

- 6.12 Determine the corresponding stream function for each of the velocity potential functions in Problem 6.10

$$\left[(a) y - x + \text{constant}; (c) -\frac{1}{2} \log(x^2 + y^2) + \text{constant}; \right. \\ \left. (d) -\frac{Ay}{(x^2 + y^2)} + \text{constant} \right]$$

Note : The velocity potential functions (b), (e), (f) do not represent the flow field and therefore no explicit stream functions corresponding to these can be obtained. The velocities of (b), (e), (f) determined in Problem 6.10 above have therefore no physical significance.

- 6.13 Describe the use and limitations of flow nets.
 6.14 Discuss briefly the different methods of drawing the flow nets.
 6.15 For a two dimensional flow of an incompressible fluid the velocity components V_r and V_θ are given as

$$V_r = akr^n e^{[-k(n+1)\theta]}, \text{ and}$$

$$V_\theta = a r^n e^{[-k(n+1)\theta]}.$$

Determine the stream function ψ and show that the fluid velocity at any point is given by

$$V = [(n+1) \psi \sqrt{(1+k^2)}/r]$$

$$\left[\psi = \frac{1}{n+1} \{ ar^{(n+1)} e^{-k(n+1)\theta} \} \right]$$

Equations of Motion and Energy Equation

7.1 INTRODUCTION

A fluid in motion is subjected to several forces which results in the variation of the acceleration and the energies involved in the flow phenomenon of the fluid. As such in the study of the fluid motion the forces and energies that are involved in the flow are required to be considered. This aspect of fluid motion is known as dynamics of fluid flow.

The various forces acting on a fluid mass may be classified as (i) body or volume forces, (ii) surface forces, and (iii) line forces. The body or volume forces are the forces which are proportional to the volume of the body, the examples of which are weight, centrifugal force, magnetic force, electromotive force, etc. The surface forces are the forces which are proportional to the surface area which may include pressure force, shear or tangential force, force of compressibility, force due to turbulence etc. The linear forces are the forces which are proportional to the length and the example of which is surface tension.

Alike mechanics of solids, the dynamics of fluid is also governed by Newton's second law of motion. Newton's second law of motion states that the resultant force on any fluid element must equal the product of the mass and the acceleration of the element and the acceleration vector has the direction of the resultant force vector. In the mathematical form this law may be expressed as

$$\Sigma F = Ma \quad \dots(7.1)$$

where ΣF represents the resultant external force acting on the fluid element of mass M and a is the total acceleration. Obviously both the acceleration and the resultant external force must be along the same line of action. Therefore the force and the acceleration vectors can be resolved along the three reference directions x , y and z and the corresponding equations may be expressed as

$$\left. \begin{aligned} \Sigma F_x &= Ma_x \\ \Sigma F_y &= Ma_y \\ \Sigma F_z &= Ma_z \end{aligned} \right\} \quad \dots(7.1a)$$

where ΣF_x , ΣF_y and ΣF_z are the components of the resultant force in the x , y and z directions respectively and a_x , a_y and a_z are the components of the total acceleration in the x , y and z directions respectively.

Often in the study of the fluid motion the forces per unit volume of the fluid element are required to be considered. The equations 7.1 and 7.1 (a) may then be expressed as

$$\Sigma f = \rho a \quad \dots(7.2)$$

and

$$\left. \begin{aligned} \Sigma f_x &= \rho a_x \\ \Sigma f_y &= \rho a_y \\ \Sigma f_z &= \rho a_z \end{aligned} \right\} \quad \dots(7.2 a)$$

in which f is the resultant force per unit volume acting on the fluid mass, ρ is the mass density of the fluid, and f_x, f_y and f_z are the components of the resultant force per unit volume, f , along x, y and z directions.

7.2 FORCES ACTING ON FLUID IN MOTION

The various forces that may influence the motion are due to gravity, pressure, viscosity, turbulence, surface tension and compressibility. The *gravity force* F_g is due to the weight of the fluid and it is equal to ' Mg '. The *gravity force* per unit volume is equal to ' ρg '. The *pressure force* F_p is exerted on the fluid mass if there exists a pressure gradient between the two points in the direction of flow. The *viscous force* F_v is due to the viscosity of the flowing fluid and thus exists in the case of all real fluids. The *turbulent flow* F_t is due to the turbulence of the flow. In the turbulent flow the fluid particles move from one layer to other and therefore, there is a continuous momentum transfer between adjacent layers, which results in developing additional stresses (called Reynolds stresses) for the flowing fluid. The *surface tension force* F_s is due to the cohesive property of the fluid mass. It is, however, important only when the depth of flow is extremely small. The *compressibility force* F_c is due to the elastic property of the fluid and it is important only either for compressible fluids or in the cases of flowing fluids in which the elastic properties of fluids are significant.

