameter and wheel diameter** if d/D = 12.

E 14.13. A Francis turbine running at 700 rpm works under a head of 160 m. The water enters the runner at 28 m/s, the outlet angle of the guide blade being 12°. The inner and outer diameter are 1.3 m and 0.8 m. Assuming the discharge to be axial and constant flow velocity **determine the runner blade angles. Also calculate the hydraulic efficiency.**

E 14.14. In a Francis turbine the runner diameter ratio is ID/OD = 0.5. The runner width is 0.1 OD, Flow ratio is 0.18. The power is 300 kW under a head of 65 m when running at 700 rpm. η_H = 94%, η_o = 85%.

Determine the diameter and blade angles assuming radial flow at exit

(**OD = 0.52 m, ID = 0.26 mm, b = 0.052 m, α_1 = 11.6°, β_1 = 27.8°, β_2 = 34°**).

E 14.15. In an inward flow reaction turbine the available head is 60 m. The blade angle at inlet is 100°. The flow ratio is 0.3. Hydraulic efficiency is 87%. If the speed is 475 rpm, **determine the runner diameter.** **(0.09 m)**

E 14.16. An inward flow reaction turbine develops 185 kW under a head of 30 m. Guide vane outlet angle is 20°. Vane outlet angle is 25° and outlet is radial. The ratio of inlet to outlet area is 3 : 4. Loss in guide vane is 10% of velocity head at inlet. Loss in the runner is 20% of outlet radial velocity head. **Determine the pressure at inlet of the wheel, flow rate, and area at outlet of the guide vanes. (Q = 0.7256 m^3/s, 0.1163 m^2)**

E 14.17. A Francis turbine works under a head of 95 m, the flow rate being 6 m^3/s. The flow velocity remains constant at 18 m/s. The blade inlet is at 90°. The width of the runner at inlet is 0.16 times the diameter and the inner diameter is 0.6 times the outer diameter. **Determine the runner diameter** and hydraulic efficiency. Assume that blade thickness reduces the flow area by 10%. Also calculate **the guide blade and runner outlet angles.** Assume zero whirl at exit.

E 14.18. An inward flow reaction turbine has diameters 45 cm and 30 cm the widths being 5 cm and 7.5 cm. Vanes occupy 8% of flow area. Guide vane angle is 24° and runner blade angles are 95° and 30°. Head available is 52.8 m. Hydraulic efficiency is 88%. Mechanical efficiency is 96%. **Determine the turbine speed for smooth flow.** Also find the **power** and **angle the absolute velocity at exit makes** with tangential direction. (**900 rpm, 455 Nm/s/kg, 79.6°**)

E 14.19. An inward flow reaction turbine works under a head of 25 m. The hydraulic efficiency is 80%. The turbine speed is 300 rpm. The peripheral velocity is 30 m/s. The flow velocity is 4 m/s. **Determine the guide blade outlet angle, the runner inlet angle and the runner diameter. (31.4°, 170.3°, 1.91 m)**

E 14.20. In a Francis turbine, the wheel peripheral velocity is 24 m/s. The whirl velocity at inlet is 18 m/s. The flow velocity is 6 m/s. The whirl at exit is zero. Assuming no other losses, **determine the head on the turbine. Also find the vane angle at inlet. (45.8 m, 135°).**

E 14.21. A Francis turbine works under a head of 14 m. The guide vane outlet angle is 20° and the blade angle at inlet is 90°. The ratio of diameters is 3 : 2. The flow velocity is constant at 4 m/s. **Determine the peripheral velocity of the runner.** Assume zero whirl at exit.

(**11 m/s, 28.6°**)

E 14.22. A Francis turbine with OD = 1 m and b = 0.135 m runs at 180 rpm. The flow velocity is 3 m/s. The flow is **radial both at inlet and outlet.**

Determine (*i*) Work done per kg (*ii*) Power developed and (*iii*) Hydraulic efficiency. (88.7 Nm/kg/s, 113 kW, 90.7%)

E 14.23. In a Francis turbine the guidevane outlet angle is 10°. The inlet blade angle is 90°. The runner diameters are 1 m and 0.5 m. Whirl at exit is zero. The flow velocity is 3 m/s both at inlet and outlet. **Determine the speed of the wheel and the outlet angle of the runner. (324 rpm, 19.4°).**

E 14.24. In a Francis turbine installation the total head at entry to the spiral casing is 120 m. The distance to the tailrace from this level is 3 m. The tangential velocity at entrance is 30 m/s. The flow velocity is constant at 9 m/s. The discharge is without whirl. The estimated hydraulic losses are (*i*) 4.8 m in the guide vanes. (*ii*) 8.8 m in the rotor. (*iii*) 0.79 m in the draft tube (*iv*) kinetic head at exil is 0.46 m. **Determine the guide vane angle at exit, the runner blade angle at inlet and the pressure heads at the runner entry and exit.** Assume η_H = 0.9.

(**14.3°, 59.46°, 47.4 m, – 5.9 m)**

E 14.25. A Franics turbine runs at 268 rpm. The outer diameter is 1.2 m. The blade angle at inlet is 90°. If the flow rate is 1 kg/s, determine the power developed. If the head is 30 m determine the hydraulic efficiency. If the guide blade outlet angle is 15° and if the flow velocity is constant, **determine the runner outlet angle.** Whirl at exit is zero. **(92.7%, 28.2°)**

E 14.26. A Kaplan turbine runner hub and tip diameters are 2.5 m and 5 m respectively. When running at 150 rpm under a head of 30 m, it develops 30 MW. The overall efficiency is 85% and hydraulic efficiency is 90%. **Determine the runner blade angles at inlet and outlet both at the tip and the hub.** Assume zero whirl at exit, and the product of whirl and tip speed is constant at all diameters. **(α_{1t} = 14°, β_{1h} = 52.9°, α_{2t} = 11.7°, β_{2h} = 22.5°)**

E 14.27. A Kaplan turbine delivers with an overall efficiency of 90%. 25 MW, the head available being 40 m. The speed ratio and flow ratio are 2 and 0.6 respectively. The hub to tip ratio is 0.4. **Determine the diameter and speed of the turbine. (2.5 m, 428.6 rpm)**

E 14.28. A Kaplan turbine produces 12 MW under a head of 28 m. The hub and tip diameters of the runner are 1.1 m and 2.9 m respectively. If the overall efficiency is 85% and if the inlet and outlet velocity triangles are right angled, **calculate the speed and guide blade and runner outlet angles. Hydraulic efficiency is 94%.**

E 14.29. A Kaplan turbine develops 35000 kW while running at 240 rpm. The dimensional specific speed is 320. **Determine the head available.** If the overall efficiency is 84.5% and the hub tip ratio is 0.3 **determine the speed ratio based on outer diameter.** Assume a flow ratio of 0.35. **(21.27 m, 2.525)**

E 14.30 An axial flow turbine has a tip diameter of 4.5 m and hub diameter of 2.5 m. The power developed is 21 MW. The running speed is 140 rpm. The net head is 20 m. The hydraulic and overall efficiencies are 94% and 80% respectively. **Calculate the guide vane outlet angle and the blade inlet angle. (63.2°, 22.05°)**

E 14.31. An axial flow turbine of the Kalpan type develops 20 MW when working under a head of 20 m. The speed ratio is 2. The flow ratio is 0.6. The hub diameter is 35% of the tip diameter. **Calculate the turbine speed and diameter. (198 rpm, 3.8 m)**

E 14.32. A Kaplan turbine operates under a net head of 20 m and develops 16 MW with a hydraulic efficiency of 90 percent and overall efficiency of 80 percent. The runner outer diameter is 4.2 m. The hub diameter is 2 m. The dimensionless specific speed is 0.8. **Determine the blade inlet and outlet angles at the tip if** $V_{u2} = 0$. **(9.2°, 18°)**

E 14.33. An axial flow turbine develops 7.5 MW under a head of 5.5 m with an overall efficiency of 85%. The hub diameter is 1/3 of the tip diameter. **Determine the diameter and speed of the runner. (5.58 m, 75 rpm)**

E 14.34. In a draft tube fixed to a reaction turbine the inlet diameter is 3 m and the outlet area is 20 m^2. The velocity at inlet is 5 m/s. The turbine exit is 5 m above the tail race level. The loss in the draft tube is 50% of the velocity head at outlet, **Determine the pressure at the top of the draft tube.** Also find the head lost in the draft tube. **(6.03 m vacuum, 0.08 m)**

E 14.35. The following data refer to the draft tube of a large Francis turbine. Inlet area 25 m^2, Area at exit 75 m^2, velocity at inlet : 12 m/s. The level above tailrace level of turbine outlet = 0.9 m. If the loss in the draft tube due to friction is 20% of the inlet kinetic energy **determine the vacuum at the turbine outlet. Also find the power wasted in the exit kinetic energy. (5.95 m, vacuum, 2000 kW)**

E 14.36. The inlet diameter of a draft tube is 3 m. The height above tailrace is 2.5 m. The pressure measured at the top was 3.1 m vacuum. The flow rate is 30 m^3/s. **Determine the efficiency of the draft tube. (65.5%)**

E 14.37. A Franics turbine for a power plant is to be designed for a power of 30 MW. The head available is 190 m. The speed is to be 180 rpm. A model for the unit is to be designed. The power available in the laboratory is 40 kW. A one sixth scale model is to be adopted. **Determine the speed, head and flow rate** for the model. Assume an overall efficiency of 98%.

E 14.38. The speed ratio in a Pelton turbine is 0.46. The head available is 300 m. The flow rate available is 2.5 m^3/s. The coefficient of velocity is 0.98. The diameter should not exceed 3 m. The jet is deflected by 167°. The blade friction coefficient is 0.1. **Determine the hydraulic efficiency. Also determine the speed and power delivered. Mechanical efficiency is 87%.**

E 14.39. A Pelton turbine is to deliver 12 MW. The mechanical and generator efficiencies are 0.85 and 0.95. The head available is 700 m. $C_v = 0.98$. Blade speed ratio is 0.46. The jet is deflected by 165°. Blade friction reduces the relative velocity by 12%.

Determine the overall efficiency, flow rate and jet diameter. If the speed is 180 rpm, **determine the wheel diameter.**

15 Rotodynamic Pumps

15.0 INTRODUCTION

Liquids have to be moved from one location to another and one level to another in domestic, agricultural and industrial spheres. The liquid is more often water in the domestic and agriculture spheres. In industries chemicals, petroleum products and in some cases slurries have to be moved, by pumping. **Three types of pumps** are in use.

(1) **Rotodynamic pumps** which move the fluid by dynamic action of imparting momentum to the fluid using mechanical energy. (2) **Reciprocating pumps** which first trap the liquid in a cylinder by suction and then push the liquid against pressure. (3) **Rotary positive displacement pumps** which also trap the liquid in a volume and push the same out against pressure.

Reciprocating pumps are limited by the **low speed of operation** required and **small volumes it can handle.**

Rotary positive displacement pumps are limited **by lower pressures** of operation and **small volumes** these can handle. Gear, vane and lobe pumps are of these type. **Rotodynamic pumps** *i.e.* centrifugal and axial flow pumps can be operated at **high speeds** often directly coupled to electric motors. These can handle from **small volumes to very large volumes.** These pumps **can handle corrosive and viscous**, **fluids and even slurries**. The **overall efficiency is high** in the case of these pumps. Hence these are found to be the most popular pumps in use. Rotodynamic pumps can be of **radial flow, mixed flow and axial flow** types according to the flow direction. **Radial flow** or purely centrifugal pumps generally handle **lower volumes at higher pressures. Mixed flow** pumps handle comparatively **larger volumes** at **medium range of pressures. Axial flow pumps** can handle **very large volumes,** but the **pressure** against which these pumps operate is **limited**. The overall efficiency of the three types are nearly the same.

15.1 CENTRIFUGAL PUMPS

These are so called because energy is imparted to the fluid by centrifugal action of moving blades from the inner radius to the outer radius. The **main components** of centrifugal pumps are (1) **the impeller,** (2) **the casing** and (3) **the drive shaft with gland and packing.**

Additionally suction pipe with one way valve (foot valve) and delivery pipe with delivery valve completes the system.

The liquid enters the eye of the impeller axially due to the suction created by the impeller motion. The impeller blades guide the fluid and impart momentum to the fluid, which increases the total head (or pressure) of the fluid, causing the fluid to flow out. The fluid comes out at a high velocity which is not directly usable. The casing can be of simple volute type or a diffuser can be used as desired. The volute is a spiral casing of gradually increasing cross section. A part of the kinetic energy in the fluid is converted to pressure in the casing.

Figure 15.1.1 shows a sectional view of the centrifugal pump.

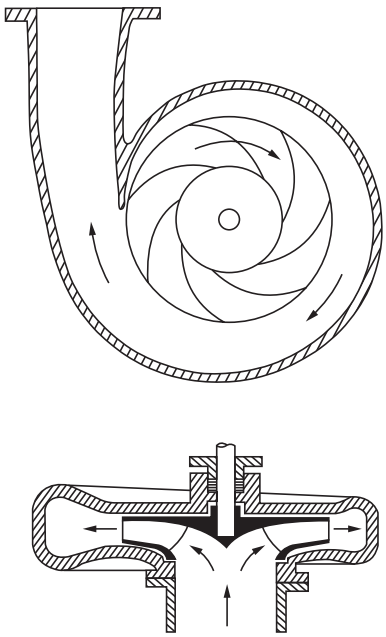


Figure 15.1.1 Volute type centrifugal pump.

Gland and packing or so called stuffing box is used to reduce leakage along the drive shaft. By the use of the volute only a small fraction of the kinetic head can be recovered as useful static head. A diffuser can diffuse the flow more efficiently and recover kinetic head as useful static head. A view of such arrangement is shown in figure 15.1.2. Diffuser pump are also called as turbine pumps as these resembles Francis turbine with flow direction reversed.

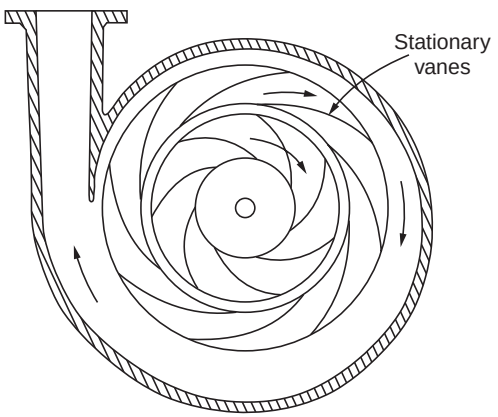


Figure 15.1.2 Diffuser pump.

15.1.1 Impeller

The impeller consists of a disc with blades mounted perpendicularly on its surface. The blades may of three different orientations. These are **(*i*) Radial, (*ii*) Backward curved, and (*iii*) Forward curved.** Backward and forward refers to the direction of motion of the disc periphery. Of these the most popular one is the backward curved type, due to its desirable characteristics, which reference to the static head developed and power variation with flow rate. This will be discussed in detail later in this chapter.

A simple disc with blades mounted perpendicularly on it is called **open impeller.** If another disc is used to cover the blades, this type is called **shrouded impeller.** This is more popular with water pumps. Open impellers are well adopted for use with dirty or water containing solids. The **third type** is just the **blades spreading out from the shaft.** These are used to pump slurries. Impellers may be of cast iron or bronzes or steel or special alloys as required by the application. In order to maintain constant radial velocity, the width of the impeller will be wider at entrance and narrower at the exit. This may be also noted from figure 15.1.1.

The blades are generally cast integral with the disc. Recently even plastic material is used for the impeller. **To start delivery of the fluid the casing and impeller should be filled with the fluid without any air pockets. This is called priming.** If air is present the there will be only compression and no delivery of fluid. In order to release any air entrained an air valve is generally provided **The one way foot value keeps the suction line and the pump casing filled with water.**

15.1.2 Classification

As already mentioned, centrifugal pumps may be classified in several ways. On the **basis of speed** as low speed, medium speed and high speed pumps. On the basis of **direction of flow** of fluid, the classification is **radial flow**, **mixed flow** and **radial flow.** On the basis of **head** pumps may be classified as **low head** (10 m and below), **medium head** (10-50 m) **and high head** pumps. Single entry type and double entry type is another classification. Double entry pumps have blades on both sides of the impeller disc. This leads to reduction in axial thrust and increase in flow for the same speed and diameter. Figure 15.1.3 illustrates the same. When the head required is high and which cannot be developed by a single impeller, **multi staging** is used. In deep well submersible pumps the diameter is limited by the diameter of the bore well casing. In this case multi stage pump becomes a must. In multi stage pumps several impellers are mounted on the same shaft and the outlet flow of one impeller is led to the inlet of the next impeller and so on. The total head developed equals the sum of heads developed by all the stages.

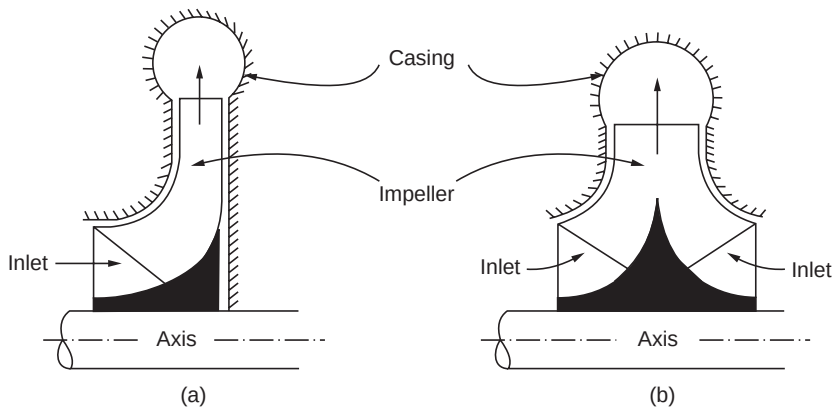


Figure 15.1.3 Single and double entry pumps

Pumps may also be operated in parallel to obtain large volumes of flow. The characteristics under series and parallel operations is discussed later in the chapter. The classification may also be based on the **specific speed** of the pump. In chapter 9 the dimensionless parameters have been derived in the case of hydraulic machines. The same is also repeated in example 15.1. The expression for the dimensionless specific speed is given in equation 15.1.1.

$$N_s = \frac{N\sqrt{Q}}{(gH)^{3/4}} \tag{15.1.1}$$

More often dimensional specific speed is used in practise. In this case

$$N_s = \frac{N\sqrt{Q}}{H^{3/4}} \tag{15.1.1a}$$

The units used are : N in rpm, Q in m^3/s, and H in meter.

Typical values are given in table 15.1

Table 15.1 Specific speed classification of pumps.

Flow direction	speed	Dimensional specific speed	Non Dimensional specific speed
Radial	Low	10 – 30	1.8 – 5.4
	Medium	30 – 50	5.4 – 9.0
	High	50 – 80	9.0 – 14.0
Mixed flow		80 – 160	14 – 29
Axial flow		100 – 450	18 – 81

The best efficiency is obtained for the various types of pumps in this range of specific speeds indicated.

15.2. PRESSURE DEVELOPED BY THE IMPELLER

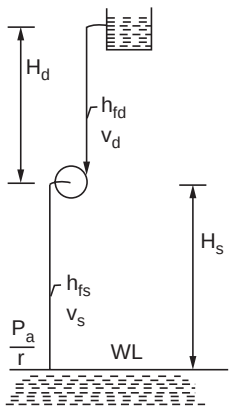
The general arrangement of a centrigugal pump system is shown in Figure 15.2.1.

H_s—Suction level above water level.

H_d—Delivery level above the impeller outlet.

h_{fd}, h_{fs}—frictionless $\dot{m}$, m.

V_s, V_d—pipe velocities.



Applying Bernoullis equation between the water level and pump suction,

$$\frac{P_a}{\gamma} + H_s + h_{fs} + \frac{V_s^2}{2g} = \frac{P_s}{\gamma} \tag{15.2.1}$$

$$\therefore \quad \frac{P_s}{\gamma} + \frac{P_a}{\gamma} H_s + h_{fs} + \frac{V_s^2}{2g} \tag{15.2.2}$$

Similarly applying Bernoulli's theorem between the pump delivery and the delivery at the tank,

$$\frac{P_d}{\gamma} + \frac{V_d^2}{2g} = \frac{P_a}{\gamma} + H_d + h_{fd} + \frac{V_d^2}{2g} \tag{15.2.3}$$

or
$$\frac{P_d}{\gamma} = \frac{P_a}{\gamma} + H_d + h_{fd} \tag{15.2.3a}$$

where P_d is the pressure at the pump delivery. From 15.2.2 and 15.2.3*a*

$$\frac{P_d}{\gamma} - \frac{P_s}{\gamma} = \frac{P_a}{\pi} + H_d + h_{fd} - \frac{P_a}{\gamma} + \frac{V_s^2}{2g} + H_s + h_{fs}$$

$$= H_d + H_s + h_f + \frac{V_s^2}{2g} = H_e + \frac{V_s^2}{2g} \tag{15.2.4}$$

where H_e is the effective head.

15.2.1 Manometric Head

The official code defines the head on the pump as the difference in total energy heads at the suction and delivery flanges. This head is defined as manometric head.

The total energy at suction inlet (expressed as head of fluid)

$$\frac{P_s}{\gamma} + \frac{V_s^2}{2g} + Z_s$$

where Z_s is the height of suction gauge from datum.

The total energy at the delivery of the pump

$$= \frac{P_d}{\gamma} + \frac{V_d^2}{2g} + Z_d$$

Z_2 is the height of delivery gauge from datum.

$\therefore$ The difference in total energy is defined as H_m

$$= \left(\frac{P_d}{\gamma} - \frac{P_s}{\gamma}\right) + \frac{V_d^2 - V_s^2}{2g} + (Z_d - Z_s)$$

From equation 15.2.4,

$$\frac{P_d}{\gamma} - \frac{P_s}{\gamma} = H_e + \frac{V_s^2}{2g}$$

Substituting
$$H_m = H_e + \frac{V_d^2}{2g} + (Z_d - Z_s) \tag{15.2.5}$$

As $(Z_d - Z_s)$ is small and $\frac{V_d^2}{2g}$ is also small as the gauges are fixed as close as possible.

$\therefore$ **H_m = Static head + all losses.**

15.3 ENERGY TRANSFER BY IMPELLER

The energy transfer is given by Euler Turbine equation applied to work absorbing machines,

$$W = -(u_1 V_{u1} - u_2 V_{u2}) = (u_2 V_{u2} - u_1 V_{u1})$$

This can be expressed as ideal head imparted as

$$H_{\text{ideal}} = \frac{u_2 V_{u2} - u_1 V_{u1}}{g} \tag{15.3.1}$$

The velocity diagrams at inlet and outlet of a backward curved vaned impeller is shown in figure 15.3.1. **The inlet whirl is generally zero.** There is no guide vanes at inlet to impart whirl. So the **inlet triangle is right angled.**

$V_1 = V_{f1}$ and are radial

$$\tan \beta_1 = \frac{V_1}{u_1} \quad \text{or} \quad \frac{V_f}{u_1}$$

$$V_{u1} = 0$$

$$\therefore \quad H_{\text{ideal}} = \frac{u_2 V_{u2}}{g} \tag{15.3.2}$$

From the outlet triangle,

$$u_2 = \pi D_2 N/60,$$

$$V_{u2} = u_2 - \frac{V_{f2}}{\tan \beta_2}$$

$$\therefore \quad H_{\text{ideal}} = \frac{u_2}{g}\left[u_2 - \frac{V_{f2}}{\tan \beta_2}\right] \tag{15.3.3}$$

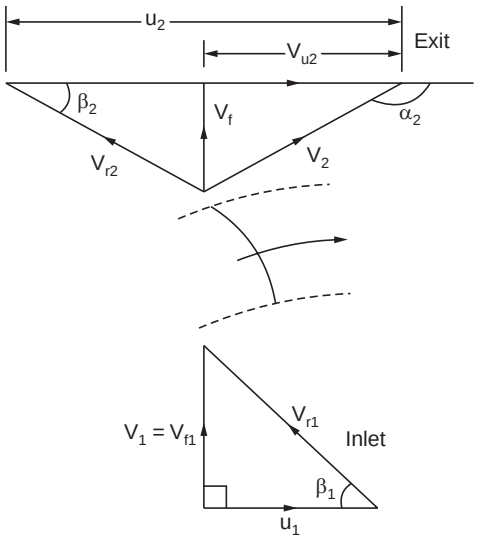


Figure 15.3.1 Velocity triangles for backward curved bladed pump.