If a certain mass of fluid in the motion is influenced by all the above mentioned forces, then according to Newton's second law of motion the following equation of motion may be written

$$Ma = F_g + F_p + F_v + F_t + F_s + F_c \quad \dots(7.3)$$

Further by resolving the various forces and the acceleration along the x, y and z directions the following equations of motion may be obtained.

$$\left. \begin{aligned} Ma_x &= F_{gx} + F_{px} + F_{vx} + F_{tx} + F_{sx} + F_{cx} \\ Ma_y &= F_{gy} + F_{py} + F_{vy} + F_{ty} + F_{sy} + F_{cy} \\ Ma_z &= F_{gz} + F_{pz} + F_{vz} + F_{tz} + F_{sz} + F_{cz} \end{aligned} \right\} \quad \dots(7.3 a)$$

The subscripts x, y, z are introduced to represent the components of each of the forces and the acceleration in the respective directions.

In most of the problems of the fluids in motion the surface tension forces and the compressibility forces are not significant. Hence these forces may be neglected. Then equations 7.3 and 7.3(a) become

$$Ma = F_g + F_p + F_v + F_t \quad \dots(7.4)$$

and

$$\left. \begin{aligned} Ma_x &= F_{gx} + F_{px} + F_{vx} + F_{tx} \\ Ma_y &= F_{gy} + F_{py} + F_{vy} + F_{ty} \\ Ma_z &= F_{gz} + F_{pz} + F_{vz} + F_{tz} \end{aligned} \right\} \quad \dots(7.4a)$$

Equation 7.4 (a) are known as *Reynolds' equations of motion* which are useful in the analysis of the turbulent flows.

Further for laminar or viscous flows the turbulent forces also become less significant and hence these may be neglected. The equations 7.4 and 7.4 (a) may then be modified as

$$Ma = F_g + F_p + F_v \quad \dots(7.5)$$

and

$$\left. \begin{aligned} Ma_x &= F_{gx} + F_{px} + F_{vx} \\ Ma_y &= F_{gy} + F_{py} + F_{vy} \\ Ma_z &= F_{gz} + F_{pz} + F_{vz} \end{aligned} \right\}$$

Equations 7.5 (a) are known as *Navier-Stokes' equations* which are useful in the analysis of viscous flow.

Further if the viscous forces are also of little significance in the problems of fluid flows, then these forces may also be neglected. The viscous forces will become insignificant if the flowing fluid is an ideal fluid. However, in the case of real fluids also the viscous forces may be considered to be insignificant if the viscosity of the flowing fluid is small. In such cases the equations 7.5 and 7.5(a) may be further modified as

$$Ma = F_g + F_p \quad \dots(7.6)$$

and

$$\left. \begin{aligned} Ma_x &= F_{gx} + F_{px} \\ Ma_y &= F_{gy} + F_{py} \\ Ma_z &= F_{gz} + F_{pz} \end{aligned} \right\} \quad \dots(7.6a)$$

Equations 7.6 (a) are known as *Euler's equations of motion*.

A detailed study of the Reynolds equations of motion and the Navier-Stokes equations are beyond the scope of this book. The Euler's equations of motion are however dealt with in detail in the next section.

7.3 EULER'S EQUATION OF MOTION

As stated above in the Euler's equations of motion only two forces, namely the pressure force and the fluid weight or in general the body force, are assumed to be acting on the mass of fluid in motion. Further in the derivation of these

equations. It is assumed that the fluid is non-viscous, so that the pressure forces may be considered to be acting in the direction normal to the surface.

Consider a point $P(x, y, z)$ in a flowing mass of fluid at which let u, v and w be the velocity components in the directions x, y and z respectively; ρ be the mass density of the fluid and p be the pressure intensity. Further let X, Y and Z be the components of the body force per unit mass at the same point.

A parallelepiped of fluid with its edges of length $\delta x, \delta y$ and δz parallel to x, y , and z axes is considered in the fluid mass with the point $P(x, y, z)$ as one of its corners, as shown in Fig. 7.1.