Manometric efficiency is defined as the ratio of manometric head and ideal head.

$$\eta_m = \frac{H_m \times g}{u_2(u_2 - V_{f2}/\tan \beta_2)}$$

H_m = Static head + all losses (for practical purposes).

$$\text{Mechanical efficiency} = \eta_{\text{mech}} = \frac{\text{Energy transferred to the fluid}}{\text{Work input}}$$

$$= \frac{(u_2 V_{u2}) Q\rho}{\text{power input}} \tag{15.3.4}$$

$$\text{Overall efficiency} = \eta_o = \frac{\text{Static head} \times Q \times \rho \times g}{\text{Power input}} \tag{15.3.5}$$

There are always some leakage of fluid after being imparted energy by the impeller.

$$\text{Volumetric efficiency} = \frac{\text{Volume delivered}}{\text{Volume passing through impeller}}$$

Thus $$\boldsymbol{\eta_o = \eta_m \, m_{mech} \cdot \eta_{vol}.} \tag{15.3.6}$$

$\frac{V_d^2}{2g}$ is not really useful as output of the pump. Hence the useful amount of energy transfer (as head, is taken as (H_a)

$$H_a = \frac{u_2 V_{u2}}{g} - \frac{V_d^2}{2g}$$

By algebraic manipulation, this can be obtained as

$$H_a = (u_2^2 - V_f^2 \operatorname{cosec}^2 \beta_2)/2g \tag{15.3.7}$$

15.3.1 Slip and Slip Factor

In the analysis it is assumed that all the fluid between two blade passages have the same velocity (both magnitude of direction). Actually at the leading edge the pressure is higher and velocity is lower. On the trailing edge the pressure is lower and the velocity is higher. This leads to a circulation over the blades. Causing a non uniform velocity distribution. **The average angle at which the fluid leaves the blade is less than the blades angle.** The result is a reduction in the exit whirl velocity V_{u2}. This is illustrated in the following figure. The solid lines represent the velocity diagram without slip. The angle β_2 is the blade angle. The dotted lines represent the velocity diagram after slip. The angle $\beta_2' < \beta_2$. It may be seen that $V_{u2}' < V_{u2}$. The ratio V_{u2}'/V_{u2} is known as slip factor. The result of the slip is that the energy transfer to the fluid is less than the theoretical value.

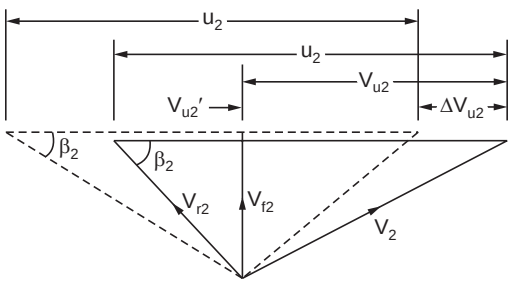


Figure 15.3.2 Velocity triangle with slip

$$H_{th} = \sigma_s \cdot \frac{u_2 V_{u2}}{g} \tag{15.3.8}$$

where σ_s is the slip coefficient or slip factor.

15.3.3 Losses in Centrifugal Pumps

Mainly there are three specific losses which can be separately calculated. These are

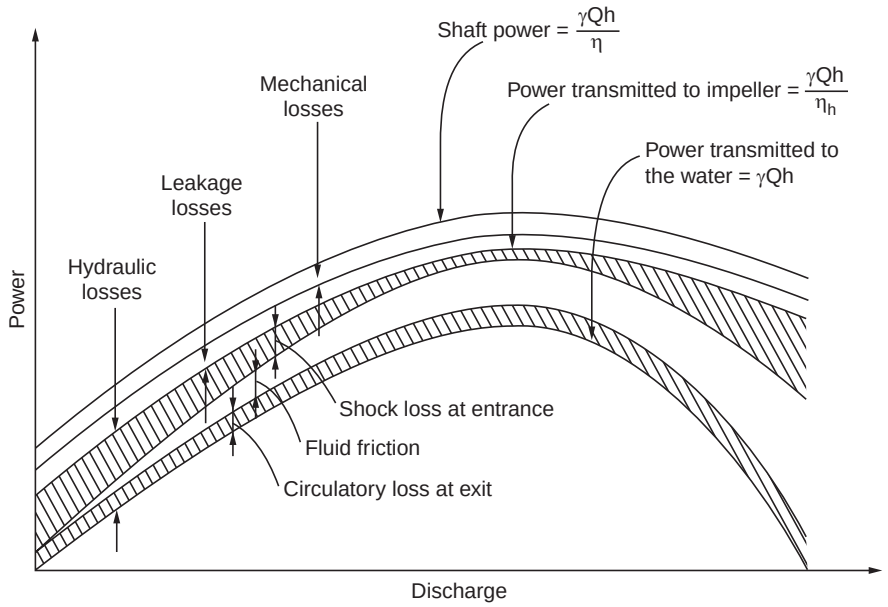


Figure 15.3.3 Losses in pump

(*i*) **Mechanical friction** losses between the fixed and rotating parts in the bearings and gland and packing.

(*ii*) **Disc friction** loss between the impeller surfaces and the fluid.

(*iii*) **Leakage and recirculation** losses. The recirculation is along the clearance between the impeller and the casing due to the pressure difference between the hub and tip of the impeller. The various losses are indicated in figure 15.3.3.

15.3.4 Effect of Outlet Blade Angle

There are three possible orientation of the blade at the outlet. These are : forward curved, radial and Backward curved arrangements. The velocity triangles for the three arrangements are shown in Figure 15.3.4. In the case of forward curved blading $V_{u2} > u_2$ and V_2 is larger comparatively. In the case of radial blades $V_{u2} = u_2$. In the case of backward curved blading, $V_{u2} < u_2$.

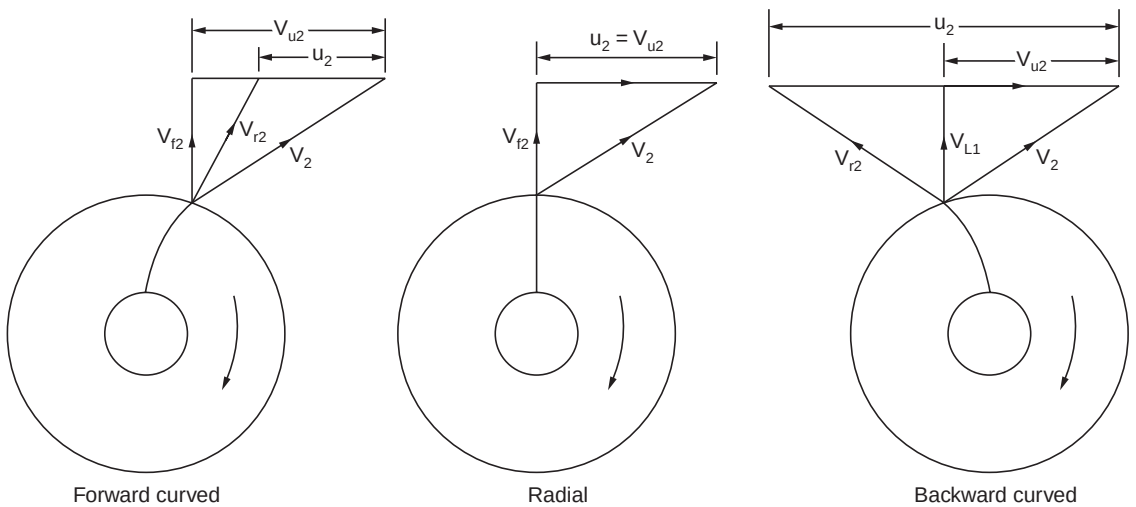


Figure 15.3.4 Different blade arrangements

The head-flow rate curves are shown in Figure 15.3.5. The theoretical head variation can be expressed as

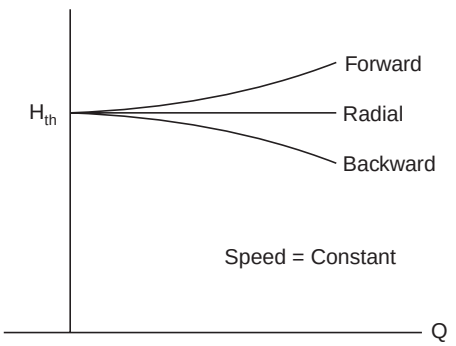


Figure 15.3.5 Head variation

$$\mathbf{H_{th} = k_1 - k_2 \,.\, \cot \beta_2 \, Q}$$

where k_1 and k_2 are constants and β_2 is the outlet blade angle. cot β_2 becomes negative for forward curved blading. So head increases with flow rate. For radial blading cost $\beta_2 = 0$, and hence the head is constant with flow rate. In the case of backward curved blading, the head decreases with flow rate.

The rising characteristics of the forward curved blading leads to increase of power input with increase of Q. The power curve is not self limiting and damage to motor is possible. The forward curved blading is rarely used.

The backward curved blading leads to self limiting power characteristics and reduced losses in the exit kinetic energy.

So the backward curved blading is almost universally used. The radial blading also leads to rising power characteristics and it is used only in small sizes.

15.4 PUMP CHARACTERISTICS

We have seen that the theoretical head

$$H_{th} = \frac{u_2 \, V_{u2}}{g} \quad \text{and} \quad V_{u2} = V_{f2} \cot \beta_2$$

$$V_{f2} = \frac{Q}{A}, \text{ where } A \text{ is the circumferential area.}$$

$$u_2 = \pi \, DN.$$

Substituting these relations in the general equation. We can write

$$H_{th} = \pi^2 D^2 N^2 - \left(\frac{\pi DN}{A} \,.\, \cot \beta_2 \right) Q \,.$$

For a given pump, D, A, β_2 and N are fixed . So at constant speed we can write

$$\mathbf{H_{th} = k_1 - k_2 \, Q} \qquad (15.4.1)$$

where k_1 and k_2 are constants and

$$k_1 = \pi^2 D^2 N^2 \text{ and } k_2 = \left(\frac{\pi DN}{A} \,.\, \cot \beta_2 \right)$$

Hence at constant speed this leads to a drooping linear characteristics for backward curved blading. This is shown by curve 1 in Figure 15.4.1.

The slip causes drop in the head, which can be written as $\sigma \, V_{u2} \, u_2/g$. As flow increases this loss also increases. Curve 2 shown the head after slip. The flow will enter without shock only at the design flow rate. At other flow rates, the water will enter with shock causing losses. This lose can be expressed as

$$\mathbf{h_{shock} = k_3 \, (Q_{th} - Q)^2}$$

The reduced head after shock losses is shown in curve 5. The shock losses with flow rate is shown by curve 3. The mechanical losses can be represented by $\mathbf{h_f = k_4 \, Q^2}$**.** The variation is

shown by curve 4. With variation of speed the head characteristic is shifted near paralley with the curve 5 shown in Figure 15.4.1.

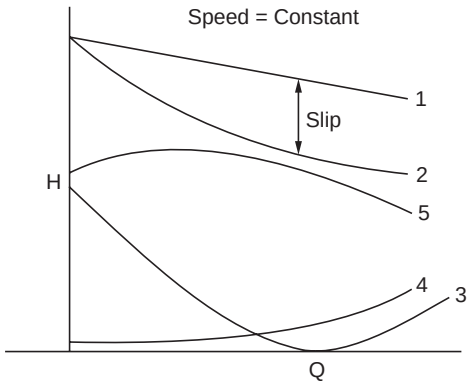


Figure 15.4.1 Characteristics of a centrifugal pump

The characteristic of a centrigugal pump at constant speed is shown in Figure 15.4.2. It may be noted that the power increases and decreases after the rated capacity. In this way the pump is self limiting in power and the choice of the motor is made easy. The distance between the brake power and water power curves gives the losses.

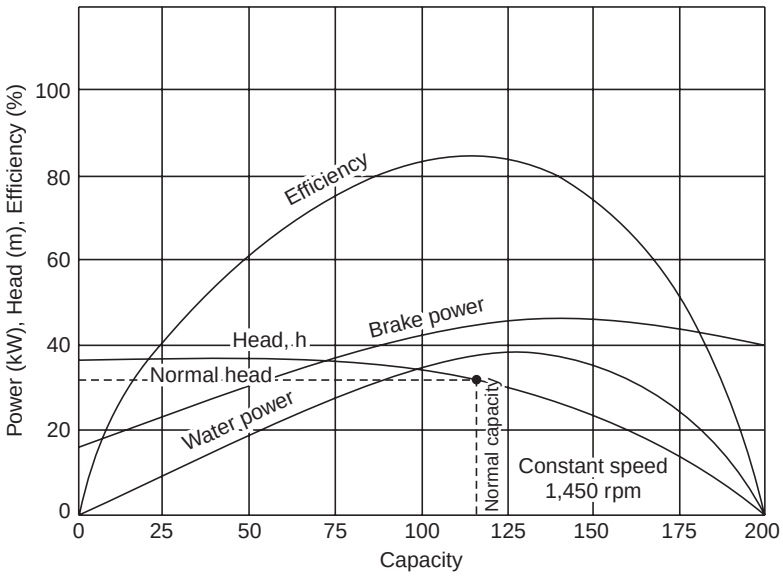


Figure 15.4.2 Centrifugal pump characteristics at constant speed

The pump characteristics at **various speeds** including **efficiency contours** in shown in Figure 15.4.3. Such a plot helps in the development of a pump, particularly in specifying the head and flow rates.

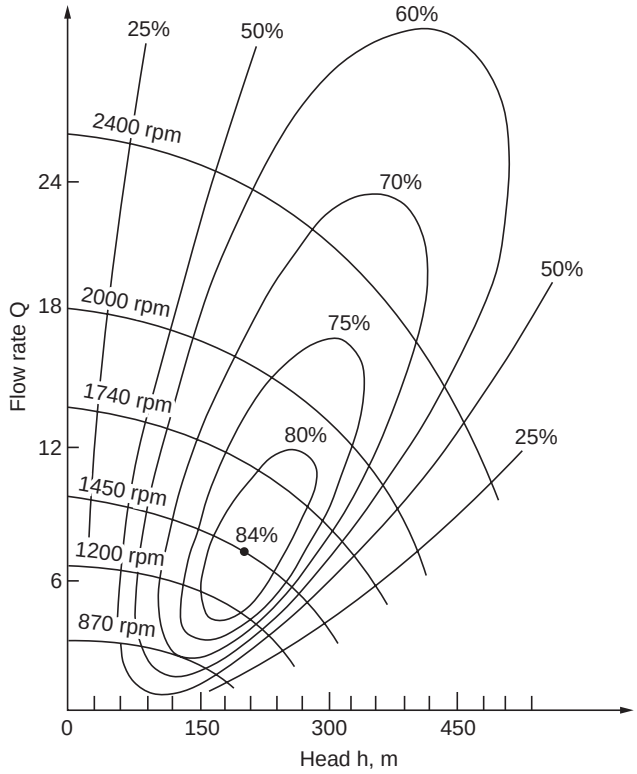


Figure 15.4.3 Pump charateristics at various speeds

15.5 OPERATION OF PUMPS IN SERIES AND PARALLEL

Pumps are chosen for particular requirement. The requirements are not constant as per example the pressure required for flow through a piping system. As flow increases, the pressure required increases. In the case of the pump as flow increases, the head decreases. The operating condition will be the meeting point of the two curves representing the variation of head required by the system and the variation of head of the pump. This is shown in Figure 15.5.1. The operating condition decides about the capacity of the pump or selection of the pump.

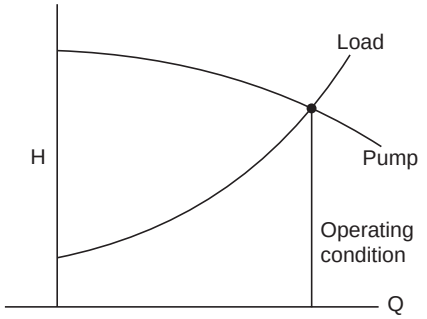


Figure 15.5.1 Pump-load characteristics

If in a certain setup, there is a need for increased load, either a completely new pump may be chosen. This may be costlier as well as complete revamping of the setup. An additional pump can be the alternate choice. If the head requirement increases the old pump and the new pump can operate in series. In case more flow is required the old pump and the new pump will operate in parallel. There are also additional advantages in two pump operation. When the

load is low one of the pump can operate with a higher efficiency when the load increases then the second pump can be switched on thus improving part load efficiency. The characteristics of parallel operation is depicted in Figure 15.5.2.

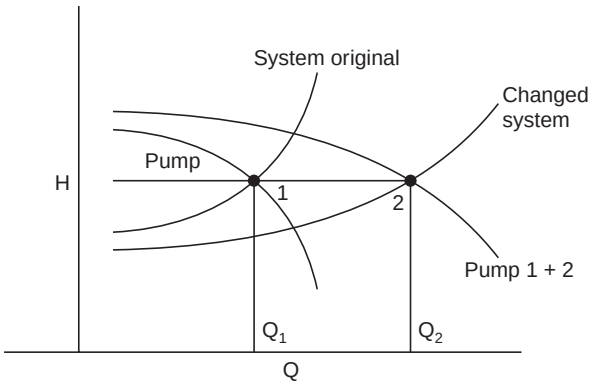


Figure 15.5.2 Pumps in parallel

The original requirement was Q_1 at H_1. Pump 1 could satisfy the same and operating point is at 1. When the flow requirement and the system characteristic is changed such that Q_2 is required at head H_1, then two pumps of similar characteristics can satisfy the requirement. Providing a flow volume of Q_2 as head H_1. It is not necessary that similar pumps should be used. Suitable control system for switching on the second pump should be used in such a case.

When the head requirement is changed with flow volume being the same, then the pumps should work in series. The characteristics are shown in Figure 15.5.3.

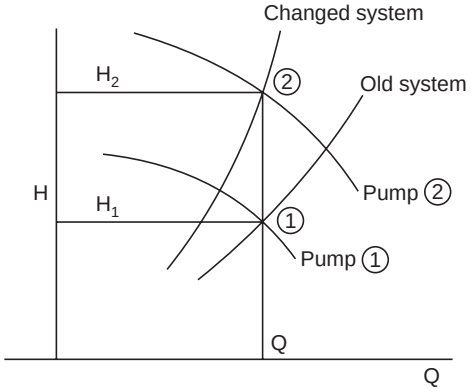


Figure 15.5.3 Pumps in series

The flow requirement is Q. Originally head requirement was H_1 met by the first pump alone. The new requirement is flow rate Q and head H_2. This can be met by adding in series the pump 2, which meets this requirement. It is also possible to meet changes in both head and flow requirements by the use of two pumps. Suitable control system should be installed for such purposes.

15.6 SPECIFIC SPEED AND SIGNIFICANCE

Some of the dimensionless parameters pertaining to pumps have been derived in the chapter on Dimensional analysis. These are derived from basics below :

1. Flow coefficient :

$$Q \propto V_f A \propto u\, Db \propto u\, DD \propto DN\, DD \propto ND^3$$

$$\therefore \quad \frac{\mathbf{Q}}{\mathbf{ND^3}} = \textbf{constant} \tag{15.6.1}$$

For similar machine and also the **same machine.** In the case of same machine D is constant.

$$\therefore \quad \frac{Q}{N} = \text{constant or } \frac{Q_1}{N_1} = \frac{Q_2}{N_2}, \text{ unit quantity}$$

2. Head parameter :

$$H \propto u^2/g \propto D^2N^2/g$$

$$\therefore \quad \frac{\mathbf{gH}}{\mathbf{N^2D^2}} = \text{constant} \tag{15.6.2}$$

The head parameter is constant for similar machines. For the same machine

$$\frac{H_1}{{N_1}^2} = \frac{H_2}{{N_2}^2}, \text{ unit head}$$

3. Power parameter :

Multiplying the two parameters,

$$\frac{gH}{N^2D^2} \cdot \frac{Q}{ND^3} = \frac{\rho\, Q\, g\, H}{\rho\, N^3D^5} = \frac{P}{\rho\, N^3D^5} \tag{15.6.3}$$

4. Specific speed :

$$\text{Specific speed} = \frac{\sqrt{\text{Flow parameter}}}{3/4\sqrt{\text{Head parameter}}}$$

$$= \frac{\sqrt{Q}}{N^{1/2}D^{1.5}} \cdot \frac{N^{1.5}D^{1.5}}{(gH)^{3/4}}$$

$$N_s = \frac{N\sqrt{Q}}{(gH)^{3/4}} \tag{15.6.4}$$

This quantity is known as the specific speed of pumps. This is dimensionless. In practise $N_s = \dfrac{N\sqrt{Q}}{H^{3/4}}$ is in usage. **One definitions for the specific speed is the speed at which the pump will operate delivering unit flow under unit head.**

Actually the significance of the specific speed is its indication of the flow direction, width etc. of the impeller. This is illustrated in Figure 15.6.1. It is seen that different types of pumps have best efficiency at different specific speeds.

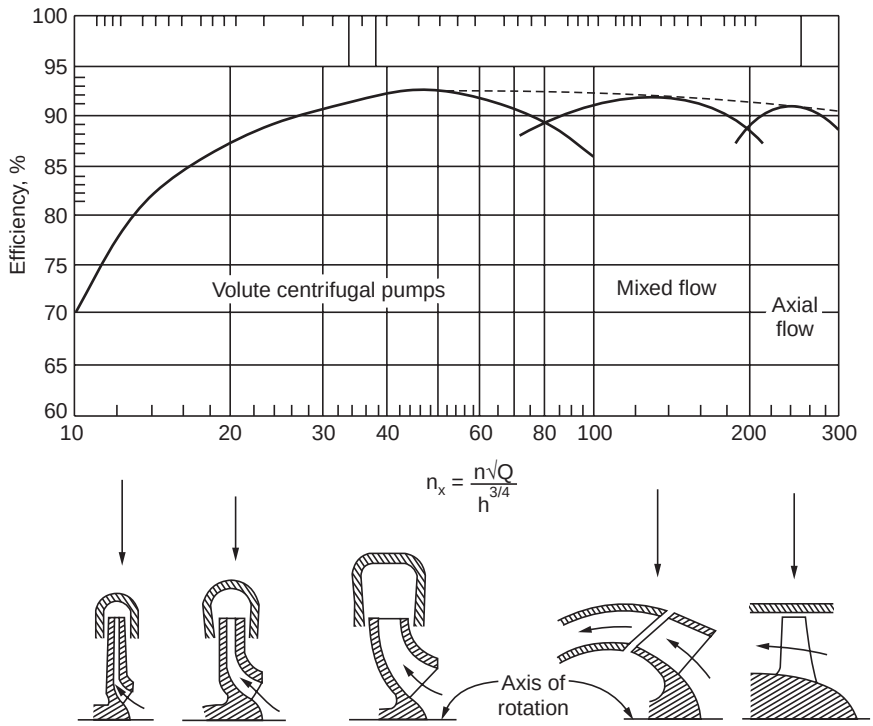


Figure 15.6.1 Efficiency-specific speed and impeller shape relations

15.7 CAVITATION

What is cavitation and where and why it occurs has been discussed in the chapter on turbines. **In the case of pumps, the pressure is lowest at the inlet and cavitation damage occurs at the inlet.** For cavitation to occur the pressure at the location should be near the vapour pressure at the location.

Applying the energy equation between sump surface and the pump suction,

$$\frac{P_s}{\gamma} + \frac{V_s^2}{2g} + Z = \frac{P_a}{\gamma} - h_{fs} \tag{15.7.1}$$

where Z is the height from sump surface and pump suction. The other terms have their usual significance. The term h_{fs} should include all losses in the suction line.

Net Positive Suction Head (NPSH) is defined as the available total suction head at the pump inlet above the head corresponding to the vapour pressure at that temperature.

$$NPSH = \frac{P_s}{\gamma} + \frac{V_s^2}{2g} - \frac{P_v}{\gamma} \tag{15.7.2}$$

where P_v is the vapour pressure.

From 15.7.1,

$$NPSH = \frac{P_a}{\gamma} - \frac{P_v}{\gamma} - Z - h_{fs} \tag{15.7.3}$$

Thoma cavitation parameter is defined by

$$\sigma = \frac{(NPSH)}{H} = \frac{(P_A/\gamma) - (P_v/\gamma) - Z - h_{fs}}{H}$$

At cavitation conditions,

$$\sigma = \sigma_c \quad \text{and} \quad \frac{P_s}{\gamma} = \frac{P_v}{\gamma}$$

$$\therefore \qquad \sigma_c = \frac{(P_a/\gamma) - (P_v/\gamma) - Z - h_{fs}}{H} \tag{15.7.4}$$

The height of suction, the frictional losses in the suction line play an important role for avoiding cavitation at a location. When pumps designed for one location is used at another location, atmospheric pressure plays a role in the onset of cavitation. Some authors use the term "suction specific speed, 'n_s". Where H in the general equation is substituted by *NPSH*. One correlation for critical cavitation parameter for pumps is given as

$$\sigma_c = \left(\frac{n_s}{175}\right)^{4/3} \tag{15.7.5}$$

These equations depend upon the units used and should be applied with caution.

15.8 AXIAL FLOW PUMP

A sectional view of axial flow pump is shown in Figure 15.8.1.

The flow in these machines is purely axial and axial velocity is constant at all radii. The blade velocity varies with radius and so the velocity diagrams and blade angles will be different at different radii. Twisted blade or airfoil sections are used for the blading. Guide vanes are situated behind the impeller to direct the flow axially without whirl. In large pumps inlet guide vanes may also be used. Such pumps are also called as propeller pumps. The head developed per stage is small, but due to increased flow area, large volumes can be handled.

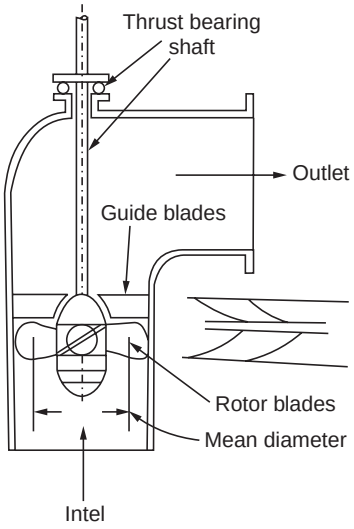


Figure 15.8.1 Axial flow pums

A comparison of values of parameters is given in table 15.8.1.

Table 15.8.1 Comparative values of parameter for different types of pumps

Type of pump	Axial	Mixed flow	Centrifugal	
			Diffuser pumps	Volute pumps
Flow ratio $(V_f/\sqrt{2gH})$	0.25 – 0.6	0.3	0.15	0.2
Speed ratio $\frac{u_2}{\sqrt{2gH}}$	2 – 2.7	1.35	0.9 – 1.05	1 – 1.2
Specific speed $N\sqrt{Q}/H^{3/4}$	150 – 800	85 – 175	15 – 20	20 – 90

The whirl at inlet is zero. **The velocity triangles are given in Figure 15.8.2.**

V_a is constant at all sections both at inlet and outlet.

u varies with radius. Hence β_1 and β_2 will vary with radius.

$H_{th} = \frac{u_2 V_{u2}}{g}$ as in the case of centrifugal pumps.

All other efficiencies are similar to the contrifugal pump.

The angle turned by the fluid during the flow over the blades is about 10 – 15°. Hence whirl imparted per

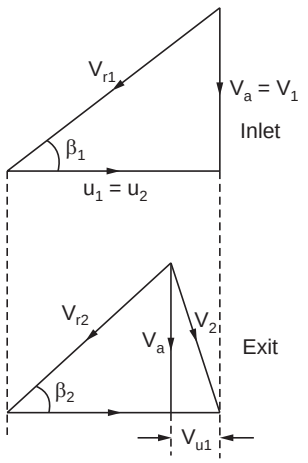


Figure 15.8.2 Velocity triangles-axial pump

stage is small. The number of blades is limited as in the case of Kaplan turbine ranging between 2 and 8. The hub to tip ratio is in the range 0.3 to 0.6. Generally the blades are fixed. In rare designs the blades are rotated as in the case of Kaplan turbine by suitable governing mechanism.

15.9 POWER TRANSMITTING SYSTEMS

Ordinarily power is transmitted by mechanical means like gear drive or belt drive. In the case of gear drive there is a rigid connection between the driving and driven shafts. The shocks and vibrations are passed on from one side to the other which is not desirable. Also gear drives can not provide a stepless variation of speeds. In certain cases where the driven machine has a large inertia, the driving prime mover like electric motor will not be able to provide a large starting torque. Instead of the mechanical connection if fluids can be used for such drives, high inertia can be met. Also shock loads and vibration will not be passed on. Smooth speed variation is also possible. The power transmitting systems offer these advantages.

There are two types power transmitting devices. These are **(*i*) Fluid coupling and (*ii*) Torque converter or torque multiplier.**

15.9.1 Fluid Coupling

A sectional view of a fluid coupling is shown in figure 15.9.1.

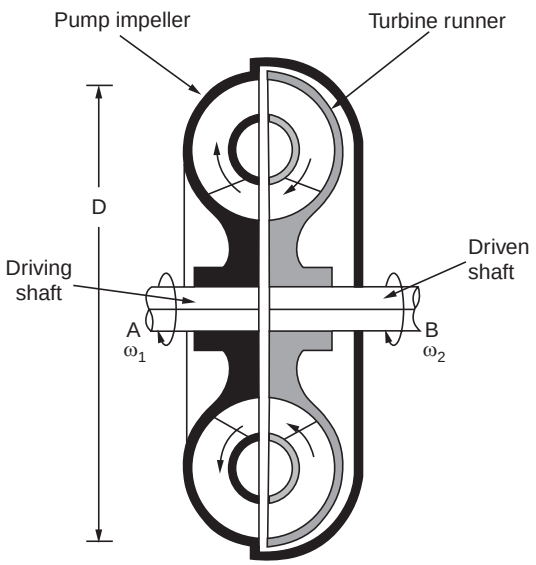


Figure 15.9.1 Fluid coupling

In this device the driving and driven shafts are not rigidly connected. The drive shaft carries a pump with radial vanes and the driven shaft carries a turbine runner. Both of these are enclosed in a casing filled with oil of suitable viscosity. The pump accelerates the oil by imparting energy to it. The oil is directed suitably to hit the turbine vanes where the energy is absorbed and the oil is decelerated. The decelerated oil now enters the pump and the cycle is repeated. There is no flow of fluid to or from the outside. The oil transfers the energy from the

drive shaft to the driven shaft. As there is no mechanical connection between the shafts, stock loads or vibration will not be passed on from one to the other. The turbine will start rotating only after a certain level of energy picked up by the oil from the pump.

Thus the prime mover can pick up speed with lower starting torque before the power is transmitted. In this way heavy devices like power plat blowers can be started with motors with lower starting torque. The pump and turbine can not rotate at the same speeds. In case these do run at the same speed, there can be no circulation of oil between them as the centrifugal heads of the pump and turbine are equal, and no energy will be transferred from one to the other. The ratio of difference in speeds to the driver speed is known as slips, S.

$$S = \frac{\omega_p - \omega_T}{\omega_p} \tag{15.9.1}$$

where ω_p is the pump speed and ω_T is the turbine speed. The variation of slip with pump speed is shown in figure 15.9.2.

As shown up to the pump speed ω_{ps} the turbine will not run and slip is 100%. As the driver speed increases slip rapidly decreases and at the operating conditions reaches values of about 2 to 5%.

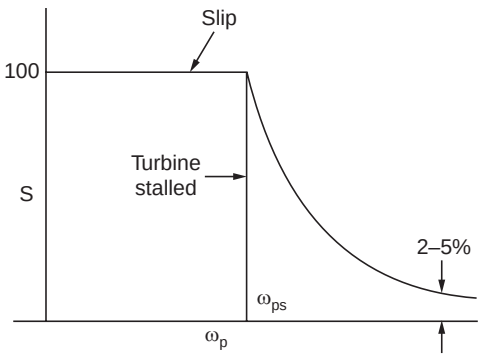


Figure 15.9.2 Slip variation with pump speed

The efficiency of transmission

$$= \frac{\tau_t \, . N_t}{\tau_p \, N_p}$$

In the absence of mechanical friction $\tau_t = \tau_p$

So, $$\eta = \frac{N_t}{N_p}$$

As slip, $$S = \frac{N_p - N_t}{N_p} = 1 - \frac{N_t}{N_p}$$

$$\eta = (1 - S) \tag{15.9.2}$$

15.9.2 Torque Converter

In the case of fluid coupling the torque on the driver and driven members are equal. The application is for direct drives of machines. But there are cases where the torque required at the driven member should be more than the torque on the driver. Of course the speeds in this case will be in the reverse ratio. Such an application is in automobiles where this is achieved in steps by varying the gear ratios. The desirable characteristics is a stepless variation of torque. This is shown in figure 15.9.3. The torque converter is thus superior to the gear train with few gear ratios. A sectional view of torque converter is shown in figure 15.9.4. Torque

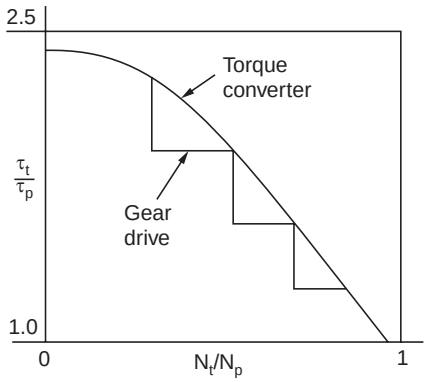


Figure 15.9.3 Torque variation in torque converter and gear train

converter consists of **three elements namely pump impeller, a turbine runner and a fixed guide wheel** as shown in figure 15.9.4. The pump is connected to the drive shaft. The guide vanes are fixed. The turbine runner is connected to the driven shaft. All the three are enclosed in a casing filled with oil. The oil passing through the pump impeller receives energy. Then it passes to the turbine runner where energy is extracted from the oil to turn the shaft. Then the oil passes to the stationary guide vanes where the direction is changed. This introduces a reactive torque on the pump which increases the torque to be transmitted. The shape and size and direction of the guide vanes controls the increase in torque. More than three elements have also been used in advanced type of torque converters. It may be noted that the speed ratio will be the inverse of torque ratio. The efficiency is found to be highest at speed ratio of about 0.6.

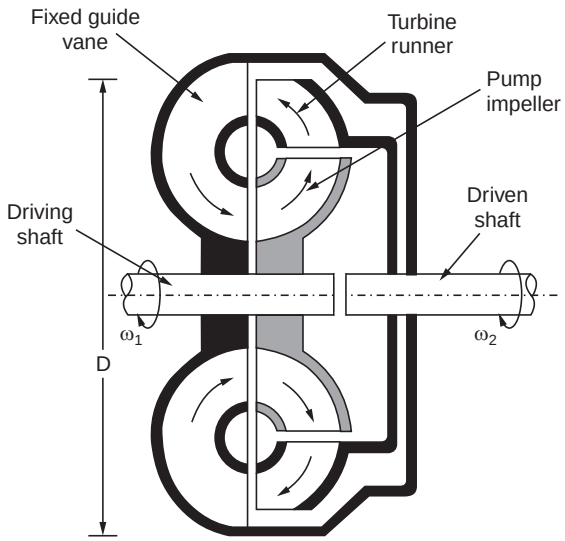


Figure 15.9.4 Torque converter

SOLVED EXAMPLES

Problem 15.1. *The following details refer to a centrifugal pump. Outer diameter : 30 cm. Eye diameter : 15 cm. Blade angle at inlet : 30°. Blade angle at outlet : 25°. Speed 1450 rpm. The flow velocity remains constant. The whirl at inlet is zero.* ***Determine the work done per kg****. If the manometric efficiency is 82%,* ***determine the working head****. If width at outlet is 2 cm,* ***determine the power*** $\eta_o = 76\%$.

$$u_1 = \frac{\pi \times 0.3 \times 1450}{60} = 22.78 \text{ m/s}$$

$$u_2 = 11.39 \text{ m/s}.$$

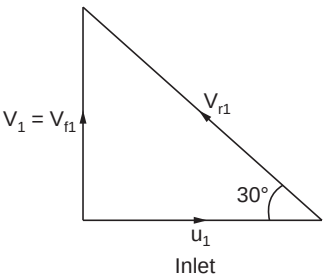


Figure P. 15.1(a)

From inlet velocity diagram.

$$V_{f1} = u_1 \tan \beta_1$$
$$= 11.39 \times \tan 30 = 6.58 \text{ m/s}$$

From the outlet velocity diagram,

$$V_{u2} = u_1 - \frac{V_{f2}}{\tan \beta_2} = 22.78 - \frac{6.58}{\tan 25} = 8.69 \text{ m/s}$$

Work done per kg $= u_2 V_{u2} = 22.78 \times 8.69$

$= \mathbf{197.7 \text{ Nm/kg/s}}$

$$\eta_m = 0.82 = \frac{g H}{197.7}$$

$\therefore$ **H = 16.52 m**

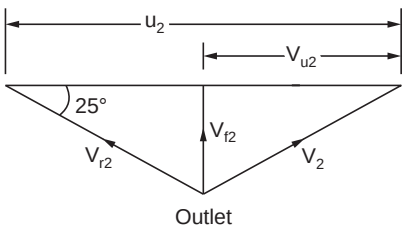


Figure P. 15.1(b)

Flow rate $= \pi \times 0.3 \times 0.02 \times 6.58 = 0.124 \text{ m}^3/\text{s}$

$$\textbf{Power} = \frac{0.124 \times 10^3 \times 9.81 \times 16.52}{0.76 \times 10^3} = \textbf{26.45 kW.}$$

Problem 15.2 *A homologus model of a centrifugal pump runs at 600 rpm against a head of 8 m, the power required being 5 kW. If the prototype 5 times the model size is to develop a head of 40 m* ***determine its speed, discharge and power****. The overall efficiency of the model is 0.8 while that of the prototype is 0.85.*

(1) $Q \propto D^2 H^{1/2}$ (as $Q = AV_f$, $A \propto Db$, $b \propto D$, $V_f \propto u \propto \sqrt{H}$)

(2) $$u \propto DN \propto \sqrt{H} \quad \therefore \quad \frac{ND}{\sqrt{H}} = \text{const.}$$

$$Q_m = \frac{P_m \times \eta_m}{\rho g H_m} = \frac{5 \times 10^3 \times 0.8}{10^3 \times 9.81 \times 8} = 0.05097 \text{ m}^3/\text{s}$$

From (1) $$\mathbf{Q_p} = Q_m \cdot \frac{D_p^{\,2}}{D_m^{\,2}} \cdot \frac{H_p^{\,1/2}}{H_m^{\,1/2}}$$

$$= 0.05097 \times 5^2 \cdot \left(\frac{40}{8}\right)^{1/2} = \mathbf{2.8492\ m^3/s}$$

From (2) $$\mathbf{N_p} = N_m \cdot \left(\frac{H_p}{H_m}\right)^{1/2} \cdot \frac{D_m}{D_p} = 600 \cdot 5^{1/2} \cdot \frac{1}{5} = \mathbf{268.32\ rpm}$$

$$\mathbf{Power} = \frac{2.8492 \times 9.81 \times 40 \times 10^3}{0.85 \times 10^3} = \mathbf{1315.3\ kW.}$$

Problem 15.3 *The diameter and width of a contrifugal pump impeller are 50 cm and 2.5 cm. The pump runs at 1200 rpm. The suction head is 6 m and the delivery head is 40 m. The frictional drop in suction is 2 m and in the delivery 8 m. The blade angle at out let is 30°. The manometric efficiency is 80% and the overall efficiency is 75%.* ***Determine the power required to drive*** *the pump. Also* ***calculate the pressures at the suction and delivery side of the pump***.

Inlet swirl is assumed as zero.

Total head against the pump is

$$40 + 6 + 2 + 8 = 56 \text{ m.}$$

$$u_2 = \pi \times 0.5 \times 1200/60 = 31.42 \text{ m/s}$$

$$\eta_m = \frac{g\,H}{u_2\,V_{u2}} = 0.8$$

$\therefore$ $$\frac{9.81 \times 56}{31.42 \times V_{u2}} = 0.8, \text{ solving } V_{u2} = 21.86 \text{ m/s}$$

To calculate V_f, the velocity triangle is used.

$$\tan \beta_2 = \frac{V_{f2}}{u_2 - V_{u2}}$$

$\therefore$ $$V_{f2} = \tan 30\ (31.42 - 21.86) = 5.52 \text{ m/s}$$

Flow rate $$= \pi\, D_2\, b_2\, V_{f2} = \pi \times 0.5 \times 0.15 \times 5.52$$

$$= 0.13006 \text{ m}^3/\text{s}$$

$\therefore$ $$\mathbf{Power} = \frac{0.13006 \times 10^3 \times 9.81 \times 56}{0.75 \times 10^3} = \mathbf{95.3\ kW}$$

Considering the water level and the suction level as 1 and 2

$$\frac{P_1}{\gamma} + 0 + 0 = \frac{P_2}{\gamma} + Z + \frac{V_2^{\,2}}{2g} + \text{losses}$$

$$10 = \frac{P_2}{\gamma} + 6 + \frac{5.52^2}{2 \times 9.81} + 2, \text{ solving,}$$

$$\frac{\mathbf{P_2}}{\gamma} = \mathbf{0.447\ m\ absolute\ (vacuum)}$$

Consider suction side and delivery side, as 2 and 3

$$\frac{P_2}{\gamma} + \frac{V_2^{\,2}}{2g} + \frac{u_2\, V_{u2}}{g} = \frac{P_3}{\gamma} + \frac{V_3^{\,2}}{2g}$$

$$V_3 = \sqrt{21.86^2 + 5.52^2} = 22.55 \text{ m/s}$$

$$\frac{P_3}{\gamma} = 0.447 + \frac{5.52^2}{2 \times 9.81} + \frac{31.42 \times 21.86}{9.81} - \frac{22.55^2}{2 \times 9.81} = \mathbf{40.1\ m\ absolute}$$

Problem 15.4 *It is proposed to design a homologous model for a centrifugal pump. The prototype pump is to run at 600 rpm and develop 30 m head the flow rate being 1 m^3/s. The model of 1/4 scale is to run at 1450 rpm.* ***Determine the head developed discharge and power required for the model.*** *Overall efficiency = 80%.*

In this case the speeds and diameter ratios are specified.

$$Q = AV_f,\ A = \pi\, Db,\ b \propto D, \quad \therefore \quad A \propto D^2$$

$$V_f \propto u \propto DN \propto \sqrt{H}$$

$\therefore$ $\mathbf{Q \propto D^3\, N}$ ...(1)

Also $u \propto \sqrt{H} \propto DN$

$\therefore$ $\mathbf{Q \propto D^2\, H^{1/2}}$ ...(2)

$\mathbf{P \propto Q\, H}$ ...(3)

As $u \propto DN \propto \sqrt{H}$

$\therefore$ $\mathbf{H \propto N^2\, D^2}$ ...(4)

$$\mathbf{P_p} = \frac{30 \times 1 \times 10^3 \times 9.81}{0.8 \times 1000} = \mathbf{367.9\ kW}$$

Using (1) $$\mathbf{Q_m} = Q_p \left(\frac{D_m}{D_p}\right)^3 \cdot \frac{N_m}{N_p} = 1 \times \left(\frac{1}{4}\right)^3 \times \frac{1450}{600} = \mathbf{0.03776\ m^3/s}$$

Using (4) $$\mathbf{H_m} = H_p \left(\frac{D_m}{D_p}\right)^2 \cdot \left(\frac{N_m}{N_p}\right)^2 = 30 \times \left(\frac{1}{4}\right)^2 \cdot \left(\frac{1450}{600}\right)^2 = \mathbf{10.95\ m}$$

Using (3) $$\mathbf{P_m} = P_p \frac{Q_m}{Q_p} \cdot \frac{H_m}{H_p} = 367.9 \times \frac{0.03776}{1} \cdot \frac{10.95}{30} = \mathbf{5.07\ kW}$$

Check : $$P_m = \frac{0.03776 \times 10^3 \times 9.81 \times 10.95}{0.8} = \mathbf{5.07\ kW}$$

Problem 15.5 *A centrifugal pump has been designed to run at 950 rpm delivering 0.4 m^3/s against a head of 16 m. If the pump is to be coupled to a motor of rated speed 1450 rpm.* ***Calculate the discharge, head and power input.*** *Assume that the overall efficiency is 0.82 remains constant.*

For a given pump, diameter, blade angles and physical parameters remain the same.

Hence, we can derive the following relations. (Similar to unit quantities).

$$Q = AV_f, A \text{ is constant} \quad \therefore \quad Q \propto V_f$$

$$V_f \propto u \text{ and } u \propto N \quad \therefore \quad Q \propto N \quad \text{or } Q/N = \text{constant}$$

$$\therefore \quad \frac{Q_2}{Q_1} = \frac{N_2}{N_1} \tag{1}$$

For centrifugal pump, $H \propto u^2 \propto N^2$

$$\therefore \quad \frac{H}{N^2} = \text{constant}$$

$$\therefore \quad \frac{H_2}{H_1} = \left(\frac{N_2}{N_1}\right)^2 \tag{2}$$

Power $\propto QH \propto N\,N^2 \propto N^3$

$$\therefore \quad \frac{P_2}{P_1} = \left(\frac{N_2}{N_1}\right)^3 \tag{3}$$

Using the equation (1), (2), (3)

$$P_1 = \frac{1000 \times 0.4 \times 9.81 \times 16}{1000 \times 0.82} = 76.57 \text{ kW}$$

$$\mathbf{Q_2} = Q_1 \cdot \frac{N_2}{N_1} = 0.4 \times \frac{1450}{950} = \mathbf{0.61\ m^3/s}$$

$$\mathbf{H_2} = H_1 \cdot \left(\frac{N_2}{N_1}\right)^2 = 16 \times \left(\frac{1450}{950}\right)^2 = \mathbf{37.27\ m}$$

$$\mathbf{P_2} = 76.57 \times \left(\frac{1450}{950}\right)^3 = \mathbf{272\ kW}$$

Check : $$\mathbf{P_2} = \frac{1000 \times 0.61 \times 37.27 \times 9.81}{1000 \times 0.82} = \mathbf{272\ kW.}$$

Problem 15.6 *A centrifugal pump running at 1450 rpm has an impeller diameter of 0.4 m. The backward curved blade outlet angle is 30° to the tangent. The flow velocity at outlet is 10 m/s.* ***Determine the static head*** *through which water will be lifted. In case a diffuser reduces the outlet velocity to 40% of the velocity at the impeller outlet, what will be the* ***increase in the static head.***

The whirl at inlet is assumed as zero.

The velocity diagram at outlet is shown.

$$u_2 = \frac{\pi DN}{60} = \frac{\pi \times 0.4 \times 1450}{60} = 30.37 \text{ m/s}$$

From velocity triangle

$$V_{u2} = u_2 - V_{f2} / \tan \beta_2 = 30.27 - \frac{10}{\tan 30} = \mathbf{13.05\ m/s}$$

Total head developed by the impeller

$$= \frac{u_2 V_{u2}}{g} = \frac{30.37 \times 13.05}{9.81} = 40.4 \text{ m}$$

Absolute velocity at exit $= (V_f^2 + V_{u2}^2)^{0.5}$

$= (10^2 + 13.05^2)^{0.5}$

$= 16.44$ m/s

Kinetic head $= \frac{V_2^2}{2g} = \frac{16.44^2}{2 \times 9.81} = 13.77$

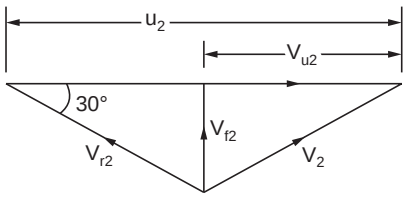


Figure P. 15.6

If diffuser is not used, static lift

$= 40.4 - 13.77 =$ **26.63 m**

The diffuser outlet velocity $= 0.4 \times 16.44 = 6.576$ m/s

Kinetic head at outlet $= 6.576^2/2 \times 9.81 = 2.2$ m

With diffuser use, the static lift $= 40.4 - 2.2 =$ **38.2 m**

Increase in static head $= 38.2 - 26.63$ m = **11.57 m**

Problem 15.7 *A form stage centrifugal pump running at 600 rpm is to deliver 1 m^3/s of water against a manometric head of 80 m. The vanes are curved back at 40° to the tangent at outer periphery. The velocity of flow is 25% of the peripheral velocity at outlet. The hydraulic losses are 30% of the velocity head at the outlet of the impeller.* ***Determine the diameter of the impeller and the manometric efficiency.***

$$u_2 = \frac{\pi D_2 N}{60} = \frac{\pi \times D_2 \times 600}{60} = 31.42\ D_2 \text{ m/s}$$

Velocity of flow at outlet $= \frac{1}{4} \times 31.42 = 7.85\ D_2$ m/s

$$V_{u2} = 31.42\ D_2 - \frac{7.85\ D_2}{\tan 40} = 2.06\ D_2 \text{ m/s}$$

$$V_2 = [V_{u2}^2 + V_{f2}^2]^{0.5} = [22.06^2 D_2^2 + 7.85^2 D_2^2]^{0.5}$$

$$= 23.41\ D_2 \text{ m/s}$$

$$\frac{0.3 V_2^2}{2g} = 8.383\ D_2^2$$

Energy imparted to impeller

$$= \frac{u_2 V_{u2}}{g} = \frac{31.42 \times 22.06}{9.81} D_2^2 = 70.655 D_2^2$$

$$70.655 D_2^2 = \text{manometric head per stage + losses.}$$

$$= (80/4) + 8.838 D_2^2$$

$\therefore$ $62.272 D_2^2 = 20$

$\therefore$ **Solving** $\mathbf{D_2 = 0.57\ m}$

$$\eta_m = \frac{20}{70.655 \times 0.57^2} = \mathbf{0.8712 \text{ or } 87.12\%}$$

Problem 15.8 *A centrifugal pump works at 900 rpm and is required to work against a head of 30m. The blades are curved back at 25° to the tangent at outlet. The flow velocity is 2.5 m/s.* ***Determine the diameter (i) If all the kinetic energy is lost (ii) It the velocity is reduced to 50%*** *converting kinetic energy to pressure energy.*

Energy imparted to the impeller $= u\left(u - \dfrac{2.5}{\tan 25}\right)/g$

Kinetic head at exit $= \dfrac{V_2^2}{2g} = \left\{\left(u - \dfrac{2.5}{\tan 25}\right)^2 + 2.5^2\right\}/2g$

The difference is the head against which the pump works (simplyfying)

$$2(u^2 - 5.361\,u) - (u^2 + 5.361^2 - 2 \times 5.361\,u + 6.25) = 30 \times 2 \times 9.81$$

This reduces to

$$u^2 = 623.59 \quad \therefore \quad u = 24.97\ m/s$$

$$u = \frac{\pi DN}{60},\ 24.97 = \frac{\pi \times D \times 900}{60},$$

Solving $\mathbf{D = 0.53\ m}$

In case the velocity at exit is reduced to half its value,

$$\frac{u(u - 5.361\,u)}{g} - \frac{(u - 5.361)^2 + 2.5^2}{8g} = 30$$

This reduces to

$$7u^2 - 32.166\,u - 2392.35 = 0$$

Solving $u = 20.93$ m/s

This leads to D = 0.44 m.