The various external forces acting on the parallelepiped may now be evaluated. The mass of the fluid contained in the parallelepiped is $(\rho \delta x \delta y \delta z)$. Therefore the total component of the body force acting on the parallelepiped in the x direction is equal to $X(\rho \delta x \delta y \delta z)$.

Similarly the components of the body force acting on the parallelepiped in the y and z directions are obtained as $Y(\rho \delta x \delta y \delta z)$ and $Z(\rho \delta x \delta y \delta z)$ respectively.

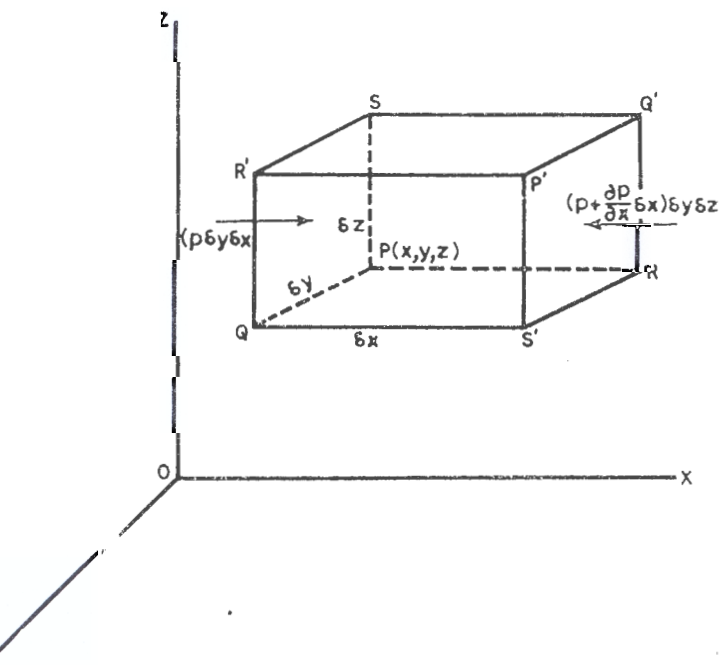


Fig. 7.1. Elementary parallelepiped subjected to the pressure and body forces

The pressure intensity at the point P is p . Since the length of the edges of the parallelepiped are extremely small, it may be assumed that the pressure intensity on the face $PQR'S$ is uniform and equal to p . Therefore the total pressure force acting on the face $PQR'S$ is in the direction normal to the surface

$$= (p \delta y \delta z)$$

Since the pressure p varies with x, y and z , the pressure intensity on the face $R'S'P'Q'$ will be

$$= \left(p + \frac{\partial p}{\partial x} \delta x \right)$$

Therefore the total pressure force acting on the face $R'S'P'Q'$ in the direction normal to the face

$$= \left(p + \frac{\partial p}{\partial x} \delta x \right) \delta y \delta z$$

Since there is a difference of pressure between the two end faces of the parallelepiped, a net pressure force F_{px} acts on the fluid mass in the x direction, the magnitude of which is obtained as

$$F_{px} = p \delta y \delta z - \left(p + \frac{\partial p}{\partial x} \delta x \right) \delta y \delta z$$

or

$$F_{px} = - \frac{\partial p}{\partial x} \delta x \delta y \delta z \tag{7.7 a}$$

In the same manner the components of the pressure force in the y and z directions may also be obtained as

$$F_{py} = \left(- \frac{\partial p}{\partial y} \delta x \delta y \delta z \right) \tag{7.7 b}$$

and

$$F_{pz} = \left(- \frac{\partial p}{\partial z} \delta x \delta y \delta z \right) \tag{7.7 c}$$

The components of the pressure force per unit volume may be obtained by dividing both sides of equation 7.7 by the volume of the parallelepiped ($\delta x \delta y \delta z$) as

$$\left. \begin{aligned} f_{px} &= - \frac{\partial p}{\partial x} \\ f_{py} &= - \frac{\partial p}{\partial y} \\ f_{pz} &= - \frac{\partial p}{\partial z} \end{aligned} \right\} \tag{7.8}$$

Now adding the body force and the pressure forces and equating to the product of the mass and the acceleration in the x direction, in accordance with the Newton's second law of motion, we get

$$X(\rho \delta x \delta y \delta z) - \frac{\partial p}{\partial x} \delta x \delta y \delta z = (\rho \delta x \delta y \delta z) a_x$$

where a_x is the component of the total acceleration in the x direction.