Problem 15.9 *The dimensionless specific speed of a centrifugal pump is 0.06. Static head is 30 m. Flow rate is 50 l/s. The suction and delivery pipes are each of 15 cm diameter. The friction factor is 0.02. Total length is 55 m other losses equal 4 times the velocity head in the pipe. The vanes are forward curved at 120°. The width is one tenth of the diameter. There is a 6% reduction in flow area due to the blade thickness. The manometric efficiency is 80%.* ***Determine the impeller diameter.*** *Inlet is radial.*

Frictional head is calculated first. Velocity in the pipe

$$= \frac{0.05 \times 4}{\pi \times 0.15^2} = 2.83 \text{ m/s}$$

Total loss of head $$= \frac{fl\,V^2}{2g\,D} + \frac{4\,V^2}{2g}$$

$$= \frac{0.02 \times 55 \times 2.83^2}{2 \times 9.81 \times 0.15} + \frac{4 \times 2.83^2}{2 \times 9.81} = 4.63 \text{ m}$$

Total head against which pump operates = 34.63 m

Speed is calculated from specific speed $N_s = N\sqrt{Q}\,/(gH)^{3/4}$

$$N = \frac{0.06 \times (9.81 \times 34.63)^{3/4}}{0.05^{1/2}} = 21.23 \text{ rps}$$

Flow velocity is determined :

Flow area $= \pi \times D \times \frac{D}{10} \times 0.94 = 0.2953\,D^2$

$$V_{f2} = \frac{0.05}{0.2953\,D^2} = 0.1693/D^2 \qquad ...(1)$$

$$u_2 = \pi\,DN = 21.23 \times \pi \times D = 66.7\,D \qquad ...(2)$$

$$\eta_m = 0.8 = \frac{9.81 \times 34.63}{66.7\,D \times V_{u2}}$$

$$\therefore \qquad V_{u2} = \frac{6.367}{D} \qquad ...(3)$$

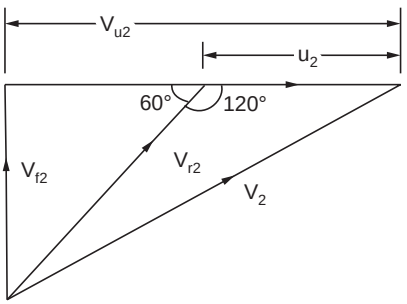


Figure P. 15.9 Outlet velocity diagram (forwerd curved)

From velocity diagram,

$$\tan 60 = \frac{V_{f2}}{V_{u2} - u_2} = \frac{0.1693}{D^2} \cdot \frac{1}{\left(\frac{6.367}{D} - 66.7\,D\right)}$$

Rearranging, $115.52\,D^3 - 11.028\,D + 0.1693 = 0$

Solving, **D = 0.3 m.**

Problem 15.10 *The* ***head developed*** *by a centrifugal pump running at 900 rpm is 27 m. The flow velocity is 3 m/s. The blade angle at exit is 45°.* ***Determine the impeller diameter.***

The head developed $= \frac{u_2\,V_{u2}}{g} = 27\ m$

$$V_{u2} = u_2 - \frac{V_f}{\tan 45} = u_2 - 3$$

$\therefore \quad u_2 \times (u_2 - 3) = 27 \times 9.81$

$u_2^2 - 3u_2 - 264.87 = 0$

$u_2 = 17.84$ (or – 14.84, trivial)

$$\frac{\pi DN}{60} = 17.84,\ N = 900$$

$$\therefore \quad \mathbf{D} = \frac{17.84 \times 60}{\pi \times 900} = \mathbf{0.397\ m\ or\ 379\ mm.}$$

Problem 15.11 *A radial vaned centrifugal compressor delivers 0.3 m^3/s against a head of 20 m. The flow velocity is constant at 3 m/s. The manometric efficiency is 80%. If the width is 1/10th of the diameter* ***Calculate the diameter, width and speed.*** *The eye diameter is 0.5 of outer diameter. Calculate the dimensions of inlet.*

Assume zero whirl at inlet. The velocity diagram at outlet is shown

$$\eta_m = \frac{g\,H}{u_2\,V_{u2}} \text{ here } u_2 = V_{u2}$$

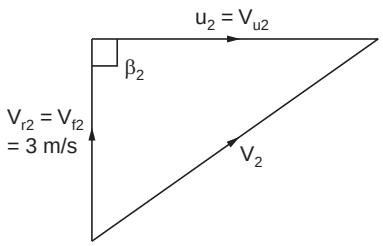


Figure P. 15.11

$$\therefore \quad u_2^2 = \frac{g\,H}{\eta_m} = \frac{9.81 \times 20}{0.8} = 245.25$$

$\therefore \quad u = 15.66$ m/s

$Q = \pi\, D_2\, b_2\, V_{f2}$

$\therefore \quad 0.3 = \pi \times D_2 \times 0.1\, D_2 \times 3$

$$D_2^2 = \frac{0.3}{\pi \times 0.1 \times 3},$$

Solving, $\mathbf{D_2 = 0.5642\ m}$

$\therefore \quad \mathbf{b_2 = 0.05642\ m}$

$\mathbf{D_1 = 0.2821\ m}$

$\mathbf{b_1 = 0.1128\ m}$

$$u_2 = \frac{\pi DN}{60},\ 15.66 = \frac{\pi \times 0.5642}{60} \times N$$

Solving $\mathbf{N = 530\ rpm.}$

Problem 15.12 *A centrifugal pump running at 900 rpm and delivering 0.3 m^3/s of water against a head of 25 m, the flow velocity being 3 m/s. If the manometric efficiency is 82%* ***determine the diameter and width of the impeller.*** *The blade angle at outlet is 25°.*

The velocity diagram at outlet is as shown. The inlet whirl is generally assumed as zero unless mentioned.

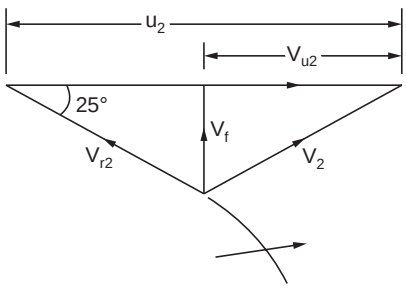


Figure P. 15.12

$$\eta_m = \frac{g\,H}{u_2\,V_{u2}},$$

$$\therefore \quad u_2 V_{u2} = \frac{9.81 \times 25}{0.82} = 299.09 \qquad \text{(A)}$$

$$\tan 25 = \frac{V_f}{u_2 - V_{u2}} \quad \therefore \quad u_2 - V_{u2} = \frac{3}{\tan 25} = 6.43352 \qquad \text{(B)}$$

$$V_{u2} = u_2 - 6.43352$$

$$\therefore \quad u_2 \times (u_2 - 6.43352) = 299.09$$

or $$u_2^2 - 6.43352\, u_2 - 299.09 = 0$$

Solving $$u_2 = 20.808 \text{ m/s (the other solution being negative).}$$

$$u_2 = \frac{\pi DN}{60},$$

$$\therefore \quad \mathbf{D} = \frac{u_2 \times 60}{\pi N} = \frac{20.808 \times 60}{\pi \times 900} = \mathbf{0.4416 \text{ m or } 44.16 \text{ cm}}$$

$$Q = \pi D_2 b_2 V_{f2}$$

$$\therefore \quad \mathbf{b_2} = \frac{Q}{\pi D_2 V_{f2}} = \frac{0.3}{\pi \times 0.4416 \times 3} = \mathbf{0.0721 \text{ m or } 7.21 \text{ cm.}}$$

Problem 15.13. *An axial flow pump running at 600 rpm deliver 1.4 m^3/s against a head of 5 m. The speed ratio is 2.5. The flow ratio is 0.5. The overall efficiency is 0.83.* ***Determine the power required*** *and the* ***blade angles at the root and tip and the diffuser blade inlet angle.*** *Inlet whirl is zero.*

β_1 – Blade angle at inlet

β_2 – Blade angle at outlet

α_2 – Diffuser blade inlet angle

$$\textbf{Power} = \frac{1.4 \times 10^3 \times 5 \times 9.81}{0.83 \times 10^3} = \mathbf{82.73 \text{ kW}}$$

$$u_t = 2.5\sqrt{2g\,H}$$

$$= 2.5\sqrt{2 \times 9.81 \times 5}$$

$$= 24.76 \text{ m/s},$$

$$V_f = 0.5\sqrt{2g\,H} = 4.95 \text{ m/s}$$

$$D_0 = \frac{24.76 \times 60}{\pi \times 600} = 0.788 \text{ m}$$

$$Q = \frac{\pi (D_0^2 - D_1^2) V_f}{4}$$

$$\therefore \quad 1.4 = \pi(0.788^2 - D_1^2)\, 4.95$$

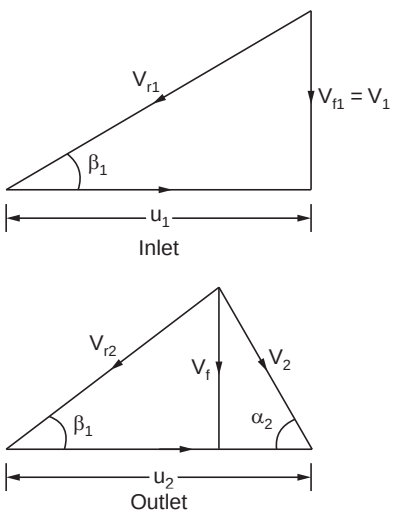


Figure P. 15.13

Solving, $D_1 = 0.51$ m

At Tip : From inlet triangle,

$$\beta_{1t} = \tan^{-1}\left(\frac{4.95}{24.76}\right) = \mathbf{11.3°}$$

From outlet triangle, $\beta_{2t} = V_f/(u - V_{u2})$

$$u_2 V_{u2} = gH,\ 24.76 \times V_{u2} = 9.81 \times 5,\ V_{u2} = 1.981 \text{ m/s}$$

$$\therefore \quad \beta_{2t} = \tan^{-1}\left(\frac{4.95}{24.76 - 1.981}\right) = \mathbf{12.26°}$$

$$\alpha_{2t} = \tan^{-1}\left(\frac{4.95}{1.981}\right) = 68.2°, \text{ can also be given as } (180 - 68.2°)$$

At Root : $u_2 V_{u2} = gH,\ u_2 = \pi \times 0.51 \times 600/60 = 16.02$ m/s

$\therefore$ $V_{u2} = 5 \times 9.81/16.02 = 3.06$ m/s, V_f = constant

$$\beta_{1R} = \tan^{-1}\left(\frac{4.95}{16.02}\right) = \mathbf{17.17°}$$

$$\beta_{2R} = \tan^{-1}(4.95/(16.02 - 3.06) = \mathbf{20.9°}$$

$$\alpha_{2R} = \tan^{-1}\left(\frac{4.95}{3.06}\right) = \mathbf{58.3°}$$

α values can also be given as (180 – 58.3)°.

Problem 15.14 *A centrifugal pump of impeller diameter 0.4 m runs at 1450 rpm. The blades are curved back at 30° to the tangent at the outlet. The velocity of flow is 3 m per second.* ***Determine the theoretical maximum lift*** *if the outlet velocity is reduced by the diffuser by 50%.*

Inlet whirl is assumed to be zero

$$u_2 = \frac{\pi \times 0.4 \times 1450}{60} = 30.37 \text{ m/s}$$

$$V_{u2} = 30.37 - \frac{3}{\tan 30} = 25.17 \text{ m/s}$$

$$V_2 = (25.17^2 + 3^2)^{0.5} = 25.35 \text{ m}$$

Head imparted $= \dfrac{30.37 \times 25.17}{9.81} = \mathbf{77.92\ m}$

Static head $= 77.92 - \dfrac{25.35^2}{2 \times 9.81} = 45.17$ m

Without diffuser the pump can pump to a head of 45.17 m theoretically.

If velocity is reduced to 50% of the value

New velocity = 12.675 m/s

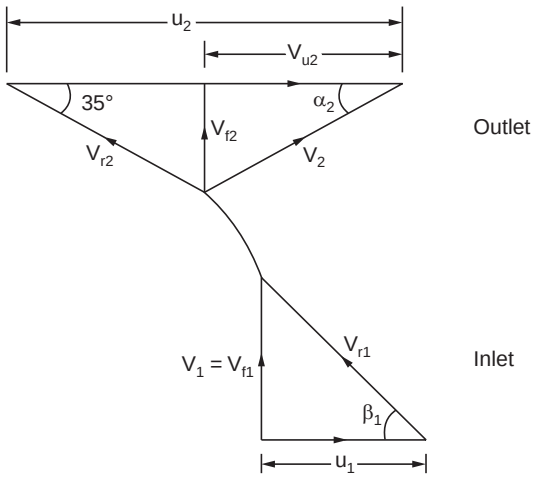
$\therefore$ Head recovered $= \dfrac{25.35^2 - 12.675^2}{2 \times 9.81} = 24.57$ m

$\therefore$ **Theoretical maximum lift**

$= 45.17 + 24.57 =$ **69.74 m**

Problem 15.15 *A centrifugal pump running at 900 rpm has an impeller diameter of 500 mm and eye diameter of 200 mm. The blade angle at outlet is 35° with the tangent. Determine assuming zero whirl at inlet,* ***the inlet blade angle. Also calculate the absolute velocity at outlet and its angle with the tangent.*** *The flow velocity is constant at 3 m/s. Also calculate the manometric head.*

The velocity diagrams are as shown.



Figire P. 15.15

Consider inlet

$$u_1 = \frac{\pi \times 0.2 \times 900}{60} = 9.42 \text{ m/s}$$

$V_{f1} = 3$ m/s

Blade angle at inlet

$$\tan \beta_1 = \frac{V_{f1}}{u_1} = \frac{3}{9.42}$$

$\therefore$ $\boldsymbol{\beta_1 = 17.66°}$

Considering outlet

$$u_2 = \frac{\pi \times 0.5 \times 900}{60} = 23.56 \text{ m/s}$$

$$V_{u2} = u_2 - \frac{V_{f2}}{\tan 35} = 23.56 - \frac{3}{\tan 35} = 19.28 \text{ m/s}$$

$$\tan \alpha_2 = \frac{3}{19.28}$$

$\therefore$ $\boldsymbol{\alpha_2 = 8.85°}$

$$\boldsymbol{V_2 = \sqrt{3^2 + 19.28^2} = 19.51 \text{ m/s}}$$

The outlet velocity is 19.51 m/s at an angle of 8.85° to the tangent. (taken in the opposite direction of u).

Manometric head $= \dfrac{23.56 \times 19.28}{9.81} =$ **46.3 m**

Problem 15.16 *A centrifugal pump running at 900 rpm delivers 800 l/s against a head of 70 m. The outer diameter of the impeller is 0.7 m and the width at outlet is 7 cm. There is recirculation of 3% of volume delivered. There is a mechanical loss of 14 kW. If the manometric efficiency is 82%* ***determine the blade angle at outlet, the motor power and the overall efficiency.***

The velocity diagram at outlet is as shown. The inlet whirl is assumed as zero. Backward curved vane is assumed (This is the general case).

$$u_2 = \frac{\pi D_2 N}{60} = \frac{\pi \times 0.7 \times 900}{60} = 32.99 \text{ m/s}$$

From manometric efficiency, V_{u2} is determined

$$\eta_m = \frac{g H}{u_2 V_{u2}},$$

$$0.82 = \frac{9.81 \times 70}{32.99 \times V_{u2}},$$

$$\therefore \quad V_{u2} = 25.385 \text{ m/s}$$

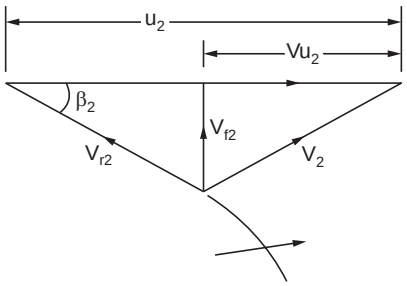


Figure P. 15.16

The flow through the impeller is increased by 3%

$$\therefore \quad V_{f2} = \frac{Q \times 1.03}{\pi D_2 b_2} = \frac{0.8 \times 1.03}{\pi \times 0.7 \times 0.07} = 5.353 \text{ m/s}$$

From velocity triangle,

$$\tan \beta_2 = \frac{V_{f2}}{u_2 - V_{u2}} = \frac{5.353}{32.99 - 25.385}$$

Solving $\beta_2 = 35.14°$

Motor Power $= Q \times 1.03 \times u_2 V_{u2}$ + Mechanical losses

$= (0.8 \times 1.03 \times 32.99 \times 25.385) + 14$

$= 690.06 + 14 =$ **704.06 kW**

Overall efficiency $= 1.03 \times 0.8 \times 9.81 \times 70 \times \frac{1}{704.06}$

$=$ **0.8036 or 80.36%.**

Problem 15.17 *The pressure difference between the suction and delivery sides of a pump is 25 m. The impeller diameter is 0.3 m and the speed is 1450 rpm. The vane angle at outlet 30° with the tangent. The velocity of flow is 2.5 m.* ***Determine the manometric efficiency.*** *If frictional losses in the impeller is 2 m* ***calculate the fraction of total energy converted to pressure energy*** *in the impeller. Also calculate the* ***pressure rise in the pump casing.***

$$u_2 = \frac{\pi \times 0.3 \times 1450}{60} = 22.78 \text{ m/s}$$

$$V_{u2} = 22.78 - \frac{2.5}{\tan 30} = 18.45 \text{ m/s}$$

$$V_2 = (18.45^2 + 2.5^2)^{0.5} = 18.62 \text{ m/s}$$

Energy input $= \dfrac{22.78 \times 18.45}{9.81} = 42.84$ m

$$\eta_m = \frac{25}{42.84} = \mathbf{0.5835 \text{ or } 58.35\%}$$

Kinetic energy at exit of impeller

$= 18.62^2/2 \times 9.81 = 17.67$ m

$\therefore$ Energy conversion in the impeller

$= 42.84 - 17.67 = 25.17$ m

Frictional loss $= 2$ m

Energy fraction for pressure

$$= \frac{25.17 - 2}{42.84} = \mathbf{0.5409 \text{ or } 54.09\%}$$

Total pressure rise $= 25$ m

Pressure rise in the impeller $= (25.17 - 2) = 23.17$ m

$\therefore$ **Pressure rise in the pump casing = 1.83 m.**

Problem 15.18 *A centrifugal pump running at 1450 rpm has impeller of 350 mm OD and 150 mm ID. The blade angles as* ***measured with the radial direction*** *are 60° and 65° at inlet and outlet. The hydraulic efficiency is 85%. The width of the impeller at inlet is 50 mm. Flow velocity is constant.*

Determine the static and stagnation pressure risc across *the impeller and the power input to the impeller.*

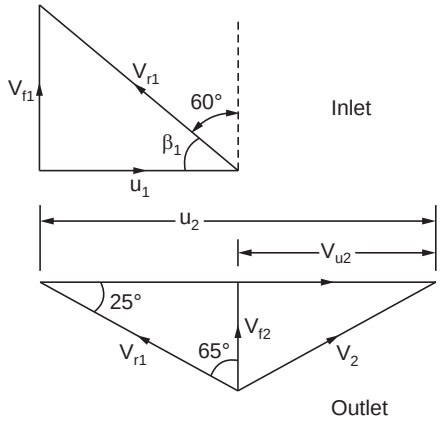


Figure P. 15.18

$$u_1 = \frac{\pi \times 0.15 \times 1450}{60} = 11.39 \text{ m/s}$$

$$V_{f1} = u_1 \,.\, \tan (90 - 60)$$

$$= 11.39 \times \tan 30 = 6.575 \text{ m/s}$$

$$u_2 = \frac{\pi \times 0.35 \times 1450}{60} = 26.572 \text{ m/s}$$

$$V_{u2} = u_2 - \frac{V_{f2}}{\tan \beta_2} = 26.572 - \frac{6.575}{\tan 25^\circ} = 12.472 \text{ m/s}$$

Energy input to the pump $= \dfrac{u_2 \, V_{u2}}{g} = 33.78$ m

Energy to the fluid $= 0.85 \times 33.78 = 28.72$ m

Stagnation pressure rise through the impeller

$$= \left[\frac{P_2 - P_1}{\gamma} + \frac{V_2^2 - V_1^2}{2g} \right] = \mathbf{28.72\ m}$$

Static pressure rise $= 28.72 - \left[\dfrac{V_2^2 - V_1^2}{2g} \right]$

$$V_2 = (V_{f2}^2 + V_{u2}^2)^{1/2} = (6.575^2 + 12.472^2)^{1/2} = 16.1 \text{ m/s}$$

$\therefore$ **Static pressure rise**

$$= 28.72 - \frac{16.1^2 - 6.575^2}{2 \times 9.81} = \mathbf{17.71\ m}$$

(Heads can be converted to pressure using $H\gamma = P$)

Flow rate $= \pi \times 0.15 \times 0.05 \times 6.575 = 0.1549\ m^3/s$

$\therefore$ **Power input** $= \dfrac{33.78 \times 01549 \times 10^3 \times 9.81}{10^3} = \mathbf{51.3\ kW.}$

Problem 15.19 *A centrifugal pump with OD = 0.6 m and ID = 0.3 m runs at 900 rpm and discharges 0.2 m^3/s of water against a head of 55 m. The flow velocity remains constant along the flow. The peripheral area for flow is 0.0666 m^2. The vane angle at outlet is 25°. The entry is radial.* ***Determine the manometric efficiency and the inlet vane angle.***

Radial entry means zero whirl at inlet. The velocity triangles are as shown.

$$u_2 = \frac{\pi D_2 N}{60} = \frac{\pi \times 0.6 \times 900}{60} = 28.27 \text{ m/s}$$

From outlet triangle,

$$V_{u2} = u_2 - \frac{V_{f2}}{\tan 25}$$

$$V_{f2} = 0.2/0.0666 = 3 \text{ m/s}$$

$$\therefore \quad V_{u2} = 28.27 - \frac{3}{\tan 25} = 21.84 \text{ m/s}$$

$$\therefore \quad \eta_m = \frac{g H}{u_2 V_{u2}} = \frac{55 \times 9.81}{28.27 \times 21.84} = 0.8739$$

or 87.39%

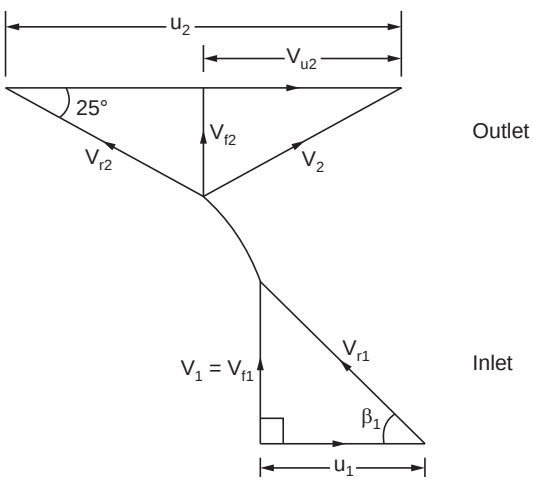


Figure P. 15.19

From inlet triangle,

$$\tan \beta_1 = \frac{V_{f1}}{u_1},\ u_1 = \frac{\pi \times 0.3 \times 900}{60} = 14.14 \text{ m/s}$$

$$\therefore \qquad \tan \beta_1 = \frac{3}{14.14} \quad \therefore \quad \boldsymbol{\beta_1 = 12°}$$

Problem 15.20 *A five stage centrifugal pump with blades radial at outlet runs at 500 rpm delivering 0.25 m^3/s against a total head of 100 m. The diameter of the impellers is 0.6 m and the flow velocity is 5 m/s.* ***Determine the manometric efficiency and the width of the impellers.***

Inlet whirl is assumed zero. The velocity diagram is shown.

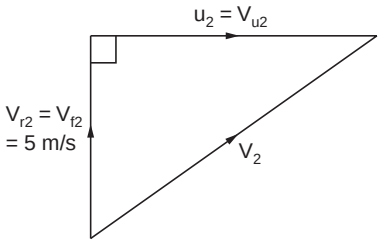


Figure P. 15.20

Here $\mathbf{u_2 = V_{u2}}$ $\qquad u_2 = \dfrac{\pi D N}{60} = \dfrac{\pi \times 0.6 \times 500}{60}$

$= 15.708$ m/s

Manometric head $\qquad = \dfrac{u_2 V_{u2}}{g} = \dfrac{u_2^2}{g} = \dfrac{15.708^2}{9.81}$

$= 25.15$ m

Head delivered by each impeller is 100/5 = 20 m

$\therefore$ **Manometric efficiency** = 20/25.15 = 0.7952 = **79.52%**

Flow rate $= \pi D b V_f = 0.25 = \pi \times 0.6 \times 5 \,.\, b_2$

Solving $\mathbf{b_2}$ = 0.0265 m or **26.5 mm.**

Problem 15.21 *A centrifugal pump with an impeller diameter of 0.4 m runs at 1450 rpm. The angle at outlet of the backward curved vane is 25° with tangent. The flow velocity remains constant at 3 m/s. If the manometric efficiency is 84%* ***determine the fraction of the kinetic energy at outlet recovered as static head.***

The whirl at inlet is zero. The velocity triangle is as shown.

$$u_2 = \frac{\pi D N}{60} = \frac{\pi \times 0.4 \times 1450}{60}$$

$= 30.37$ m/s

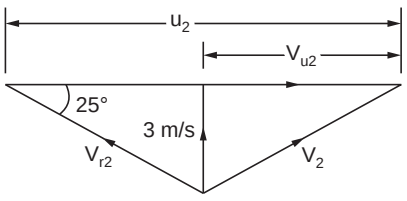


Figure 15. P.21

$$V_{u2} = u_2 - \frac{V_{f2}}{\tan \beta_2}$$

$= 30.37 - (3/\tan 25) = 23.94$ m/s

$$V_2 = \sqrt{V_{u2}^2 + V_{f2}^2} = \sqrt{23.94^2 + 3^2} = 24.12 \text{ m/s}$$

Total head developed $\qquad = \dfrac{30.37 \times 23.94}{9.81} = 74.11$ m

Kinetic head $= \dfrac{24.12^2}{2 \times 9.81} = 29.65$ m

Kinetic head at inlet $= \dfrac{3^2}{2 \times 9.81} = 0.46$ m

$\therefore$ Static head at impeller exit (using Bernoulli equation between inlet and outlet)

$= 74.11 + 0.46 - 29.65 = 44.92$ m

Actual static head $= \eta_m \times \dfrac{u_2\, V_{u2}}{9.81} = 0.84 \times 74.11 = 62.25$ m

Static head recovered $= 62.25 - 44.92 = 17.33$ m

Let the fraction be ϕ

$$\frac{\phi \times 24.12^2}{2 \times 9.81} = 17.33,$$

Solving $\phi = 0.5823$

Problem 15.22 *A centrifugal pump running at 900 rpm delivers 1 m^3/s against a head of 12 m. The impeller diameters are 0.5 m and 0.3 m respectively. The blade angle at outlet is 20° to the tangent.* ***Determine the manometric efficiency and the power required.*** *Mechanical efficiency = 98%. Also* ***estimate the minimum*** *speed at which the pump will* ***start delivery. The impeller width at outlet is 10 cm.***

Whirl at inlet is zero is assumed

$$u_2 = \frac{\pi \times 0.5 \times 900}{60} = 23.56 \text{ m/s}$$

$$V_{f2} = 1/\pi \times 0.5 \times 0.1 = 6.366 \text{ m/s}$$

$$V_{u2} = u_2 - \frac{V_{f2}}{\tan 20} = 23.56 - \frac{6.366}{\tan 20} = 6.07 \text{ m/s}$$

$$\frac{u_2\, V_{u2}}{g} = \frac{23.56 \times 6.07}{9.81} = 14.58 \text{ m}$$

$\therefore$ Manometric efficiency

$$= \frac{12}{14.58} = 0.8232 = 83.32\%$$

Power required $= \dfrac{1 \times 10^3 \times 14.58 \times 9.81}{0.98 \times 10^3} = 145.9$ kW

Starting speed is given by the expression

$$\frac{{u_2}^2 - {u_1}^2}{2g} \geq \frac{u_2\, V_{u2}}{g}$$

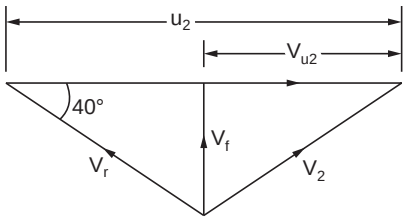
$$\frac{\pi^2 N^2 (0.5^2 - 0.3^2)}{60^2 \times 2 \times 9.81} = 14.58$$

Solving **N = 801.6 rpm**

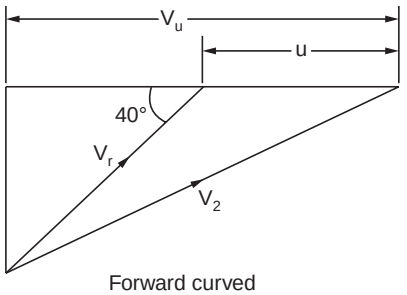
The pump will start delivering at 801.6 rpm.

Problem 15.23 *If the backward curved bladed impeller of 40° outlet angle, running an 1440 rpm is operated in the opposite direction,* ***find the ratio of power and exit velocities.*** *The diameter of the impeller is 0.3 m and the flow velocity is 0.20 of blade velocity.*

The respective velocity diagrams are shown.



Backward curved



Forward curved

Figure P. 15.23

Backward curved

$$u_2 = \frac{\pi \times 0.3 \times 1440}{60} = 22.62 \text{ m/s}$$

$$V_{f2} = 22.62 \times 0.2 = 4.52 \text{ m/s}$$

$$V_{u2} = 22.62 - \frac{4.52}{\tan 40} = 17.23 \text{ m/s}$$

Input head $= \frac{u_2 V_{u2}}{g} = \frac{22.62 \times 17.23}{9.81}$

$= \mathbf{39.72\ m}$

$\mathbf{V_2} = [17.23^2 + 4.52^2]^{0.5} = \mathbf{17.81\ m/s}$

Static pressure rise in the impeller

$$= 39.72 - \frac{V_2^2}{2g} = 39.72 - \frac{17.81^2}{2 \times 9.81} = \mathbf{23.55\ m}$$

Forward curved

$u_2 = 22.62$ m/s, $V_{f2} = 4.52$ m/s

$$V_{u2} = u_2 + \frac{V_{f2}}{\tan 40} = 28 \text{ m/s}$$

Input head $= \frac{u_2 V_{u2}}{g} = \frac{22.62 \times 28}{9.81} = \mathbf{64.58\ m}$

$\mathbf{V_2} = [28^2 + 4.52^2]^{0.5} = \mathbf{28.36\ m/s}$

Static pressure rise in the impeller

$$= 64.58 - \frac{28.36^2}{2 \times 9.81} = 64.58 - 41 = \mathbf{25.58\ m}$$

Ratio of power : $FB/BB = \frac{64.58}{39.72} = \mathbf{1.63}$

Ratio of velocities at outlet : $FB/BB = \frac{28.36}{17.81} = \mathbf{1.59}$

Static pressure rise is found to be nearly equal.

Problem 15.24 *A centrifugal pump when tested with Brine of density 1190 kg/m³ discharged 60 l/s against a pressure of 300 kPa. It is desired to investigate the change in power when a similar pump is used to pump petrol of density 700 kg/m³ against the same pressure. It is desired to keep the speed the same.* ***Check whether any change in the drive motor is required.***

Assume an overall efficiency of 70% in both cases.

With Brine :

$$\text{Head developed} = \frac{300 \times 10^3}{1190 \times 9.81} = 25.7 \text{ m}$$

$$\textbf{Power} = \frac{60 \times 1.19 \times 9.81 \times 25.7}{0.7 \times 10^3} = \textbf{25.71 kW}$$

With petrol :

$$\text{Head developed} = \frac{300 \times 10^3}{700 \times 9.81} = 43.69 \text{ m}$$

$$\textbf{Power} = \frac{60 \times 0.7 \times 9.81 \times 43.69}{0.7 \times 10^3} = \textbf{25.71 kW}$$

There is no need to change the motor.

As long as γH is the same, other conditions remaining constant, the power will be the same.

Problem 15.25 *A centrifugal pump was tested for cavitation initiation. Total head was 40 m and flow rate was 0.06 m³/s. Cavitation started when the total head at the suction side was 3 m. The atmospheric pressure was 760 mm Hg and the vapour pressure at this temperature was 2 kPa. It was proposed to install the pump where the atmospheric pressure is 700 mm Hg and the vapour pressure at the location temperature is 1 kPa. If the pump develops the same total head and flow, can the pump be fixed as the same height as the lab setup ?* ***What should be the new height.***

It is necessary to consider the suction point

Total head = Vapour pressure + velocity head.

$\therefore$ Velocity head = Total head – Vapour pressure in head of water

$$\therefore \quad \frac{V_s^2}{2g} = 3 - \frac{2 \times 10^3}{10^3 \times 9.81} = 2.769 \text{ m}$$

Cavitation parameter σ is defined by

$$\sigma = \frac{V_s^2}{2g\,H} = 2.796 / 40 = 0.0699$$

Considering the point at the sump level and the suction point

$$\frac{P_1}{\gamma} + \frac{V_1^2}{2g} + Z_1 = \frac{P_{atm}}{\gamma} - h_f$$

$$\therefore \qquad (Z_1 + h_f) = \frac{P_{atm}}{\gamma} - \sigma H - \frac{P_v}{\gamma}$$

$$= \left(\frac{760 \times 13.6}{1000}\right) - 2.796 - \frac{2 \times 10^3}{10^3 \times 9.81} = 7.336 \text{ m}$$

At the new location (head and flow being the same, friction loss will be the same)

$$(Z' + h_f') = \frac{700 \times 13.6}{1000} - 2.796 - \frac{1 \times 10^3}{10^3 \times 9.81} = 6.622 \text{ m}$$

As $h_f = h_f'$

$Z - Z' = 0.716$ m

The pump cannot be set at the same height.

It should be lowered by **0.716 m**

The new height is 6.622 m.

REVIERW QUESTIONS

1. What are the types of casings used in centrifugal pumps?
2. What are the advantages of centrifugal pumps over reciprocating pumps?
3. What are the advantages of double suction pumps?
4. Define manometric head and manometric efficiency of a centrifugal pump.
5. Explain why priming is necessary to start pumping by centrifugal pump.
6. Define shut off head.
7. List the types of impellers and indicate where each of them are used.
8. Explain the functions of a foot value. Indicate how it works.
9. Explain what is meant by slip. What are the effects of slip?
10. What is cavitation? Where does it occur in centrifugal pumps?
11. Define critical cavitation parameter and write down the expression for the same.
12. Explain why backward curved blades are more popularly used.

OBJECTIVE QUESTIONS

I. Choose the correct answer

1. Manometric head of a centrifugal pump is given by
   (*a*) Static head + losses (*b*) Static head
   (*c*) $u_2 V_{u2} / g$ (*d*) Static head + losses + exit kinetic head.
2. The dimensionless specific speed of a centrifugal pump

   (*a*) $\frac{N\sqrt{P}}{H^{3/4}}$ (*b*) $\frac{N\sqrt{Q}}{H^{5/4}}$ (*c*) $\frac{N\sqrt{Q}}{(gH)^{3/4}}$ (*d*) $\frac{N\sqrt{Q}}{H^{3/4}}$.

3. The manometric efficiency of a centrifugal pump is given by
   (*a*) $u_2 V_{u2}/g$ (*b*) $2H/u_2 V_{u2}$
   (*c*) $gH/u_2 V_{u2}$ (*d*) $u_2 V_{u2}/gH$.
4. The shut off head of a centrifugal pump is
   (*a*) $\frac{u_2^2 - u_1^2}{2g}$ (*b*) $\frac{u_2^2 - u_1^2}{g}$
   (*c*) $\frac{u_2^2}{2g}$ (*d*) $\frac{u_2 - V_{u2}}{g}$.
5. Slip in the case of a centrifugal pump.
   (*a*) Reduces the flow rate. (*b*) Reduces the energy transfer.
   (*c*) Reduces the speed. (*d*) Increases cavitation.

Answers

(1) *d* (2) *c* (3) *c* (4) *a* (5) *b*.

EXERCISE PROBLEMS

E 15.1 A centrifugal pump with 30 cm impeller delivers 30 l/s against a head of 24 m when running at 1750 rpm. **What will be the delivery of a homologous pump of** 15 cm impeller diameter assuming same efficiencies and speed. **(3.75 l/s)**

E 15.2 **Determine the discharge from** centrifugal pump running at 1000 rpm, the head being 14.5 m. The vane angle at outlet is 30° to the periphery. The impeller diameter is 0.3 m and width is 0.05 m. The manometric efficiency of the pump is 85%. **(Q 1.14 m^3/s)**

E 15.3 A centrifugal pump impeller is 0.5 m in diameter and delivers 2 m^3/min of water. The peripheral velocity is 10 m/s and the flow velocity is 2 m/s. The blade outlet angle is 35°. Whirl at inlet is zero. **Determine the power and torque delivered** by the impeller.

(2.18 kW, 54.5 mN)

E 15.4 A centrifugal pump running at 750 rpm delivers 60 l/s against a head of 20 m. If the pump is speeded up so that it runs at 1200 rpm, **Determine the head and discharge. Compare the power.** **(H = 32 m, Q = 96 l/s, P_1 = 11.772 kW, P_2 = 30.13 kW)**

E 15.5 A centrifugal pump with 2.3 diameter impeller running at 327 rpm delivers 7.9 m^3/s of water. The head developed is 72.8 m. The width of the impeller at outlet is 0.22 m. If the overall efficiency is 91.7% **determine the power to drive the pump. Also determine the blade angle at exit.** **(61.52 kW, 13°)**

E 15.6 A centrifugal pump running at 1000 rpm works against a head of 80 m delivering 1 m^3/s. The impeller diameter and width are 80 cm and 8 cm respectively. Leakage loss is 3 percent of discharge. Hydraulic efficiency is 80%. External mechanical loss is 10 kW. **Calculate the blade angle at outlet, the power required and the overall efficiency.**

(15.5°, 1020 kW, 76.9%)

E 15.7 The diameters of a centrifugal pump impeller is 600 mm and that of the eye is 300 mm. The vane angle at inlet is 30° and that at outlet is 45°. If the absolute velocity of water at inlet is 2.5 m/s **determine the speed and manometric head.** The whirl at inlet is zero.

(275.8 rpm, 5.44 m)

E 15.8 The speed of a centrifugal pump was 240 rpm and it is required to develop 22.5 m head when discharging 2 m^3/s of water. The flow velocity at outlet is 2.5 m/s. The vanes at outlet are set back at 30° to the tangential direction. **Determine the manometric efficiency and the power required to drive** the pump. Impeller diameter is 1.5 m. **(η_m = 81%, 545 kW)**

E 15.9 In a three stage pump the diameter and width at outlet of each impeller is 37.5 cm and 2 cm respectively. The discharge required is 3 m^3/min at 900 rpm. The vanes are set back at 45° to the tangent at the outlet. If the manometric efficiency a 84% **determine the total head developed** by the pump. **(68.7 m)**

E 15.10 The diameter of a centrifugal pump impeller is 0.8 m. The width at outlet is 0.12 m. It delivers 1.8 m^3/s through a height of 77 m, when running at 900 rpm. The blades are curved back at an angle of 25° to the tangent at outlet. **Calculate the manometric efficiency and power required to run the pump.** **(0.8047, 1690 kW)**

E 15.11 A backward curved bladed impeller of diameter 24 cm rotates at 2400 rpm. The blade angle is 45°. The flow velocity is 20% of the peripheral velocity and is constant. If the direction is reversed it will act as a forward bladed impeller. **Determine the static pressure rise in the impeller** in both cases. Also find the **ratio of exit velocities and power.**

(42.66 in both, 1.48, 1.5)

E 15.12 The impeller of a centrifugal pump of diameters 32 cm and 16 cm rotates at 90 radians/second. The width at inlet is 5 cm. The vane angles at inlet and outlet are 25.7° and 14.3° respectively. Assuming zero whirl at inlet determine the head developed. **(11.17 m)**

E 15.13 A centrifugal pump delivers 50 l/s when running at 1500 rpm. The inner and outer diameters are 0.15 m and 0.25 m respectively. The blades are curved at 30° to the tangent at the outlet. The flow velocity is 2.5 m/s and is constant. The suction and delivery pipe diameters are 15 cm and 10 cm, respectively. The pressure head at suction is 4 m below atmosphere. The pressure at the delivery is 18 m above atmosphere. The power required was 18 kW. **Determine the vane angle at inlet for** zero whirl at inlet. Also find the manometric efficiency and overall efficiency. **(12°, 77.2%, 64.6%)**

E 15.14 The impeller diameter of a pump is 400 mm. The speed is 1450 rpm. The working head is 60 m. Velocity of flow is 3 m/s. **Determine the manometric efficiency** if the vane angle at outlet is 30°. **(77%)**

E 15.15 The inlet and outlet diameters of a centrifugal impeller are 0.2 m and 0.4 m respectively. The vane angle at outlet is 45°. The pump speed is 1000 rpm. The flow velocity is constant at 3 m/s. The entry of the water is at radial direction. **Determine the vane angle at inlet, the work done for 1 kg, the absolute velocity at outlet and its direction α_2.**

(β_1 = 15.9°, 374 Nm/kg, 18.1 m/s, 9.5°)

E 15.16 A centrifugal pump discharges 125 *l* of water per second against a head of 35 m while running at 11 revolutions per second. The inner and outer diameters of the impeller are 0.6 m and 0.3 m. The flow area remains constant at 0.06 m^2. The vane angle at outlet is 45°. Water enters the impeller radially. **Determine the vane angle at inlet and the manometric efficiency.**

(11.5°, 0.8918)

E 15.17 A centrifugal pump running at 1450 rpm delivers 0.11 m^3/s of water against a head of 23 m. The impeller diameter is 250 mm and the width is 50 mm. The manometric efficiency is 75% **determine vane angle at outlet.** **(42.1°)**

E 15.18 The diameters of a centrifugal pump impeller are 750 mm and 400 mm respectively. The vane is backward curved and the outlet angle is 35°. If the speed is 1000 rpm **determine the angle at inlet. Also calculate the work done per kg** velocity of flow is 6 m/s.

(16°, 1250 Nm/kg/s)

E 15.19 A multi stage pump is required to deliver of 2 l/s water against a maximum discharge head of 240 m. The diameter of radial bladed impeller should not be more than 15 cm. Assume a speed of 2800 rpm. **Determine the impeller diameter, number of stages and power.** Overall efficiency is 0.7. **(5 stages, 148 mm, 6.7 kW)**

E 15.20 A radial bladed centrifugal pump running at 1440 rpm is to deliver 30 l/min of water against a head of 20 m. Assuming flow velocity as 3 m/s. **Determine the diameter and width of the impeller** at the outlet. **(186 mm, 28.5 mm)**

E 15.21 A centrifugal pump running at 1000 rpm delivers 250 l/s. The flow velocity at outlet is 3 m/s. The blades are swept back at 30° to the tangent at outlet. The hydraulic efficiency is 80%. **Determine the diameter and width of the impeller at outlet. (41.9 cm, 6.33 cm)**

E 15.22 The impeller diameters of a centrifugal pump are 30 cm and 15 cm. The width at outlet is 6 mm. Vanes at outlet are curved back by 45° to the tangent. **Determine the increase in pressure as** the water flows through the impeller. **(28.1 m)**

E 15.23 **Calculate the least diameter** of a centrifugal pump impeller to just start delivering against a head of 40 m when running at 1450 rpm if the inner diameter is 0.4 times the outer diameter. Manometric efficiency is 0.9. **(0.4244 m)**

E 15.24 A centrifugal pump started cavitating when the total pressure at suction was 3.26 m. The barometric pressure was 750 mm Hg and the vapour pressure at that condition was 1.8 kPa. **Determine the value of critical cavitation parameter for the pump.**

The total head across the pump was 36.5 m and the discharge 0.048 m^3/s. The pump is to be relocated where the barometric pressure is 622 mm Hg and the vapour pressure was 830 Pa, **how much must the hight above the sump be reduced to** avoid cavitation. Assume the same total head and discharge. **(σ_c = 0.084, 1.65 m)**

16 Reciprocating Pumps

16.0 INTRODUCTION

There are two main types of pumps namely the dynamic and positive displacement pumps. Dynamic pumps consist of centrifugal, axial and mixed flow pumps. In these cases pressure is developed by the dynamic action of the impeller on the fluid. Momentum is imparted to the fluid by dynamic action. This type was discussed in the previous chapter. Positive displacement pumps consist of reciprocating and rotary types. These types of pumps are discussed in this chapter. In these types a certain volume of fluid is taken in an enclosed volume and then it is forced out against pressure to the required application.

16.1 COMPARISON

Dynamic pumps	Positive displacement pumps
1. Simple in construction.	More complex, consists of several moving parts.
2. Can operate at high speed and hence compact.	Speed is limited by the higher inertia of the moving parts and the fluid.
3. Suitable for large volumes of discharge at moderate pressures in a single stage.	Suitable for fairly low volumes of flow at high pressures.
4. Lower maintenance requirements.	Higher maintenance cost.
5. Delivery is smooth and continuous.	Fluctuating flow.

16.2 DESCRIPTION AND WORKING

The main components are:

1. Cylinder with suitable valves at inlet and delivery.
2. Plunger or piston with piston rings.
3. Connecting rod and crank mechanism.

4. Suction pipe with one way valve.
5. Delivery pipe.
6. Supporting frame.
7. Air vessels to reduce flow fluctuation and reduction of acceleration head and friction head.

A diagramatic sketch is shown in Fig. 16.2.1.

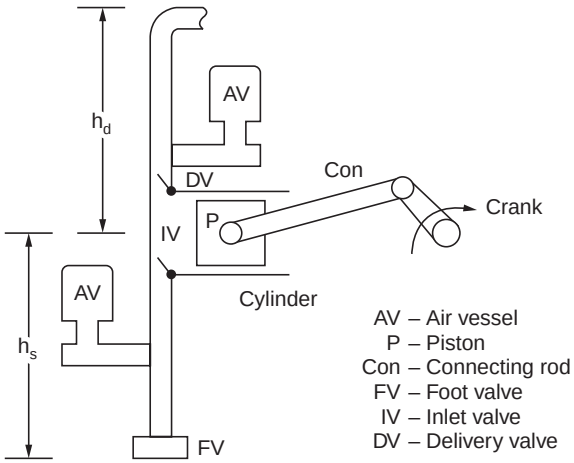


Figure 16.2.1 Diagramatic view of single acting reciprocating pump

The action is similar to that of reciprocating engines. As the crank moves outwards, the piston moves out creating suction in the cylinder. Due to the suction water/fluid is drawn into the cylinder through the inlet valve. The delivery valve will be closed during this outward stroke. During the return stroke as the fluid is incompressible pressure will developed immediately which opens the delivery valve and closes the inlet valve. During the return stroke fluid will be pushed out of the cylinder against the delivery side pressure. The functions of the air vessels will be discussed in a later section. The volume delivered per stroke will be the product of the piston area and the stroke length. In a single acting type of pump there will be only one delivery stroke per revolution. Suction takes place during half revolution and delivery takes place during the other half. As the piston speed is not uniform (crank speed is uniform) the discharge will vary with the position of the crank. The discharge variation is shown in figure 16.2.2.

In a single acting pump the flow will be fluctuating because of this operation.

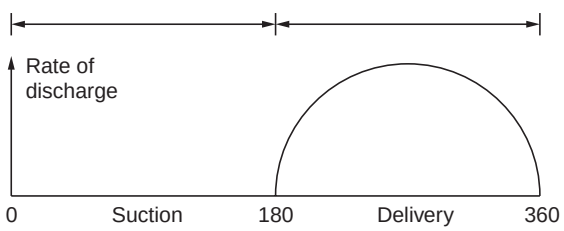


Figure 16.2.2 Flow variation during crank movement of single acting pump

Fluctuation can be reduced to some extent by double acting pump or multicylinder pump.

The diagramatic sketch of a double acting pump is shown in figure 16.2.3.

In this case the piston cannot be connected directly with the connecting rod. A gland and packing and piston rod and cross-head and guide are additional components. There will be nearly double the discharge per revolution as compared to single acting pump. When one side of the piston is under suction the other side will be delivering the fluid under pressure. As can be noted, the construction is more complex.

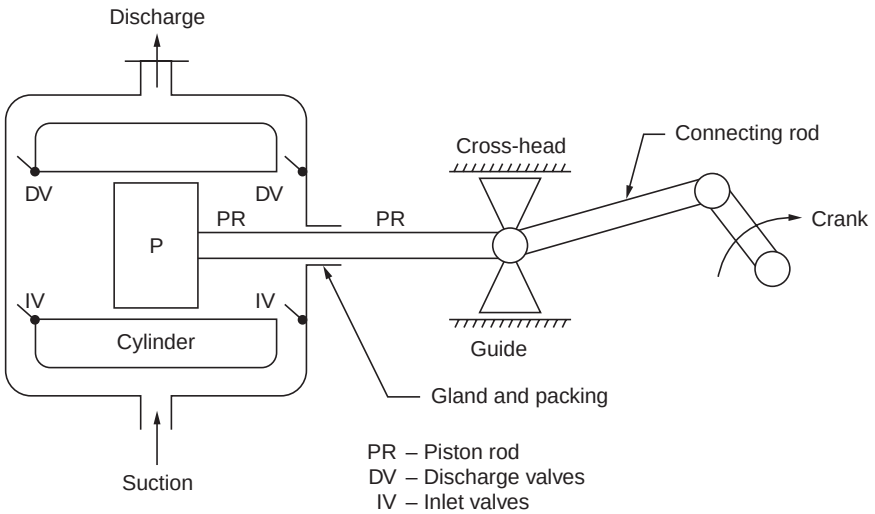


Figure 16.2.3 Diagramatic view of a double action pump

16.3 FLOW RATE AND POWER

Theoretical flow rate per second for single acting pump is given by, $Q_{SA} = \dfrac{L\,A\,N}{60}$ m^3/s

(16.3.1)

Where L is the length of stroke, A is the cylinder or piston area and N is the revolution per minute. It is desirable to express the same in terms of crank radius and the angular velocity as simple harmonic motion is assumed.

$$\omega = \frac{2\pi N}{60},\ N = \frac{60\,\omega}{2\pi},\ r = \frac{L}{2}$$

$$Q_{SA} = \frac{2r\,.\,A \times 60\,\omega}{2\pi \times 60} = \frac{A\omega r}{\pi}\ \text{m}^3/\text{s} \qquad (16.3.1a)$$

In double acting pumps, the flow will be nearly twice this value. If the piston rod area is taken into account, then

$$Q_{DA} = \frac{A\,L\,N}{60} + (A - A_{pr})\,\frac{L\,N}{60}\ \text{m}^3/\text{s} \qquad (16.3.2)$$

Compared to the piston area, the piston rod area is very small and neglecting this will lead to an error less than 1%.

$$\therefore \qquad Q_{DA} = \frac{2A\,L\,N}{60} = \frac{2A\,w\,r}{\pi} \text{ m}^3\text{/s.} \qquad (16.3.2a)$$

For multicylinder pumps, these expressions, (16.3.1), (16.3.1*a*), (16.3.2), and (16.3.2*a*) are to be multiplied by the number of cylinders.