Dividing both sides by the mass of the fluid ($\rho dx dy dz$) in the parallelepiped and taking the limit so that the parallelepiped tends to point P , we obtain the Euler's equation of motion in the x direction at any point in the flowing mass of fluid as

$$X - \frac{1}{\rho} \frac{\partial p}{\partial x} = a_x \quad \dots(7.9 a)$$

In the same manner the equations for the y and z directions may be obtained as

$$Y - \frac{1}{\rho} \frac{\partial p}{\partial y} = a_y \quad \dots(7.9 b)$$

and

$$Z - \frac{1}{\rho} \frac{\partial p}{\partial z} = a_z \quad \dots(7.9 c)$$

Equations 7.9 (a), 7.9 (b) and 7.9 (c) are known as Euler's equations of motion.

As indicated in chapter 6 the acceleration components a_x, a_y and a_z may be expressed in terms of the velocity components u, v and w as

$$\begin{aligned} a_x &= \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \\ a_y &= \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \\ a_z &= \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \end{aligned}$$

Substituting these expressions for the accelerations in equation 7.9, the Euler's equations of motion are obtained as

$$\left. \begin{aligned} X - \frac{1}{\rho} \frac{\partial p}{\partial x} &= \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \\ Y - \frac{1}{\rho} \frac{\partial p}{\partial y} &= \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \\ Z - \frac{1}{\rho} \frac{\partial p}{\partial z} &= \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \end{aligned} \right\} \quad \dots(7.10)$$

In the derivation of these equations, no assumption has been made that the mass density ρ is a constant. Hence these equations of motion are applicable to compressible or incompressible, non viscous fluids in steady or unsteady state of flow.

7.4 INTEGRATION OF EULER'S EQUATIONS

The Euler's equations of motion can be integrated to yield the energy equation under the following assumptions.

(i) There exists a *force potential* which is defined as that whose negative derivative with respect to any direction gives the component of the body force per unit mass in that direction. It is usually denoted by Ω and according to the definition given

$$\left. \begin{aligned} X &= - \frac{\partial \Omega}{\partial x} \\ Y &= - \frac{\partial \Omega}{\partial y} \\ Z &= - \frac{\partial \Omega}{\partial z} \end{aligned} \right\} \quad \dots(7.11)$$

and

(ii) The flow is *irrotational* i.e., the velocity potential exists or the flow may be rotational, but it is steady.

(a) **When the flow is irrotational.** Considering the Euler's equation of motion in one of the directions, we have

$$X - \frac{1}{\rho} \frac{\partial p}{\partial x} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z}$$

If the flow is irrotational then as indicated in chapter 6

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}, \quad \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}$$

and $u = -\frac{\partial \phi}{\partial x}$, since the velocity potential ϕ exists for an irrotational flow.

$$\begin{aligned} \text{Then} \quad \frac{\partial u}{\partial t} &= \frac{\partial}{\partial t} \left(-\frac{\partial \phi}{\partial x} \right) \\ &= -\frac{\partial^2 \phi}{\partial t \partial x} = -\frac{\partial^2 \phi}{\partial x \partial t} \end{aligned}$$

The order of differentiation has been changed meaning thereby that $\frac{\partial^2 \phi}{\partial t \partial x}$ is same as $\frac{\partial^2 \phi}{\partial x \partial t}$, which is however true only if ϕ is a continuous function and its second derivative exists.

By substituting these values the equation of motion in the x direction becomes

$$\begin{aligned} -\frac{\partial \Omega}{\partial x} - \frac{1}{\rho} \frac{\partial p}{\partial x} &= -\frac{\partial^2 \phi}{\partial x \partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial x} + w \frac{\partial w}{\partial x} \\ \text{or} \quad -\frac{\partial \Omega}{\partial x} - \frac{1}{\rho} \frac{\partial p}{\partial x} &= -\frac{\partial^2 \phi}{\partial x \partial t} + \frac{1}{2} \frac{\partial u^2}{\partial x} + \frac{1}{2} \frac{\partial v^2}{\partial x} + \frac{1}{2} \frac{\partial w^2}{\partial x} \\ \text{or} \quad \frac{\partial}{\partial x} \left[\Omega + \frac{p}{\rho} + \frac{u^2 + v^2 + w^2}{2} - \frac{\partial \phi}{\partial t} \right] &= 0 \end{aligned}$$

Now if it is assumed that the fluid is incompressible so that the mass density ρ is independent of pressure, then integrating the above equation with respect to x , we get

$$\Omega + \frac{p}{\rho} + \frac{u^2 + v^2 + w^2}{2} - \frac{\partial \phi}{\partial t} = F_1(y, z, t)$$

where F_1 is an arbitrary function resulting from integration.