16.3.1 Slip

There can be leakage along the valves, piston rings, gland and packing which will reduce the discharge to some extent. This is accounted for by the term slip.

$$\text{Percentage of slip} = \frac{Q_{th} - Q_{ac}}{Q_{th}} \times 100 \qquad (16.3.3)$$

Where Q_{th} is the theoretical discharge given by equation (16.3.1) and 2 and Q_{ac} is the measured discharge.

$$\text{Coefficient of discharge, } C_d = \frac{Q_{ac}}{Q_{th}} \qquad (16.3.4)$$

It has been found in some cases that $\mathbf{Q_{ac} > Q_{th}}$, due to operating conditions. In this case the slip is called **negative slip**. When the delivery pipe is short or the delivery head is small and the accelerating head in the suction side is high, the delivery valve is found to open before the end of suction stroke and the water passes directly into the delivery pipe. Such a situation leads to negative slip.

Theoretical power $= mg(h_s + h_d)\ W$

where m is given by $Q \times \delta$. (16.3.5)

Example 16.1 *A single acting reciprocating pump has a bore of 200 mm and a stroke of 350 mm and runs at 45 rpm. The suction head is 8 m and the delivery head is 20 m. Determine the theoretical discharge of water and power required. If slip is 10%, what is the actual flow rate ?*

Theoretical flow volume $Q = \dfrac{L\,A\,N}{60} = \dfrac{0.35 \times \pi \times 0.2^2}{4} \times \dfrac{45}{60}$

$= 8.247 \times 10^{-3}$ m^3/s or 8.247 l/s or 8.247 kg/s

Theoretical power = (mass flow/s) × head in $m \times g$ Nm/s or W

$= 0.9 \times 8.247 \times (20 + 8) \times 9.81$

$=$ **2039 W or 2.039 kW**

$$\text{Slip} = \frac{Q_{th} - Q_{ac}}{Q_{th}},\ 0.1 = \frac{8.247 - Q_{ac}}{8.247}$$

$\therefore$ $\mathbf{Q_{actual} = 7.422\ l/s}$

The actual power will be higher than this value due to both solid and fluid friction.

Example 16.2 *A double acting reciprocating pump has a bore of 150 mm and stroke of 250 mm and runs at 35 rpm. The piston rod diameter is 20 mm. The suction head is 6.5 m and the delivery head is 14.5 m. The discharge of water was 4.7 l/s. Determine the slip and the power required.*

$$\mathbf{Q} = \frac{L\,A_1\,N}{60} + \frac{L\,A_2\,N}{60} = \frac{L\,N}{60}\,[A_1 + A_2]$$

$$= \frac{0.25 \times 35}{60}\left[\frac{\pi \times 0.15^2}{4} + \frac{\pi}{4}(0.15^2 - 0.02^2)\right]$$

$$= \frac{0.25 \times 35 \times \pi}{60 \times 4}\,[2 \times 0.15^2 - 0.02^2]$$

$$= \mathbf{5.108 \times 10^{-3}\ m^3/s \text{ or } 5.108\ l/s \text{ or } 5.108\ kg/s}$$

It piston rod area is not taken into account

$$Q = 5.154\ \text{l/s}.$$

An error of 0.9% rather negligible.

$$\textbf{Slip} = \frac{5.108 - 4.7}{5.108} \times 100 = \mathbf{7.99\%}$$

Theoretical power $= m\,g\,h = 4.7 \times 9.81 \times (14.5 + 6.5)\ \text{W} = \mathbf{968\ W}$

The actual power will be higher than this value due to mechanical and fluid friction.

16.4 INDICATOR DIAGRAM

The pressure variation in the cylinder during a cycle consisting of one revolution of the crank. When represented in a diagram is termed as indicator diagram. The same is shown in figure 16.4.1.

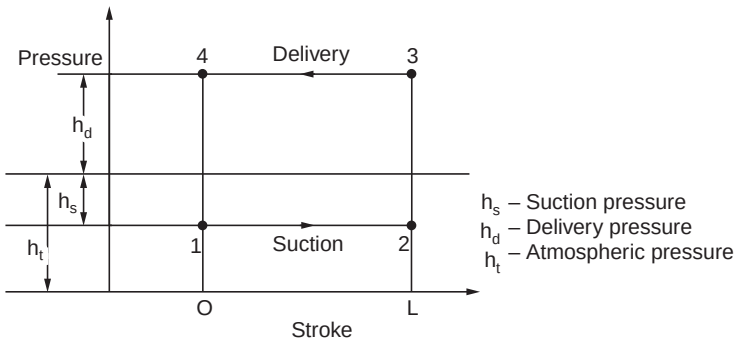


Figure 16.4.1 Indicator diagram for a crank revolution

Figure represents an ideal diagram, assuming no other effects are involved except the suction and delivery pressures. Modifications due to other effects will be discussed later in the section.

Point 1 represents the condition as the piston has just started moving during the suction stroke. 1-2 represents the suction stroke and the pressure in the cylinder is the suction pressure below the atmospheric pressure. The point 3 represents the condition just as the piston has started moving when the pressure increases to the delivery pressure. Along 3-4 representing the delivery stroke the pressure remains constant. The area enclosed represents the work done during a crank revolution to some scale

$$\text{Power} = Q\ \rho\ g(h_s + h_d) = \rho\ g\ L\ A\ N\ (h_s + h_d)/60 \tag{16.4.1}$$

16.4.1 Acceleration Head

The piston in the reciprocating pump has to move from rest when it starts the suction stroke. Hence it has to accelerate. The water in the suction pipe which is also not flowing at this point has to be accelerated. Such acceleration results in a force which when divided by area results as pressure. When the piston passes the mid point, the velocity gets reduced and so there is retardation of the piston together with the water in the cylinder and the pipe. This again results in a pressure. These pressures are called acceleration pressure and is denoted as head of fluid ($h = P/\rho g$) for convenience. Referring to the figure 16.4.2 shown below the following equations are written.

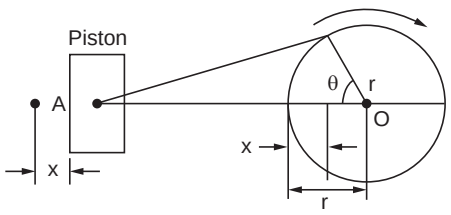


Figure 16.4.2 Piston Crank Configuration

Let ω be the angular velocity.

Then at time t, the angle travelled $\theta = \omega t$

Distance $x = r - r \cos \theta = r - r \cos \omega t$

Velocity at this point,

$$v = \frac{dx}{dt} = \omega r \sin wt \tag{16.4.2}$$

The acceleration at this condition

$$\ddot{x} = \frac{dv}{dt} = \omega^2 r \cos wt \tag{16.4.3}$$

This is the acceleration in the cylinder of area A. The acceleration in the pipe of area a is

$$= \frac{A}{a}\omega^2\ r \cos \omega t. \tag{16.4.4}$$

Accelerating force = mass × acceleration

$$\text{mass in the pipe} = \rho\, al \text{ kg} = \frac{\gamma al}{g}$$

$$\therefore \quad \text{Acceleration force} = \frac{\gamma al}{g} \times \frac{A}{a} \omega^2 r \cos \omega t \tag{16.4.5}$$

Pressure = force/area

$$= \frac{ral}{g} \cdot \frac{1}{a} \cdot \frac{A}{a} \omega^2 r \cos \omega t$$

$$= \frac{rl}{g} \cdot \frac{A}{a} \omega^2 r \cos \theta$$

Head = Pressure/γ

$$h_a = \frac{l}{g} \cdot \frac{A}{a} \omega^2 r \cos \theta \tag{16.4.6}$$

This head is imposed on the piston in addition to the static head at that condition. This results in the modification of the indicator diagram as shown in figure 16.4.3.

(*i*) Beginning of suction stroke: $\theta = 0$, $\cos \theta = 1$

$$\therefore \qquad h_{as} = \frac{l_s}{g} \cdot \frac{A}{a_s} \cdot \omega^2 r$$

This is over and above the static suction head. Hence the pressure is indicated by 1′ in the diagram.

(*ii*) Middle of stroke: $\theta = 90 \quad \therefore \quad h_{as} = 0$. There is no additional acceleration head.

(*iii*) End of stroke: $\theta = 180$. $\cos \theta = -1$

$$\therefore \qquad h_{as} = -\frac{l_s}{g} \cdot \frac{A}{a_s} \cdot \omega^2 r$$

This reduces the suction head. Hence the pressure is indicated at 2′ in the diagram.

Similarly during the beginning of the delivery stroke

$$\theta = 0, \cos \theta = 1$$

$$h_{ad} = \frac{l_d}{g} \cdot \frac{A}{a_d} \cdot \omega^2 r$$

This head is over and above the static delivery pressure. The pressure is indicated by point 3′ in the diagram. At the middle stroke $h_{ad} = 0$. At the end of the stroke $h_{ad} = -\frac{l_a}{g} \cdot \frac{A}{a_d} \cdot \omega^2 r$. This reduces the pressure at this condition and the same is indicated by 4′, in the diagram.

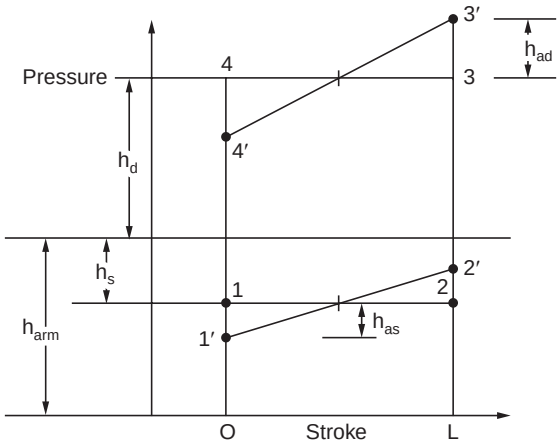


Figure 16.4.3 Modified indicator diagram due to acceleration head

The effect of acceleration head are:

1. No change in the work done.

2. Suction head is reduced. This leads to the problem of separation in suction pipe in case the pressure at 1′ is around 2.5 m of head of water (absolute). As the value depends on ω which is directly related to speed, the speed of operation of reciprocating pumps is limited. Later it will be shown than the installation of an air vessel alleviates this problem to some extent.

Example 16.3 *Calculate and tabulate the distance travelled as a fraction of stroke length as the crank rotates from 0 to 180°.*

Distance travelled = $r - r \cos \theta$

Stroke = $2r$.

$\therefore$ Distance travelled as a fraction of, L

$$= \frac{r - r \cos \theta}{2r} = \frac{1 - \cos \theta}{2}$$

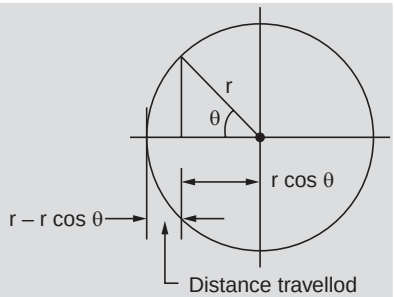


Figure Ex. 16.3 (a)

Values are calculated for 0, 30, 60, 90, 120, 150, 180° and tabulated below.

Angle	0	30	60	90	120	150	180
Distance moved	0	.0677	0.250	0.5	0.75	0.933	1

Note: The distances of piston movement is not uniform with crank angle.

For the data, speed = 40 rpm and r = 0.15 m. calculate the velocity and acceleration as the crank moves from one dead centre to the next.

Velocity = $\omega r \sin \theta = \frac{2\pi N}{60} \cdot r \sin \theta$

Acceleration = $\omega^2 r \cos \theta = \left(\frac{2\pi N}{60}\right)^2 \cdot r \cdot \cos \theta$

The values are calculated using the specified data and tabulated below.

Angle	0	30	60	90	120	150	180
Velocity m/s	0	0.314	0.544	0.628	0.544	0.314	0
Acceleration m/s^2	2.632	2.279	1.316	0	– 1.316	– 2.279	– 2.632

Note: The velocity follows sine curve and acceleration the cosine curve.

The acceleration is highest at start of stroke and decreases up to the middle of stroke and becomes zero and then decelerates at an increasing rate.

This can be illustrated as below.

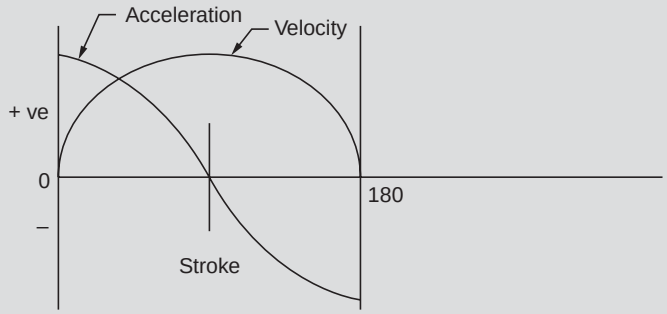


Figure Ex. 16.3 (b) Velocity and acceleration plot during a stroke

Example 16.4 *A single acting reciprocating pump of 200 mm bore and 300 mm stroke runs at 30 rpm. The suction head is 4 m and the delivery head is 15 m. Considering acceleration determine the pressure in the cylinder at the beginning and end of suction and delivery strokes. Take the value of atmospheric pressure as 10.3 m of water head. The length of suction pipe is 8 m and that of delivery pipe is 20 m. The pipe diameters are 120 mm each.*

Acceleration head on the suction side, $h_{as} = \frac{l_s}{g}\frac{A}{a_s}.\ \omega^2 r$

A – piston diameter, a_s – pipe diameter, $r = L/2$

$$\frac{A}{a_s} = \frac{\pi \times 0.2^2}{4} \times \frac{4}{\pi \times 0.12^2} = 2.78,\ \omega = \frac{2\pi N}{60}$$

$$\therefore \qquad \mathbf{h_{as}} = \frac{8}{9.81} \times 2.78 \times \left(\frac{2 \times \pi \times 30}{30}\right)^2 \times 0.15 = \mathbf{3.35\ m}$$

At start of suction, $H_{BS} = h_{atm} - h_s - h_{as} = 10.3 - 4 - 3.35 = 2.95$ m absolute or 7.05 m vacuum.

At end of suction $H_{es} = 10.3 - 4 + 3.35 =$ **9.65 m absolute or** 0.65 m vacuum.

$$h_{ad} = \frac{20}{9.81} \times 2.78 \times \left(\frac{2\pi \times 30}{60}\right)^2 \times 0.15 = \mathbf{8.38\ m\ of\ water\ column}$$

At starting of delivery, $\mathbf{H_{Bd}} = 10.3 + 15 + 8.38 =$ **33.68 m absolute or 23.38 m gauge**

At end of delivery, $\mathbf{H_{ed}} = 10.3 + 15 - 8.38 =$ **16.92 m absolute or 6.62 m gauge**

This applies for both single acting and double acting pumps.

16.4.2 Minimum Speed of Rotation of Crank

During the suction stroke, the head at the suction side is given by

$$h = h_{atm} - h_s - h_{as}$$

In case this head is below 2.5 m of head of water, water may vaporise at this point and the flow will be disrupted causing separation in the liquid column. Pumping will be discontinuous.

In order to avoid this, the acceleration head which can be changed should be limited. As this depends on the speed there is a limitation to the operating speed.

During delivery stroke also, there is a possibility of separation which may be caused by the layout of the delivery pipe. Two alternatives are shown in figure 16.4.4. The first method is to have a horizontal bend at the pump level and then to have the vertical line. In this case separation is avoided as at the bend the column of water above it exerts a pressure above 2.5 m (absolute). In the second arrangement the pressure at the bend is given by $(h_{atm} - h_{ad})$ and this may be below 2.5 m of water hence the preferred arrangement is to have a horizontal bend immediately after the pump.

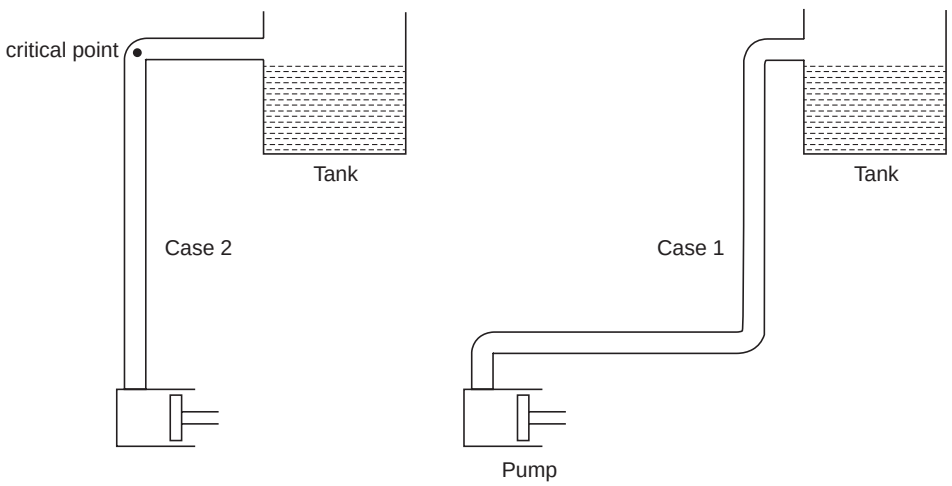


Figure 16.4.4 Delivery pipe arrangement

Example 16.5 *A single acting reciprocating pump of 200 mm plunger diameter and 300 mm stroke length has a suction head of 4 m. The suction pipe diameter is 110 mm and is 9 m long. The pressure at the beginning of suction should be above 2.5 m water column (absolute) to avoid separation. Determine the highest speed at which the pump can operate.*

At the beginning the pressure required is 2.5 m. This equals the difference between the absolute pressure and the sum of suction and accelerating head.

$$\therefore \qquad 2.5 = 10.3 - 4 - h_{as} \quad \therefore \quad h_{as} = 3.8 \text{ m}$$

$$h_{as} = \frac{l}{g} \cdot \frac{A}{a} \cdot \omega^2 r \quad (\cos \theta = 1)$$

$$3.8 = \frac{9}{9.81} \times \frac{\pi \times 0.2^2}{4} \times \frac{4}{\pi \times .011^2} . \omega^2 \times 0.15$$

Solving $\omega^2 = 8.353$, $\omega = 2.89$ radians/second.

$$\omega = \frac{2\pi N}{60}, \quad \mathbf{N} = \frac{\omega \times 60}{2\pi} = \frac{2.89 \times 60}{2 \times \pi} = \mathbf{27.6\ rpm}$$

This will be the same for double acting pump also.

Example 16.6 *The delivery pipe of a reciprocating pump is taken vertically up and then given a horizontal bend. The pump diameter is 180 mm and the stroke is 300 mm. The pipe diameter is 100 mm and the length is 18 m. The speed is 30 rpm. Check whether separation will occur at the bend. Separation is expected to take place if the absolute pressure is 2.5 m head. Atmospheric pressure may be taken as 10.3 m head of water.*

At the top of the pipe, the static head is zero. The only pressure is the accelerating head. To avoid separation $(P_{atm} - P_a) > 2.5$ m.

$$\mathbf{P_a} = \frac{l}{g} . \frac{A}{a} . \omega^2 r = \frac{18}{9.81} \times \left(\frac{\pi \times 0.18^2}{4} \times \frac{4}{\pi \times 0.1^2} \right) \times \left(\frac{2\pi \times 30}{60} \right)^2 \times 0.15$$

$$= \mathbf{8.8\ m}$$

Pressure P at the bend = 10.3 – 8.8 = 1.5 m

Hence separation will occur at the bend.

In the above problem if the pipe is taken first horizontally and then vertically, what will be the pressure at the bend. The delivery pressure is 15 m. The pressure at the bend will be the sum of the atmospheric pressure and the static pressure minus the acceleration head. The acceleration head itself is 8.8 m head.

$$P = 10.3 + 15 - 8.8 = 16.5 \text{ m head.}$$

Hence the arrangement is safe against separation.

16.4.3 Friction Head

When air vessels are not fixed in a pump, the velocity variation of water in the cylinder is given by equation (16.4.2)

$$v = \omega r \sin \omega t = \omega r \sin \theta$$

In the pipe the velocity variation will be in the ratio of areas.

$$\therefore \qquad v_p = \frac{A}{a} . \omega r \sin \theta$$

Friction head $\qquad h_f = f l v^2 / 2gd$

$$= \frac{fl}{2gd} . \left(\frac{A}{a} . \omega r \sin \theta \right)^2 \qquad (16.4.7)$$

The maximum value of friction heed

$$h_{f\,max} = \frac{fl}{2gd} . \left(\frac{A}{a} . \omega r \right)^2 \qquad (16.4.8)$$

At the beginning of the stroke $\theta = 0°$ $\therefore$ $h_f = 0$

At middle of stroke, $\theta = 90°$ and $h_f = h_{f\,max}$

At end of stroke, $\theta = 180°$ and $h_f = 0$.

The friction head leads to another modification of the indicator diagram shown in figure 16.4.5. With h_s and h_d as static suction and delivery heads, the pressure at the various locations are indicated below.

Suction stroke: At the start, head = $h_s + h_{as}$

At middle position, head = $h_s + h_{fs}$

At the end, $h = h_s - h_{as}$

Delivery stroke: At the start, head = $h_d + h_{ad}$

At middle position, head = $h_d + h_f$

At the end, head = $h_d - h_{ad.}$

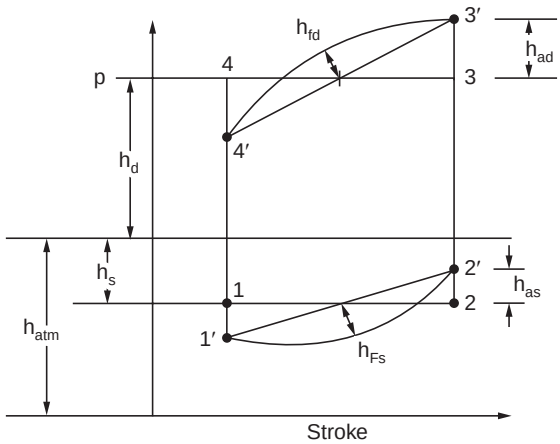


Figure 16.4.5 Variation of indicator diagram taking pipe friction into account

It can be observed that friction head increases the work done as seen by the increased area enclosed in the indicator diagram. The introduction of air vessels will reduce the friction work considerably. In order to calculate the work done, it will be desirable to calculate the average friction head. As the variation is parabolic, the average is found to be 2/3 of the maximum.

$$h_{fav} = 2/3\ h_{f\,max} \tag{16.4.9}$$

The total head against which work is done equals

$$h_{to} = h_s + h_d + 2/3\ h_{f\,max\,s} + 2/3\ h_{f\,max\,d} \tag{16.4.10}$$

The addition work due to friction is given by

$$2/3\ Q\ \rho\ g\ (h_{fs} + h_{fd}) \tag{16.4.11}$$

Later it will be seen that the use of air vessels causes the velocity in the pipe to be constant without fluctuations and this reduces the work to be overcome by friction.

Example 16.7 *A single acting pump with 200 mm bore diameter and 320 mm stroke runs at 30 rpm. The suction pipe diameter is 110 mm. The delivery pipe diameter is 100 mm. The suction and delivery pipes are 10 m and 22 m long. The friction factor is 0.01. Determine the frictional head at the suction and delivery.*

Assume no air vessels are fitted.

$$h_f = \frac{f l V^2}{2gd} = \frac{4f\,l}{2gd}\left(\frac{A}{a} \cdot \omega r \sin\theta\right)^2$$

Maximum occurs at the middle of stroke.

$$\mathbf{h_{fs\ max}} = \frac{4 \times 0.01 \times 10}{2 \times 9.81 \times 0.11}\left[\frac{200^2}{110^2} \cdot \frac{2\pi \times 30}{60} \times 0.15 \cdot \sin 90\right]^2$$

$$= \mathbf{0.45\ m\ \ head}$$

$$\mathbf{h_{fd\ max}} = \frac{4 \times 0.01 \times 22}{2 \times 9.81 \times 0.1}\left[\frac{200^2}{100^2} \times \frac{2\pi \times 30}{60} \times 0.15 \cdot \sin 90\right]^2$$

$$= \mathbf{1.59\ m\ head.}$$

Note: Two types of equations for frictional head are used:

$$\frac{4f\,lv^2}{2gd} \text{ and } \frac{f\,lv^2}{2gd}$$

When $4f$ is used, f is the coefficient of friction commonly denoted as c. When f alone is used it is called Darcy friction factor and

Darcy friction factor = $4\,c_f$

Now the popular use is the second equation using f from Moody diagram.

16.5 AIR VESSELS

Air vessel is a strong closed vessel as shown in figure 16.5.1. The top half contains compressed air and the lower portion contains water or the fluid being pumped. Air and water are separated by a flexible diaphragm which can move up or down depending on the difference in pressure between the fluids. The air charged at near total delivery pressure/suction pressure from the top and sealed. The air vessel is connected to the pipe lines very near the pump, at nearly the pump level. On the delivery side, when at the beginning and up to the middle of the delivery stroke the head equals $h_s + h_f + h_a$, higher than the static and friction heads. At this time part of the water from pump will flow into the air vessel and the remaining will flow through the delivery pipe. This will increase the compressed air pressure. At the middle stroke position the head will be sufficient to just cause flow. The whole of the flow from pump will flow to the delivery pipe. At the second half of the stroke the head will be equal to $h_s + h_f - h_a$. At the position the head will be not sufficient to cause flow. The compressed air pressure will act on the water and water charged earlier into the air vessel will now flow out. Similar situation prevails on the suction side. At the start and up to the middle of the suction stroke the head at

the pump is higher than static suction head by the amount of acceleration head. The flow will be more and part will flow into the air vessel. The second half of the stroke water will flow out of the air vessel. In this process the velocity of water in the delivery pipe beyond the air vessel is uniform, and lower than the maximum velocity if air vessel is not fitted. Similar situation prevails in the suction side also. The effect is not only to give uniform flow but reduce the friction head to a considerable extent saving work. Without air vessel the friction head increases, reaches a maximum value at the mid stroke and then decreases to zero. With air vessel the friction head is lower and is constant throughout the stroke. This is due to the constant velocity in the pipe.