However if the fluid is compressible, so that the mass density ρ depends on the pressure p , then integrating the above equation with respect to x , we get

$$\Omega + \int \frac{dp}{\rho} + \frac{u^2 + v^2 + w^2}{2} - \frac{\partial \phi}{\partial t} = F_1(y, z, t)$$

The above equation is thus a more general equation which is applicable to both compressible as well as incompressible fluids.

If V represents the resultant velocity at any point whose components in the x , y and z directions are u , v and w , then

$$V^2 = (u^2 + v^2 + w^2)$$

and the integrated equation becomes

$$\Omega + \int \frac{dp}{\rho} + \frac{V^2}{2} - \frac{\partial \phi}{\partial t} = F_1(y, z, t) \quad \dots(7.12 a)$$

By adopting the same procedure the other two Euler's equations of motion for the y and z directions may also be integrated which may be written as

$$\Omega + \int \frac{dp}{\rho} + \frac{V^2}{2} - \frac{\partial \phi}{\partial t} = F_2(x, z, t) \quad \dots(7.12 b)$$

$$\text{and} \quad \Omega + \int \frac{dp}{\rho} + \frac{V^2}{2} - \frac{\partial \phi}{\partial t} = F_3(x, y, t) \quad \dots(7.12 c)$$

In equations 7.12 (a), 7.12 (b) and 7.12 (c) it is observed that the left hand side of all these equations is same and therefore

$$F_1(y, z, t) = F_2(x, z, t) = F_3(x, y, t)$$

Since x , y and z are independent variables, the above equation will hold good only if these variables disappear from the functional terms, or in other words the arbitrary functions F_1 , F_2 and F_3 are the functions of the time alone or are mere constants.

Thus the final integrated form of the Euler's equations of motion for all the three directions becomes one equation as

$$\Omega + \int \frac{dp}{\rho} + \frac{V^2}{2} - \frac{\partial \phi}{\partial t} = F(t) \quad \dots(7.13)$$

which contains an arbitrary function of time $F(t)$.

Equation 7.13 is known as *pressure equation*. In steady flow where there is no change in the conditions with respect to time equation 7.13 reduces to

$$\Omega + \int \frac{dp}{\rho} + \frac{V^2}{2} = C \quad \dots(7.14)$$

where C is an arbitrary constant which is to be determined by the known conditions of velocity, pressure and body force potential at some point in the flow.

If the body force exerted on the flowing fluid is only due to gravity and the z -axis is so oriented that z is measured in the vertical upward direction with reference to a datum, then we have

$$-\frac{\partial \Omega}{\partial x} = 0; -\frac{\partial \Omega}{\partial y} = 0; -\frac{\partial \Omega}{\partial z} = -g$$

since the acceleration due to gravity g is force per unit mass and it is considered to be positive when acting in the vertical downward direction. By integrating the above equation, we get

$$\Omega = gz + C_1$$

where C_1 is a constant of integration. Since at $z = 0$, $\Omega = 0$, so that $C_1 = 0$ and hence $\Omega = gz$.

Thus by substituting the value of Ω in equation 7.14 it becomes

$$\int \frac{dp}{\rho} + \frac{V^2}{2} + gz = C \quad \dots(7.15)$$

Equation 7.15 is known as Bernoulli's equation which is applicable for steady irrotational flow of compressible fluids. If the flowing fluid is incompressible then since the mass density ρ is independent of pressure, equation 7.15 becomes

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = C \quad \dots(7.16)$$

Equation 7.16 is also known as Bernoulli's equation which is applicable for steady irrotational flow of incompressible fluids. Since the gravitational acceleration g is constant, by dividing both sides of equation 7.16 by g , we get

$$\frac{p}{\rho g} + \frac{V^2}{2g} + z = C'$$

or

$$\frac{p}{w} + \frac{V^2}{2g} + z = C' \quad \dots(7.17)$$

since specific weight $w = \rho g$ and C' is another constant.