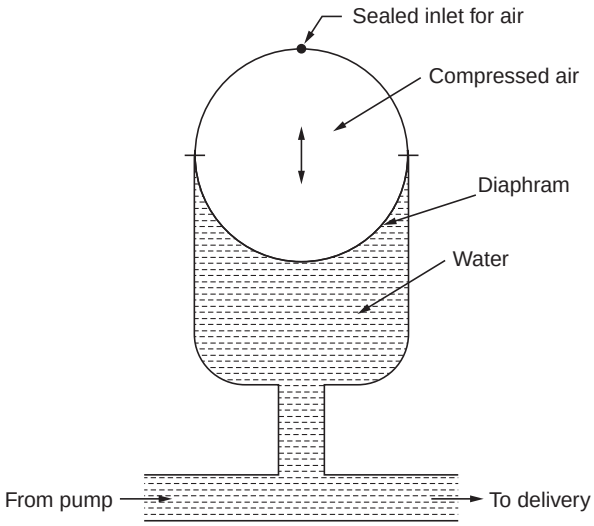


Figure 16.5.1 Air vessel

The advantages of installing air vessels are:

(*i*) The flow fluctuation is reduced and a uniform flow is obtained.

(*ii*) The friction work is reduced.

(*iii*) The acceleration head is reduced considerably.

(*iv*) Enables the use of higher speeds.

The maximum friction head of water without air vessel, refer eqn. (16.4.8),

$$h_{f\,max} = \frac{4fl}{2gd}\left(\frac{A}{a}.\omega r\right)^2$$

The average friction head = 2/3 $h_{f\,max}$. (refer eqn. 16.4.9). When the air vessel is placed near the pump, the uniform velocity,

$$v = \frac{A}{a}.\frac{LN}{60} = \frac{A}{a}.2r.\frac{60\omega}{2\pi} = \frac{A}{a}.\frac{\omega r}{\pi} \tag{16.5.1}$$

$$h_f = \frac{4fl}{2gd}.\left(\frac{A}{a}.\frac{\omega r}{\pi}\right)^2 \tag{16.5.2}$$

$$\frac{h_f}{h_f a_v} = \frac{3}{2} \cdot \frac{1}{\pi^2} = 0.152 \qquad (16.5.3)$$

$\therefore$ Reduction is 84.8%

Naturally the work done due to friction will reduce by this percentage.

Example 16.8 *In a single acting reciprocating pump with plunger diameter of 120 mm and stroke of 180 mm running at 60 rpm, an air vessel is fixed at the same level as the pump at a distance of 3 m. The diameter of the delivery pipe is 90 mm and the length is 25 m. Friction factor is 0.02. Determine the reduction in accelerating head and the friction head due to the fitting of air vessel.*

Without air vessel :

$$\mathbf{h_{ad}} = \frac{l}{g} \cdot \frac{A}{a} \cdot \omega^2 r = \frac{25}{9.81} \cdot \frac{0.12^2}{0.09^2} \cdot \left(\frac{2\pi \times 60}{60}\right)^2 \times 0.09$$

$$= \mathbf{16.097\ m}$$

With air vessel :

$$\mathbf{h'_{ad}} = \frac{3}{9.81} \cdot \frac{0.12^2}{0.09^2} \cdot \left(\frac{2\pi \times 60}{60}\right)^2 \times 0.09 = \mathbf{1.932\ m}$$

Reduction = 16.097 – 1.932 = **14.165 m**

Fitting air vessel reduces the acceleration head.

Without air vessel :

$$\text{Friction head} \quad h_f = \frac{4 f l . V^2}{2gd} = \frac{4fl}{2gd}\left[\frac{A}{a} \cdot \omega r \sin\theta\right]^2$$

At $\theta = 90°$

$$\mathbf{h_{f\,max}} = \frac{4 \times 0.02 \times 25}{2 \times 9.81 \times 0.09}\left[\frac{0.12^2}{0.09^2} \cdot \frac{2\pi \times 60}{60} \times 0.09 \times 1\right]^2 = \mathbf{1.145\ m}$$

With air vessel, the velocity is constant in the pipe.

$$\text{Velocity} = \frac{A\ L\ N}{60} \times \frac{4}{\pi \cdot d^2} = \frac{\pi \times 0.12^2}{4} \times \frac{0.18 \times 60 \times 4}{60 \times \pi \times 0.09^2}$$

$$= 0.102 \text{ m/s}$$

$$\text{Friction head} = \frac{4 \times 0.02 \times 25}{2 \times 9.81 \times 0.09} \times 0.102^2 = \mathbf{0.012\ m}$$

Percentage saving over maximum

$$= \frac{1.145 - 0.012}{1.145} \times 100 = 99\%$$

Air vessel reduces the frictional loss.

16.5.1 Flow into and out of Air vessel

Single acting pump: The flow into the delivery side is only during half a revolution. This amount has to flow during the full revolution:

The average velocity in the pipe = $\left(\frac{L\,N}{60} \times \frac{A}{a}\right)$.

Double acting pump: There are two discharger per revolution. The average velocity in this case = $\frac{2L\,N}{60} \times \frac{A}{a}$.

Hence the frictional head will be different in single acting and double acting pumps. This is illustrated in figure 16.5.2.

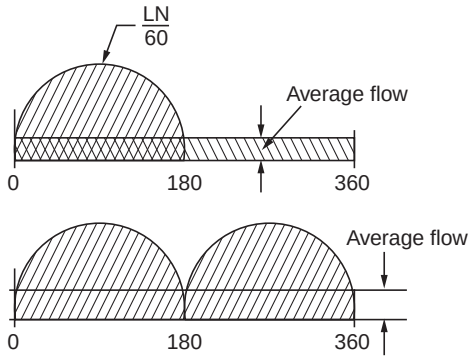


Figure 16.5.2 Pipe flow in single acting and double acting pumps

Single acting pump :

The flow from cylinder, $Q = A\ \omega r \sin\theta$

With air vessel the average velocity (refer 16.5.1) = $\frac{A}{a} \cdot \frac{\omega r}{\pi}$

Flow through pipe $= a \cdot \frac{A}{a} \cdot \frac{\omega r}{\pi} = \frac{A\ \omega r}{\pi}$

Flow into or out of the air vessel with $\theta = 0$ to $360 = A\ \omega r\ \sin\theta - \frac{A\ \omega r}{\pi}$

$$= A\ \omega r\left(\sin\theta - \frac{1}{\pi}\right) \tag{16.5.4}$$

Double acting pump: Flow through the pump = $\frac{2A\ \omega r}{\pi}$

At any point of time flow from cylinder = $A\ \omega r \sin\theta$

Flow into air vessel = $A\ \omega r\left(\sin\theta - \frac{2}{\pi}\right)$ (16.5.5)

Example 16.9 *Determine the rate of flow in and out of the air vessel on the delivery side in a single acting centrifugal pump of 200 mm bore and 300 mm stroke running at 60 rpm. Also find the angle of crank rotation at which there is no flow into or out of the air vessel.*

At any instant of time the flow rate from the pump cylinder is

$$Q = A\ \omega r \sin\theta$$

Beyond the air vessel the velocity in the pipe is constant

$$v = \frac{A}{a} \cdot \frac{\omega r}{\pi}$$

The flow rate $$= a \times \frac{A}{a} \cdot \frac{\omega r}{\pi} = \frac{A\ \omega r}{\pi}$$

Volume flow rate into the air vessel,

q = Volume flow rate from cylinder

– Volume flow rate beyond the air vessel

$$q = A\ \omega r \sin\theta - \frac{A\ \omega r}{\pi} = A\ \omega r\left[\sin\theta - \frac{1}{\pi}\right]$$

In this case $$\mathbf{q} = \frac{\pi \times 0.2^2}{4} \times \frac{2\pi \times 60}{60} \times \frac{0.3}{2}\left[\sin\theta - \frac{1}{\pi}\right]$$

$$= \mathbf{0.0296}\left[\mathbf{\sin\theta - \frac{1}{\pi}}\right].$$

	Delivery									Suction			
θ	0	30	60	90	120	150	180	210	240	270	300	330	360
q	– 0.318	0.182	0.548	.0682	0.548	0.182	– 0.318	– 0.818	– 1.184	– 1.318	– 1.118	– 0.818	– 0.318

At no flow condition, the quantity within the bracket showed be zero.

or $$\boldsymbol{\theta} = \sin^{-1}\left(\frac{1}{\pi}\right) = \mathbf{18.56^o}$$

Also $$\theta = 161.44°.$$

At two locations there is no flow into or out of the air vessel. Similar situation prevails on the suction side also.

16.6 ROTARY POSITIVE DISPLACEMENT PUMPS

In order to avoid the complexity of construction and restriction on speed of the reciprocating pumps, rotary positive displacement pumps have been developed. These can run at higher speeds and produce moderately high pressures. These are very compact and can be made for very low delivery volumes also. These are extensively used for pumping lubricant to the engine

parts and oil hydraulic control systems. These are not suited for water pumping. Some types described in this section are: (*i*) Gear pump, (*ii*) Lobe pump and Vane pump.

16.6.1 Gear Pump

These are used more often for oil pumping. Gear pumps consist of two identical mating gears in a casing as shown in figure 16.6.1. The gears rotate as indicated in the sketch. Oil is trapped in the space between the gear teeth and the casing. The oil is then carried from the lower pressure or atmospheric pressure and is delivered at the pressure side. The two sides are sealed by the meshing teeth in the middle. The maximum pressure that can be developed depends on the clearance and viscosity of the oil. The operation is fairly simple. One of the gear is the driving gear directly coupled to an electric motor or other type of drives.

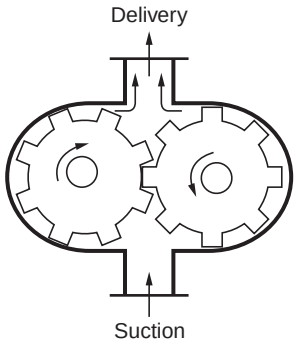


Figure 16.6.1 Gear pump

These pumps should be filled with oil before starting. The sketch shows an external gear pump. There is also another type of gear pump called internal gear pump. This is a little more compact but the construction is more complex and involved and hence used in special cases where space is a premium.

16.6.2 Lobe Pump

This type is also popularly used with oil. The diagramatic sketch of a lobe pump is shown in figure 16.6.2. This is a three lobed pump. Two lobe pump is also possible. The gear teeth are replaced by lobes. Two lobes are arranged in a casing. As the rotor rotates, oil is trapped in the space between the lobe and the casing and is carried to the pressure side. Helical lobes along the axis are used for smooth operation. Oil has to be filled before starting the pump. Lobe type of compressors are also in use. The constant contact between the lobes makes a leak tight joint preventing oil leakage from the pressure side.

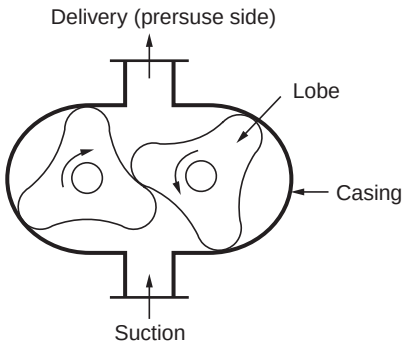


Figure 16.6.2 Lobe pump

The maximum pressure of operation is controlled by the back leakage through the clearance. This type of pump has a higher capacity compared to the gear pump.

16.6.3 Vane Pump

This is another popular type not only for oil but also for gases. A rotor is eccentrically placed in the casing as shown in figure 16.6.3. The rotor carries sliding vanes in slots along the length. Springs control the movement of the vanes and keep them pressed on the casing. Oil is trapped between the vanes and the casing. As the rotor rotates the trapped oil is carried to the pressure side. The maximum operating pressure is controlled by the back leakage.

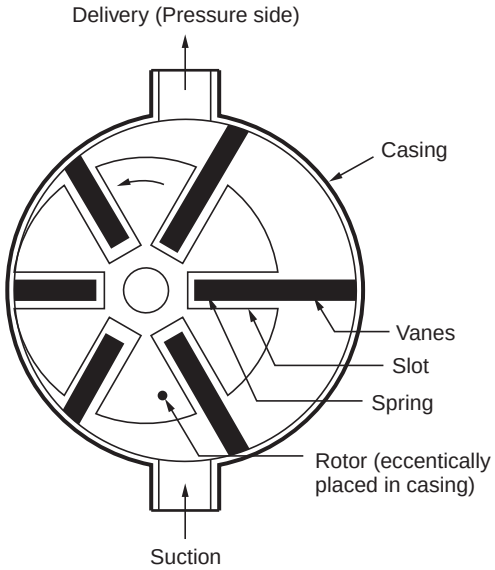


Figure 16.6.3 Vane pump

SOLVED PROBLEMS

Problem 16.1 *A single acting reciprocating water pump of 180 mm bore and 240 mm stroke operates at 40 rpm. Determine the discharge if the slip is 8%. What is the value of coefficient of discharge. If the suction and delivery heads are 6 m and 20 m respectively determine the theoretical power. If the overall efficiency was 80%, what is the power requirement.*

$$\text{Theoretical discharge} = \frac{A\,L\,N}{60}$$

$$= \frac{\pi \times 0.18^2}{4} \times 0.24 \times \frac{40}{60} = 4.0775 \times 10^{-3}\ \text{m}^3/\text{s}$$

$$= 4.0715\ \text{l/s} = 4.0715\ \text{kg/s}$$

$$\text{Slip} = 8\%$$

$$\therefore \quad \text{Actual flow} = 4.0715 \times \frac{92}{100} = 3.746\ \text{l/s or kg/s}$$

$$\text{Coefficient of discharge} = \frac{3.746}{4.0715} = 0.92$$

$$\text{Theoretical power} = mgH = 3.746 \times 9.81 \times 26$$

$$= 955.45\ \text{W}$$

$$\text{Actual Power} = 955.45/0.8$$

$$= \mathbf{1194.3\ W\ or\ 1.1943\ kW}$$

If it is a double acting pump, in case the piston rod diameter is neglected, the flow and power will be double this value. The slip and coefficient discharge and efficiency remaining the same.

Problem 16.2 *It is desired to have a discharge of water of 10 l/min using a reciprocating pump running at 42 rpm. The bore to stroke ratio is to be 1 : 1.5. It is expected that the slip will be 12%. Determine the bore and stroke for (a) single acting pump, and (b) double acting pump. If the total head is 30 m and the overall efficiency is 82%, determine the power required in both cases.*

Single acting pump :

Theoretical discharge $= \dfrac{A\,L\,N}{60}$ m^3/s

Actual discharge $= (1 - \text{slip})\,\dfrac{A\,L\,N}{60}$ m^3/s

Actual per minute $= (1 - \text{slip}) \times A\,L\,N$

$A = \dfrac{\pi D^2}{4}$, $\quad L = 1.5\,D$, $(1 - 0.12)\,\dfrac{\pi D^2}{4} \times 1.5\,D \times 42 = 0.01$

Solving $\quad D^3 = \dfrac{0.01 \times 4}{0.88 \times \pi \times 1.5 \times 42}$

Solving, $\quad$ **D = 62.7 mm** and **Stroke = 94 mm**

$$\textbf{Power} = \frac{mgh}{\eta} = \frac{10 \times 9.81 \times 30}{0.82 \times 60} = \textbf{60 W}$$

Double acting : (Neglecting piston rod diameter)

$$0.01 = (1 - 0.12)\left(\frac{2\pi D^2}{4} \times 1.5\,D \times 42\right)$$

Solving: $\quad$ **D = 48.6 mm, L = 97.2 mm**

The advantage of double acting pump is compactness and lower weight as can be seen form the values.

The power required will be double that of the single acting pump

P = 120 W.

Problem 16.3 *A reciprocating pump with plunger diameter of 120 mm and 200 mm stroke has both delivery and suction pipes of 90 mm diameter. The suction length is 9 m and the delivery length is 18 m. The atmospheric head is 10.3 m of water head. Determine the suction head and the delivery head due to acceleration at speeds 30, 40, 50, 60 rpm operating speeds.*

Delivery side: $\mathbf{h_{a\ max\ d}} = \dfrac{l}{g} \cdot \dfrac{A}{a} \cdot \omega^2 r = \dfrac{18}{9.81} \times \dfrac{0.12^2}{0.09^2}\left(\dfrac{2\pi \times N}{60}\right)^2 \times 0.1$

$$= \mathbf{3.577 \times 10^{-3}\,N^2}$$

The values are tabulated below

N rpm	30	40	50	60
$h_{a\ max\ d}$	3.22	5.72	8.94	12.88

Suction side: $\mathbf{h_{a\ max\ s}} = \frac{l}{g} \cdot \frac{A}{a} \cdot \omega^2 r = \frac{9}{9.81} \times \frac{0.12^2}{0.09^2} \left(\frac{2\pi \times N}{60} \right)^2 \times 0.1$

$= \mathbf{1.7886 \times 10^{-3}\ N^2}$.

The value are tabulated below for various speeds.

N rpm	30	40	50	60
$h_{a\ max\ d}$	1.61	2.86	4.47	6.44

As the separation limit is 2.5 m absolute, the maximum suction head at 60 rpm (for example) will be equal to (10.3 = 2.5 – 6.44) = 1.36 m only. It can be seen that this speed itself is on the higher side.

Problem 16.4 *In a reciprocating pump the bore is 180 mm and stroke is 280 mm. Water level is 5 m from the pump level. The suction pipe is 110 m diameter and 9 m long. The atmospheric pressure head is 10.3 m water. Determine the maximum speed if the head at pipe suction should not be less than 2.5 m head of water. If the suction pipe diameter is increased 125 mm and length reduced to 6 m, what will be the maximum speed ?*

Suction head = 5 m

$r = L/2 = 0.14$ m

Acceleration head available

$= 10.3 - 5 - 2.5 = 2.8$ m

$$h_{as} = \frac{l_s}{g} \cdot \frac{A}{a} \cdot \omega^2 r$$

$$2.8 = \frac{9}{9.81} \times \frac{0.18^2}{0.11^2} \omega^2 \times 0.14$$

$\therefore \quad \omega^2 = (2.8 \times 9.81 \times 0.11^2)/(9 \times 0.18^2 \times 0.14) = 8.413$

$$\omega = 2.8533 = \frac{2\pi N}{60}$$

$$\mathbf{N} = \frac{2.8533 \times 60}{2\pi} = \mathbf{27.25\ rpm}$$

Fairly low speed.

At the changed condition,

$$2.8 = \frac{6}{9.81} \times \frac{0.18^2}{0.125^2} \cdot \omega^2 \times 0.14$$

Solving, $\omega = 3.9711$

$\therefore$ **N = 37.92 rpm.**

Problem 16.5 *In a single acting reciprocating pump the bore and stroke are 90 and 160 mm. The static head requirements are 4 m suction and 15 m delivery. If the pressure at the end of delivery is atmospheric determine operating speed. The diameter of the delivery pipe is 90 mm and the length of the delivery pipe is 22 m. Determine the acceleration head at $\theta = 30$ from the start of delivery.*

In this case, the acceleration head equals the static delivery head.

$$\therefore \quad 15 = \frac{22}{9.81} \times \frac{0.09^2}{0.09^2} . \omega^2 \times 0.08$$

Solving, $\omega = 9.1437$

$$\mathbf{N} = \frac{\omega \times 60}{2\pi} = \frac{9.1437 \times 60}{2 \times \pi}$$

$$= \mathbf{87.32\ rpm}$$

At the position 30° from start of delivery,

$$\mathbf{h_a} = \frac{l}{g} . \frac{A}{a} . \omega^2 r \cos\theta$$

$$= \frac{22}{9.81} \times \frac{0.09^2}{0.09^2} . 9.1437^2 \times 0.08 \times \cos 30$$

$$= \mathbf{12.99\ m.}$$

Problem 16.6 *A reciprocating pump handling water with a bore of 115 mm and stroke of 210 mm runs at 35 rpm. The delivery pipe is of 90 mm diameter and 25 m long. An air vessel of sufficient volume is added at a distance of 2 m from the pump. Determine the acceleration head with and without the air vessel.*

Without air vessel:

$$\mathbf{h_a} = \frac{l}{g} . \frac{A}{a} . \omega^2 r$$

$$= \frac{25}{9.81} \times \frac{0.115^2}{0.09^2} \times \left(\frac{2\pi \times 35}{60} \right)^2 \times 0.105 = \mathbf{5.869\ m}$$

With air vessel l reduces to 2 m.

$$\therefore \quad \mathbf{h'} = \frac{5.869 \times 2}{25} = \mathbf{0.47\ m}$$

A considerable reduction.

Problem 16.7 *The bore and stroke of a reciprocating pump are 10 cm and 15 cm. The pump runs at 40 rpm. The suction pipe is 9 cm diameter and 12 m long. Determine the absolute pressure at suction if static suction is 3.5 m. Take $h_{atm} = 10.3$ m. If an air vessel is fitted at 1.5 m from the pump determine the absolute pressure at suction.*

Without air vessel:

$$h_{as} = \frac{l_s}{g} \cdot \frac{A}{a_s} \cdot \omega^2 r = \frac{12}{9.81} \times \frac{0.1^2}{0.09^2} \cdot \left(\frac{2\pi \times 40}{60}\right)^2 \times 0.075$$

$= 1.987$ m

Absolute pressure $= 10.3 - 3.5 - 1.987 =$ **4.812.** Safe against separation

With air vessel :

$$h_a' = 1.987 \times 1.5/12 = 0.248 \text{ m}$$

Absolute pressure $= 10.3 - 35 - 0.248 =$ **6.55 m**

The pump can be run at a higher speed.

Problem 16.8 *In a reciprocating pump delivering water the bore is 14 cm and the stroke is 21 cm. The suction lift is 4 m and delivery head is 12 m. The suction and delivery pipe are both 10 cm diameter, length of pipes are 9 m suction and 24 m delivery. Friction factor is 0.015. Determine the theoretical power required. Slip is 8 percent. The pump speed is 36 rpm.*

Volume delivered assuming single acting,

$$= A\,L\,N/60 = \frac{\pi \times 0.14^2}{4} \times 0.21 \times \frac{36}{60}$$

$= 1.9396 \times 10^{-3}$ m^3/s or 1.9396 kg/s

Slip is 8%

$\therefore$ Actual mass delivered $= 1.9396 \times 0.92 = 1.784$ kg/s

Total static head $= 4 + 12 = 16$ m head

Friction head in the delivery pipe:

Maximum velocity, $v = \frac{A}{a}\omega r = \frac{0.14^2}{0.1^2} \times \frac{2\pi \times 36}{60} \times 0.105 = 0.7758$ m/s

$$h_{fd} = \frac{flv^2}{2gd} = \frac{0.015 \times 24}{2 \times 9.81 \times 0.1} \cdot [0.7758]^2 = 0.11 \text{ m}$$

Average is, $2/3\ \mathbf{h_{fd}} =$ **0.07363 m**

Friction head in the suction pipe ;

Velocity is the same as diameters are equal

$$h_{fs} = \frac{0.015 \times 9}{2 \times 9.81 \times 0.1} \times [0.7758]^2 = 0.0414 \text{ m}$$

Average $= 2/3\ h_{fs} = 0.0414 \times 2/3 = 0.02761$ m

Total head $= 16 + 0.07363 + 0.02761 = 16.10124$ m

Theoretical Power $= 1.784 \times 9.81 \times 16.1024 =$ **282 W.**

Problem 16.9 *The bore and stroke of a single acting reciprocating water pump are 20 cm and 30 cm. The suction pipe is of 15 cm diameter and 10 m long. The delivery pipe is 12 cm diameter and 28 m long. The pump is driven at 32 rpm. Determine the acceleration heads and the friction head, f = 0.02. Sketch the indicator diagram. The suction and delivery heads from atmosphere are 4 m and 16 m respectively.*

$$\mathbf{h_{as\ max}} = \frac{l_s}{g} \cdot \frac{A}{a_s} \cdot \omega^2 r = \frac{10}{9.81} \times \frac{0.2^2}{0.15^2} \left(\frac{2\pi \times 32}{60} \right)^2 \times 0.15 = \mathbf{3.05\ m}$$

$$\mathbf{h_{ad\ max}} = \frac{l_d}{g} \cdot \frac{A}{a_d} \cdot \omega^2 r = \frac{28}{9.81} \times \frac{0.2^2}{0.12^2} \left(\frac{2\pi \times 32}{60} \right)^2 \times 0.15 = \mathbf{13.35\ m}$$

$$V_{s\ max} = \frac{A}{a} \omega r = \frac{0.2^2}{0.15^2} \times \frac{2\pi \times 32}{60} \times 0.15 = 0.8936$$

$$\mathbf{h_{fs}} = \frac{f l_s}{2 g d_s} \cdot v_s^{\,2} = \frac{0.2 \times 10 \times 0.8436^2}{2 \times 9.81 \times 0.15} = \mathbf{0.5427\ m}$$

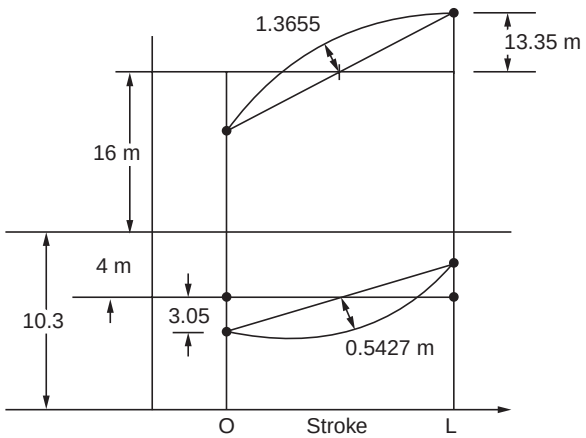


Figure P.16.9 Problem model

$$\mathbf{h_{fd}} = \frac{f l_d}{2 g d_d} \cdot V_d^{\,2},\ V_{d\ max} = \frac{A}{a} \cdot \omega r = \frac{0.2^2}{0.12^2} \times \frac{2\pi \times 32}{60} \times 0.15$$

$$= \mathbf{1.396\ m/s}$$

$$\mathbf{h_{fd\ max}} = \frac{0.02 \times 28}{2 \times 9.81 \times 0.12} \times 2.396^2 = \mathbf{1.3655\ m.}$$

Problem 16.10 *Using the data from problem 16.9 determine the theoretical power required.*

Flow rate $= \dfrac{A\,L\,N}{60} = 0.2^2 \times 0.3 \times 32/60$

$= 6.4 \times 10^{-3}$ m^3/s or 6.4 kg/s

Total head $= 16 + 4 + 2/3(1.3655 + 0.5427) = 21.272$ m

Power $= 21.272 \times 9.81 \times 6.4 =$ **1336 W.**

Problem 16.11 *A single acting reciprocating of pump handles water. The bore and stroke of the unit are 20 cm and 30 cm. The suction pipe diameter is 12 cm and length is 8 m. The delivery pipe diameter is 12 cm and length is 24 m. f = 0.02. The speed of operation is 32 rpm. Determine the friction power with and without air vessels.*

Without air vessels $V = \frac{A}{a}\omega r = \frac{0.2^2}{0.12^2} \times \frac{2\pi \times 32}{60} \times 0.15$

$= 1.3963$ m/s

$$\mathbf{h_{fs\ max}} = \frac{fl_s}{2gd_s} \times v^2 = \frac{0.02 \times 8}{2 \times 9.81 \times 0.12} \times (1.3963)^2 = \mathbf{0.1325\ m}$$

$$\mathbf{h_{fd\ max}} = \frac{0.02 \times 24}{2 \times 9.81 \times 0.12} \times \left(\frac{0.2^2}{0.12^2} \cdot \frac{2\pi \times 32}{60} \times 0.15 \right)^2 = \mathbf{0.3975\ m}$$

Total average friction head

$$= \frac{2}{3}[0.3975 + 0.1325] = 0.3533 \text{ m}$$

Flow rate $= \frac{A\,L\,N}{60} = \frac{\pi \times 0.2^2}{4} \times 0.3 \times \frac{32}{60} = 5.0265 \times 10^{-3}$ m^3/s

or $= 5.0265$ kg/s

Friction power $= 5.0265 \times 9.81 \times 0.3533 =$ **17.42 W**

When air vessels are installed,

Average velocity in suction pipe

$$= \frac{A}{a}\frac{L\,N}{60} = \frac{0.02^2}{0.12^2} \times 0.3 \times \frac{32}{60} = 0.4444 \text{ m/s}$$

$$h_{fs} = \frac{fl_s\,v^2}{2g\,d_s} = \frac{0.02 \times 8 \times 0.4444^2}{2 \times 9.81 \times 0.12} = 0.013424 \text{ m}$$

As diameters are equal velocity are equal

$$h_{fd} = \frac{0.02 \times 24 \times 0.4444^2}{2 \times 9.81 \times 0.12} = 0.040271 \text{ m}$$

Friction power $= 5.0265 \times 9.81 \times (0.013424 + 0.040271) =$ **2.65 W**

The percentage reduction is $\frac{17.42 - 2.65}{17.42} \times 100 =$ **84.8%**

By use of air vessels there is a saving of 84.8% in friction power.

Problem 16.12 *Show that in a double acting pump the work saved by fitting air vessels is about 39.2%.*

In a double acting pump during a revolution, the discharge

$$Q = \frac{2A\,L\,N}{60}$$

Velocity in the pipe with air vessel

$$= \frac{2\,A\,L\,N}{a \times 60} = \frac{2A \times 2r}{a \times 60} \times \frac{60 \times \omega}{2\pi} = \frac{2A}{a} \times \frac{\omega r}{\pi}$$

Friction head $$= \frac{fl}{2gd} \cdot v^2 = \frac{fl}{2gd} \cdot \left(\frac{2A}{a} \times \frac{\omega r}{\pi}\right)^2$$

Without air vessel,

Maximum friction head $= \frac{fl}{2gd}\left(\frac{A}{a} \cdot \omega r\right)^2$

Average value is $$= \frac{2}{3} \times \frac{fl}{2gd}\left(\frac{A}{a}\,\omega r\right)^2$$

The ratio of effective friction head is also the ratio of power as power = mgH, and mg are constant for a pump.

$$\frac{h_f \text{ with air vessel}}{h_f \text{ without air vessel}} = \frac{\dfrac{f\,l}{2gd}\left(\dfrac{2A}{a} \times \dfrac{\omega r}{\pi}\right)^2}{\dfrac{2}{3} \cdot \left(\dfrac{f\,l}{2gd}\right)\left(\dfrac{A}{a}\,\omega r\right)^2} = \frac{6}{\pi^2} = 0.608$$

$\therefore$ Reduction is **39.2%.**

Problem 16.13 *In a double acting pump, determine the angle at which there will be no flow in or out of the air vessel.*

Refer equation (16.5.5). Flow to or from air vessel

$$Q = A\omega r\left(\sin\theta - \frac{2}{\pi}\right)$$

When there is no flow in or out of the air vessel,

$$Q = 0,$$

$\therefore$ $$\sin\theta = \frac{2}{\pi} \quad \therefore \quad \theta = \sin^{-1}\left(\frac{2}{\pi}\right) = \mathbf{39.54°}$$

and $$(180 - 39.54) = \mathbf{140.46°}$$

This is as against about 18° in the case of single acting pump.

Problem 16.14 *Tabulate the flow rate as a product of Aωr for various angles of θ in the case of double acting pump.* $\left(\sin\theta - \frac{2}{\pi}\right)$ *is calculated and tabulated.*

Crank Angle θ	0	30	60	90	120	150	180
Flow rate Aωr times	– 0.637	– 0.137	0.2294	0.3634	0.2294	– 0.137	– 0.637

Note that this is different from example 16.9

For the following data speed is 30 rpm, and r = 0.15 m, bore = 20 cm.

at 30°, flow is $\left(\sin\theta - \frac{2}{\pi}\right) - 0.137$

$$\therefore \qquad \mathbf{q} = -0.137 \times \frac{\pi \times 0.2^2}{4} \times \frac{2 \times \pi \times 30}{60} \times 0.15$$

$$= -2.028 \times 10^{-3} \text{ m}^3/\text{s or } 2.028 \text{ l/s out of the air vessel.}$$

At the starting of delivery stroke, flow is zero, but there is flow in the pipe. This should come out of the air vessel.

Problem 16.15 *In a single acting pump of 16 cm bore and 24 cm stroke, the delivery pipe is 20 m long. f = 0.02. The speed 45 rpm. Determine the friction head on the delivery side, with and without air vessel for pipe diameters of 8 cm and 12 cm.*

Without air vessel : Effective friction head

$$\mathbf{h_f} = \frac{2}{3} \cdot \frac{fl}{2gd} \left(\frac{A}{a}\omega r\right)^2$$

for 8 cm pipe diameter,

$$= \frac{2}{3} \times \frac{0.02 \times 20}{2 \times 9.81 \times 0.08} \left(\frac{0.16^2}{0.08^2} \times 2\pi \times \frac{45}{60} \times 0.12\right)^2$$

$$= \mathbf{0.8692 \text{ m}}$$

For 12 cm dia, h_f = 0.1145 m (obviously, larger the pipe diameter lower the friction head)

With air vessel :

$$\mathbf{h_f} = \frac{fl}{2gd} \left(\frac{A}{a} \cdot \frac{\omega r}{\pi}\right)^2$$

$$= \frac{0.02 \times 20}{2 \times 9.81 \times 0.08} \left(\frac{0.16^2}{0.08^2} \times \frac{2\pi \times 45}{60} \times \frac{0.12}{\pi}\right)^2 = \mathbf{0.1321 \text{ m}}$$

For $\mathbf{d = 0.12 \text{ m}, h_f = 0.0174 \text{ m.}}$

REVIEW QUESTIONS

1. List some types of positive displacement pumps.
2. List the advantages and limitations of reciprocating pumps over dynamic pumps.
3. Compare single acting and double acting pumps.
4. Explain the reason for the limitation of speed of operation of reciprocating pumps.
5. Compare reciprocating and rotary type of positive displacement pumps.
6. Describe working of vane pump.
7. Describe the working of gear pump.
8. Describe the working of lobe pump.
9. Define "slip" in reciprocating pump. Can it be negative ?
10. What is the cause of negative slip ?
11. Explain the function of air vessel.
12. What are the advantages of installing air vessels in reciprocating pumps.
13. Define separation. Explain how this controls the speed of operation.
14. Explain what causes additional head than static head in reciprocating pumps.

OBJECTIVE QUESTIONS

I. Fill in the blanks :

1. Air vessels _____ the flow.
2. Reciprocating pumps are suitable for _____ pressures.
3. Reciprocating pumps are suitable for _____ volumes.
4. Air vessels _____ friction head.
5. Acceleration heads _____ affect the power required.
6. The delivery pipe _____ be bent after the vertical run to avoid separation.
7. The friction head is _____ after the installation of air vessels.
8. The friction head _____ and then _____ in reciprocating pumps without air vessel.
9. Acceleration head is affected by _____ and _____.
10. Double acting pump will give a _____ flow.

Answers

(1) Smoothens (2) High (3) Low (4) Reduce (5) Does not (6) Should not (7) Reduced (8) Increases, decreases (9) Length of pipe, speed (10) Smoother.

EXERCISE PROBLEMS

16.E.1 A single acting pump running at 30 rpm delivers 6.5 l/s of water. The bore and stroke are 20 cm and 30 cm respectively. Determine the percentage slip and coefficient of discharge.

(8.5%, 0.92)

16.E.2 A double acting water pump with bore of 30 cm and stroke of 40 cm has piston rod of 5 cm diameter. The pump runs at 60 rpm and delivers 50 l/s. Determine the slip. Consider area occupied by the piston rod. Also find the coefficient of discharge. **(10.4%, 0.896)**

16.E.3 A single acting pump has a bore and stroke of 300 mm and 400 mm respectively. The discharge is 50 l/s. If the slip is 2%, determine the speed of operation. **(54 rpm)**

16.E.4 The bore and stroke of a double acting pump running at 60 rpm are 0.15 m and 0.3 m respectively. The total head against which works 30 m. If the efficiency is 80% determine the power required. **(4050 W)**

16.E.5 The bore and stroke of a single acting reciprocating pump are 300 mm and 450 mm respectively. The static suction head is 4 m. The suction pipe is 125 mm in diameter and 8 m long. If the separation head is 2.5 m determine the maximum speed of operation of the pump. Atmospheric head is 10.3 m of water. Also calculate the discharge at this speed and the maximum friction head on the suction side. $f = 0.02$. What will be pressure at starting, middle and end of stroke ?

(34 rpm, 18 l/s, 1.4 m, 2.5 m, 4.9 m, 10.1 m)

16.E.6 Determine the maximum speed of operation of a single acting reciprocating pump to avoid separation, give the bore and stroke as 10 cm and 15 cm and static suction head as 4 m and suction pipe length as 6 m and diameter as 25 mm. **(20.9 rpm)**

16.E.7 A single acting reciprocating pump running at 24 rpm has a bore and stroke of 12.5 cm and 30 cm. The static suction 4 m. Determine the pressure at start, middle and end of suction stroke. The suction pipe of 75 mm diameter is 9 m long. Atmospheric pressure is 10.3 m of water

(3.9, 4, 8.7)

16.E.8 A single acting pump takes water from well at 4 m from pump level. The bore and stroke are 20 cm and 30 cm. The suction pipe of 10 cm diameter is 8 m long. If the pump runs at 30 rpm check whether separation will occur. **(Separation will occur as head is 1.5 m at start)**

16.E.9 A single acting reciprocating pump running at 20 rpm has a bore and stroke of 40 cm each. The suction and delivery heads are 4 and 20 m. The suction and delivery pipes of 20 cm diameter are 4 m and 20 m long. $f = 0.02$. Determine the power required if the efficiency is 80%. **(4.99 kW)**

16.E.10 A reciprocating pump running at 60 rpm has a bore of 300 mm and stroke of 450 mm. The delivery pipe of 150 mm diameter is 50 m long. Determine the saving in friction power due to fitting of an air vessel on the delivery side. $f = 0.02$. **(890 W)**

16.E.11 Determine the maximum suction head for a reciprocating pump running at 90 rpm, if bore and stroke are 10 cm and 25 cm respectively. The suction pipe of 100 mm diameter is 5 m long. Separation occurs when the pressure in the cylinder during suction reaches 2 m.

(2.65 m)

16.E.12 The bore and stroke of a reciprocating pump are 25 cm and 50 cm. The delivery pipe is of 100 mm diameter. The delivery is to a tank 15 m above the pump. Determine the speed if separation should not occur. The separation pressure is 2.3 m. The tank is at a distance of 30 m horizontally from the pump. There is no air vessel.

Case (*i*) Pipe is vertical up to 15 m and then horizontal.

Case (*ii*) Pipe is horizental for 30 m and then vertical.

[Case (*i*) 10.1 rpm, Case (*ii*) 17 rpm)

16.E.13 The bore and stroke of a double acting pump running at 40 rpm are 0.2 m and 0.4 m. The delivery pipe of 15 cm diameter is 36 m long. The static delivery head is 10 m. All air vessel is fitted at the level of the pump at a distance of 3 m from the pump. $f = 0.032$. Determine the pressure in the pump at the beginning of delivery stroke. **(12.2 m)**

16.E.14 Determine the change in the maximum speed of operation due to fitting an air vessel on the suction side of a pump of 300 mm bore and 500 mm stroke. The suction pipe of 200 mm diameter is 10 m long. The suction lift is 3.5 m. **(26.1 rpm and 48.6 rpm)**

16.E.15 Determine the friction power on the delivery side without and with fitting air vessel on a pump running at 60 rpm if the bore and stroke are 250 mm and 500 mm. The delivery pipe of 100 mm diameter is 50 m long. $f = 0.04$. **(2.53 kW)**

16.E.16 A double acting pump of 175 mm bore and 350 mm stroke runs at 150 rpm. The suction pipe is of 150 mm diameter. Determine the crank angle at which there will be no flow from or to the air vessel. **(39.5° or 40.5°).**

APPENDIX

Property Values of Dry air at one atm. pressure

Temperature t °C	*Density kg/m³*	*Coefficient of viscosity μ, kg/ms or Ns/m²*	*Kinematic Viscosity ν, m²/s*
– 50	1.584	14.61×10^{-6}	9.23×10^{-6}
– 40	1.515	15.20×10^{-6}	10.04×10^{-6}
– 30	1.453	15.69×10^{-6}	10.80×10^{-6}
– 20	1.395	16.18×10^{-6}	11.61×10^{-6}
– 10	1.342	16.67×10^{-6}	12.43×10^{-6}
0	1.293	17.16×10^{-6}	13.28×10^{-6}
10	1.247	17.65×10^{-6}	14.16×10^{-6}
20	1.205	18.14×10^{-6}	15.06×10^{-6}
30	1.165	18.63×10^{-6}	16.00×10^{-6}
40	1.128	19.12×10^{-6}	16.96×10^{-6}
50	1.093	19.61×10^{-6}	17.95×10^{-6}
60	1.060	20.10×10^{-6}	18.97×10^{-6}
70	1.029	20.59×10^{-6}	20.02×10^{-6}
80	1.000	21.08×10^{-6}	21.09×10^{-6}
90	0.972	21.48×10^{-6}	22.10×10^{-6}
100	0.946	21.87×10^{-6}	23.13×10^{-6}
120	0.898	22.85×10^{-6}	25.45×10^{-6}
140	0.854	23.73×10^{-6}	27.80×10^{-6}
160	0.815	24.52×10^{-6}	30.09×10^{-6}
180	0.779	25.30×10^{-6}	32.49×10^{-6}
200	0.746	25.99×10^{-6}	34.85×10^{-6}
250	0.674	27.36×10^{-6}	40.61×10^{-6}
300	0.615	29.71×10^{-6}	48.20×10^{-6}

1 Ns/m^2 = 0.102 kgf/m^2 = 0.1 Poise, $\beta = 1/T$, T in K

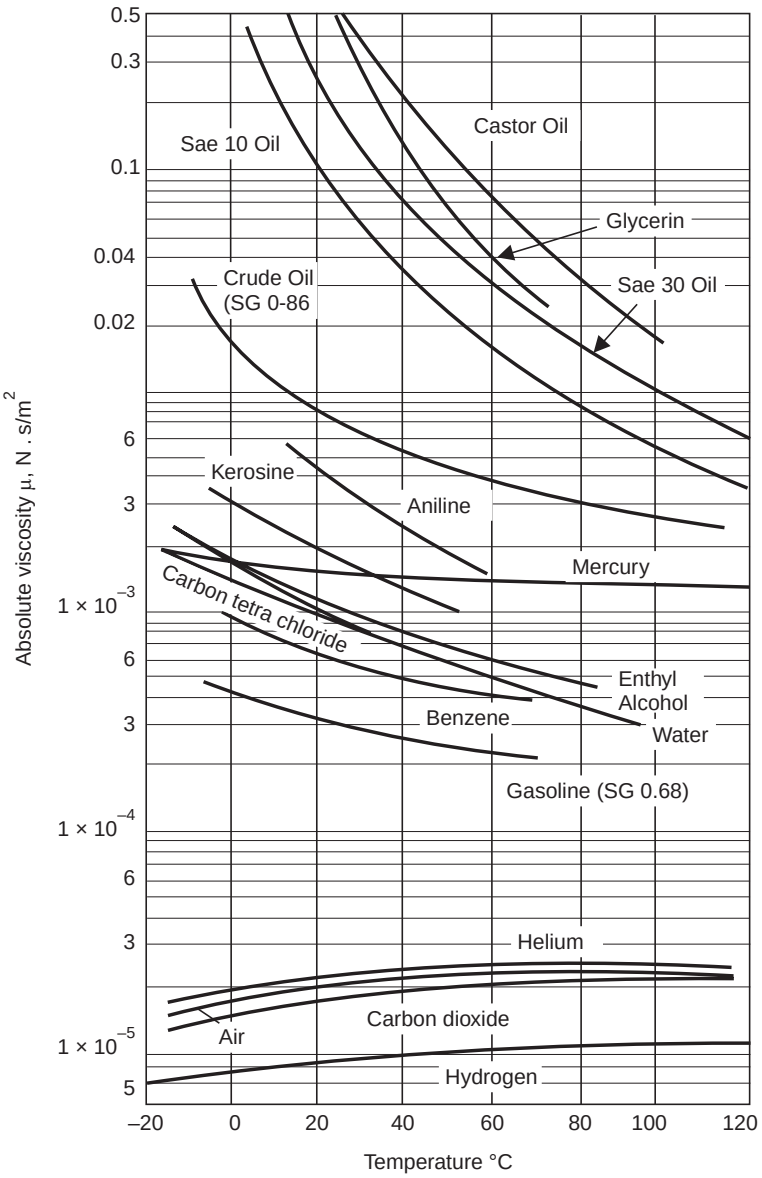
Property Values of Water in Saturated State

Temperature t °C	*Density kg/m³*	*Kinematic Viscosity ν, m²/s*
0	1002	1.788×10^{-6}
20	1000	1.006×10^{-6}
40	995	0.657×10^{-6}
60	985	0.478×10^{-6}
80	974	0.364×10^{-6}
100	961	0.293×10^{-6}
120	945	0.247×10^{-6}
140	928	0.213×10^{-6}
160	909	0.189×10^{-6}
180	889	0.173×10^{-6}
200	867	0.160×10^{-6}
220	842	0.149×10^{-6}
240	815	0.143×10^{-6}
260	786	0.137×10^{-6}
280	752	0.135×10^{-6}
300	714	0.135×10^{-6}

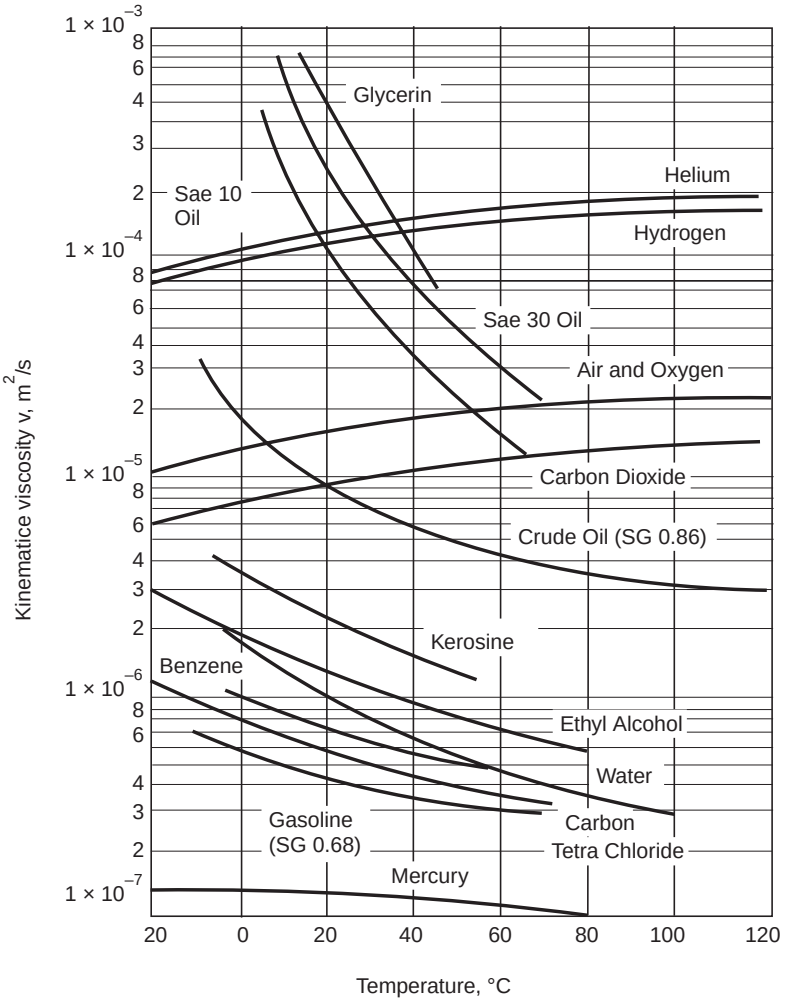
β = (change in density/change in temp.) (1/density)

μ = density × kinematic viscosity, 1 Ns/m^2 = 0.102 kgf/m^2 = 0.1 Poise

ν = 1 m^2/s = 3600 m^2/hr



Absolute Viscosity of common fluids at 1 atm.



Kinematic Viscosity of common fluids at 1 atm.

UNIT CONVERSION CONSTANTS

Quantity	*SI to Metric*	*Metric to SI*
Force	1 N = 0.1019 kg_f	1 kg_f = 9.81 N
Pressure	1 N/m^2 = 10.19 × 10^{-6} kg_f/cm^2	1 kg_f/cm^2 = 98135 N/m^2, (Pascal)
	1 bar = 1.0194 kg_f/cm^2	1 kg_f/cm^2 = 0.9814 bar
Energy	1 kJ = 0.2389 kcal	1 kcal = 4.186 kJ
(heat, work)	1 Nm (= 1J) = 0.1019 kg_f m	1 kg_f m = 9.81 Nm, (J)
	1 kW hr = 1.36 hp hr	1 hp hr = 0.736 kW hr
Power (metric)	1 W = 1.36 × 10^{-3} hp	1 hp = 736 W
Heat flow	1 W = 0.86 kcal/hr	1 kcal/hr = 1.163 W
Specific heat	1 kJ/kg K = 0.2389 kcal/kg °C	1 kcal/kg °C = 4.186 kJ/kg K
Surface Tension	1 N/m = 0.1019 kg_f/m	1 kg_f/m = 9.81 N/m
Thermal Conductivity	1 W/m K = 0.86 kcal/hr m °C	1 kcal/hr m °C = 1.163 W/m K
Convection Coefficient	1 W/m^2 K = 0.86 kcal/hr m^2 °C	1 kcal/hr m^2 °C = 1.163 W/m^2 K
Dynamic Viscosity	1 kg/ms, (Ns/m^2) = 0.1 Poise	1 Poise = 10 kg/ms, (Ns/m^2), Pa s
Kinematic Viscosity	1 m^2/s = 3600 m^2/hr,	1 m^2/hr = 2.778 × 10^{-4} m^2/s
	1 Stoke = 10^{-4} m^2/s	

Universal gas constant = 8314.41 J/kg mol K = 847.54 mkg_f/kg mol K = 1.986 kcal/kg mol K

Gas Constant for Air = 287 J/kg K, c_p = 1005 J/kg K = 0.24 kcal/kg K

Stefan-Boltzmann constant = 5.67 × 10^{-8} W/m^2K^4 = 4.876 × 10^{-8} kcal/hr m^2 K^4

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