Equations 7.15, 7.16 and 7.17 are the energy equations, since each term in these equations represents the energy possessed by the flowing fluid. Each term in equation 7.15 or 7.16 represents the energy per unit mass of the flowing

fluid and in equation 7.17 each term represents the energy per unit weight of the flowing fluid. The energy per unit weight of the fluid is expressed as N.m/N or kg(f) - m / kg(f) , that is, it has a dimension of length and therefore it is known as head. Thus in equation 7.17 the term $\frac{p}{w}$ is known as *pressure head* or *static head*.

$\frac{V^2}{2g}$ is known as *velocity head* or *kinetic head* and z is known as *potential head* or *datum head*. The sum of the pressure head, the velocity head and the potential head is known as the total head or the total energy per unit weight of the fluid. The Bernoulli's equation 7.17 thus states that in a steady, irrotational flow of an incompressible fluid the total energy at any point is constant. In other words, if the Bernoulli's equation is applied between any two points in a steady irrotational flow of an incompressible fluid then, we get

$$\frac{p_1}{w} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{w} + \frac{V_2^2}{2g} + z_2 \quad \dots(7.18)$$

where the different terms with subscripts 1 and 2 correspond to the two points considered.

The sum of the pressure head and the potential head i.e., $\left(\frac{p}{w} + z\right)$ is also termed as *piezometric head*. It is observed from equation 7.18 that if V_2 is greater than V_1 then the piezometric head $\left(\frac{p_2}{w} + z_2\right)$ must be less than the piezometric head $\left(\frac{p_1}{w} + z_1\right)$. However, if the two points considered lie along the same horizontal plane $z_1 = z_2$, in which case the changes in velocity cause corresponding change in the pressure.

Further, equation 7.18 has been derived for an ideal fluid which is non-viscous and hence there is no loss of energy. However, for the flow of real fluids since there is always some energy of the flowing fluid converted into heat due to the viscous and turbulent shear and consequently there is a certain amount of energy loss. Hence for the flow of real fluids, equation 7.18 may be modified as

$$\frac{p_1}{w} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{w} + \frac{V_2^2}{2g} + z_2 + h_L \quad \dots(7.19)$$

where h_L is the loss of energy (or head) between the points under consideration.

Equation 7.19 can be applied, between any two sections of a pipe or a channel or any other passage through which a real fluid is flowing, by considering the average values of the pressure intensity and the velocity of flow at each of these sections.

(b) When the flow is steady but may not be irrotational. If the flow is rotational then also the Euler's equation of motion can be integrated along a streamline provided the flow is steady. For this Euler's equation along a streamline may first be obtained as indicated below.

Consider an element of stream filament of cross-sectional area δa and length δs along a streamline as shown in Fig. 7.2. The various forces acting on the element are the body force and the pressure forces on the two ends. Let S be the component of the body force per unit mass in the direction s along the streamline. Further if ρ is the mass density of the fluid then the mass of the fluid contained in the element of the stream filament is $(\rho \delta a \delta s)$. Therefore, the total component of the body force acting on the element in the direction s is equal to $S (\rho \delta a \delta s)$.

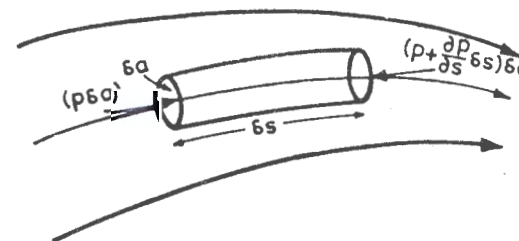


Fig 7.2. Elementary stream filament subjected to pressure and body forces

If p is the pressure intensity on the left end of the element then the total pressure force acting on this end is $(p \delta a)$.

Since the pressure p varies with the distance in the direction the pressure intensity on the right end will be $\left(p + \frac{\partial p}{\partial s} \delta s\right)$

Therefore the total pressure force on the right end of the element will be $\left(p + \frac{\partial p}{\partial s} \delta s\right) \delta a$.

The net pressure force acting on the element is therefore

$$F_{ps} = (p \delta a) - \left(p + \frac{\partial p}{\partial s} \delta s\right) \delta a$$

or

$$F_{ps} = - \frac{\partial p}{\partial s} \delta s \delta a$$

Now if V is the velocity of flowing fluid at the element in the direction tangential to the streamline then as indicated in chapter 6 for steady flow the acceleration in the direction tangential to the streamline is

$$a_s = V \frac{\partial V}{\partial s} = \frac{1}{2} \frac{\partial V^2}{\partial s}$$

Now according to the Newton's second law of motion, in the direction of the streamline, we have

$$S (\rho \delta s \delta a) - \frac{\partial p}{\partial s} \delta s \delta a = (\rho \delta s \delta a) \left(\frac{1}{2} \frac{\partial V^2}{\partial s}\right)$$

Dividing both sides by the mass of the fluid ($\rho \delta s \delta a$) in the element and taking the limit so that the element is reduced to a point, we obtain the Euler's equation of motion in the s direction at any point on the streamline in the flowing mass of fluid as

$$S - \frac{1}{\rho} \frac{\partial p}{\partial s} = \frac{1}{2} \frac{\partial V^2}{\partial s} \quad \dots(7.20)$$

The body force may be expressed in terms of the force potential as

$$S = -\frac{\partial \Omega}{\partial s}$$

then equation 7.20 becomes

$$\frac{\partial \Omega}{\partial s} + \frac{1}{\rho} \frac{\partial p}{\partial s} + \frac{1}{2} \frac{\partial V^2}{\partial s} = 0.$$

Now if it is assumed that the fluid is incompressible, then integrating the above equation with respect to s , we get

$$\Omega + \frac{p}{\rho} + \frac{V^2}{2} = C \quad \dots(7.21)$$

where C is a constant of integration, the value of which depends on the particular streamline chosen and so the constant C will in general be different for different streamlines.

However if the fluid is compressible then integrating the above equation with respect to s , we get

$$\Omega + \int \frac{\partial p}{\rho} + \frac{V^2}{2} = C \quad \dots(7.22)$$

As stated earlier if the body force exerted on the flowing fluid is due to gravity only, then we have

$$\Omega = gz$$

Thus by substituting the value of Ω in equation 7.22 it becomes

$$\int \frac{\partial p}{\rho} + \frac{V^2}{2} + gz = C \quad \dots(7.23)$$

For an incompressible fluid equation 7.23 becomes

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = C \quad \dots(7.24)$$

Equations 7.23 and 7.24 are again Bernoulli's equations for compressible and incompressible fluids respectively which are exactly same as equations 7.15 and 7.16.

Thus on the basis of the above analysis it may be concluded that if the flow is irrotational then the same Bernoulli's equation is applicable to all the points in the flow field, that is for all the streamlines the value of the constant is same. However, for the rotational flow, the Bernoulli's equation is applicable only to particular streamline that is the value of the constant is different for different streamlines.

7.5 BERNOULLI'S EQUATION DERIVED FROM THE PRINCIPLE OF CONSERVATION OF ENERGY

As stated earlier each term in the Bernoulli's equation 7.17 represents the energy possessed by the flowing fluid per unit weight of the fluid. Therefore Bernoulli's equation may be considered as an energy equation applied to flowing fluid. The general energy equation for the flow of fluids may be derived on the basis of the principle of conservation of energy and is essentially a complete accounting of the work done on the fluid and the resulting change in energies of the flowing fluid. Therefore, Bernoulli's equation may also be derived on the basis of the principle of conservation of energy as indicated below.

Consider a free body of flowing fluid occupying a portion of a stream tube between two arbitrarily chosen sections 1-1 and 2-2 at any instant of time t_1 as shown in Fig. 7.3. In a small interval of time dt the fluid moves a short distance to a new position 1'-1' and 2'-2' at another instant of time t_2 . Therefore $(t_2 - t_1)$ equals dt .

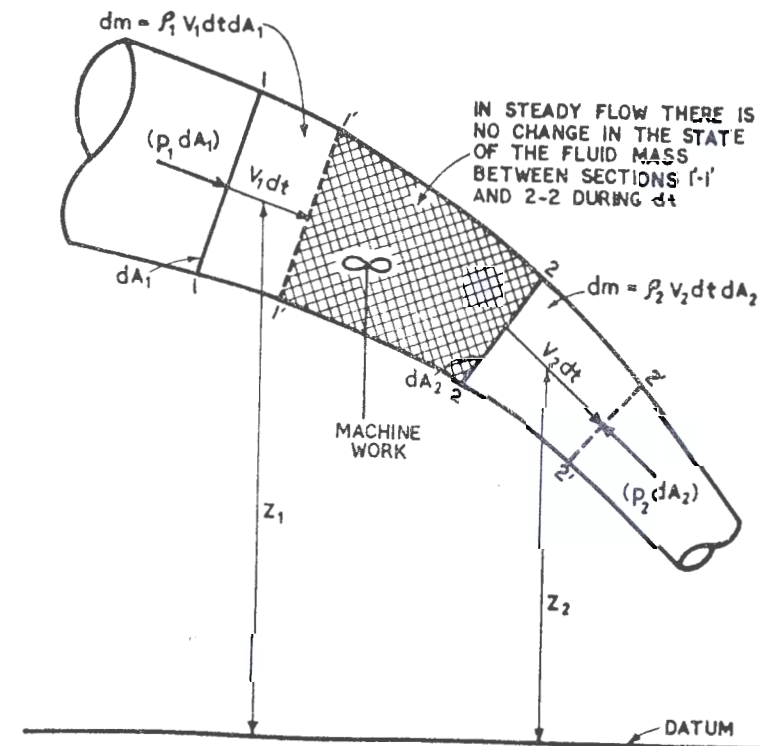


Fig. 7.3 Change of energy of flowing fluid in a stream tube

The general energy equation describing the motion of fluid in the stream tube during dt can be expressed as follows :

$$\begin{aligned}
 & \left[\begin{array}{l} \text{Work done on the} \\ \text{fluid by external} \\ \text{forces during } dt \end{array} \right] \pm \left[\begin{array}{l} \text{Mechanical} \\ \text{work performed} \\ \text{on the fluid} \end{array} \right] \\
 = & \left[\begin{array}{l} \text{Total energy of} \\ \text{fluid between} \\ \text{1'-1' and 2'-2' at } t_2 \end{array} \right] - \left[\begin{array}{l} \text{Total energy of} \\ \text{fluid between} \\ \text{1-1 and 2-2 at } t_1 \end{array} \right] \\
 \text{But} & \left[\begin{array}{l} \text{Total energy of} \\ \text{fluid between} \\ \text{1'-1' and 2'-2' at } t_2 \end{array} \right] \\
 = & \left[\begin{array}{l} \text{Total energy of} \\ \text{fluid between} \\ \text{1'-1' and 2-2 at } t_2 \end{array} \right] + \left[\begin{array}{l} \text{Total energy of} \\ \text{fluid between} \\ \text{2-2 and 2'-2' at } t_2 \end{array} \right] \\
 \text{and} & \left[\begin{array}{l} \text{Total energy of} \\ \text{fluid between} \\ \text{1-1 and 2-2 at } t_1 \end{array} \right] \\
 = & \left[\begin{array}{l} \text{Total energy of} \\ \text{fluid between} \\ \text{1-1 and 1'-1' at } t_1 \end{array} \right] + \left[\begin{array}{l} \text{Total energy of} \\ \text{fluid between} \\ \text{1'-1' and 2-2 at } t_1 \end{array} \right]
 \end{aligned}$$

For steady flow the state of the flowing fluid in the stream tube within the region bounded between sections 1' - 1' and 2 - 2 remains unchanged with respect to time. Thus

$$\left[\begin{array}{l} \text{Total energy of} \\ \text{fluid between} \\ \text{1'-1' and 2-2 at } t_2 \end{array} \right] = \left[\begin{array}{l} \text{Total energy of} \\ \text{fluid between} \\ \text{1'-1' and 2-2 at } t_1 \end{array} \right]$$

Hence the general energy equation for the steady flow of fluid is reduced to the following form :

$$\begin{aligned}
 & \left[\begin{array}{l} \text{Work done on the} \\ \text{fluid by external} \\ \text{forces during } dt \end{array} \right] \pm \left[\begin{array}{l} \text{Mechanical} \\ \text{work performed} \\ \text{on the fluid} \end{array} \right] \\
 = & \left[\begin{array}{l} \text{Total energy of} \\ \text{fluid between} \\ \text{2-2 and 2'-2' at } t_2 \end{array} \right] - \left[\begin{array}{l} \text{Total energy of} \\ \text{fluid between} \\ \text{1-1 and 1'-1' at } t_1 \end{array} \right]
 \end{aligned}$$

Further when the flow of fluid is steady, the mass flow at sections 1-1 and 2-2 during time dt being the same, the following continuity equation holds :

$$\left[\begin{array}{l} \text{Fluid mass} \\ \text{between sections} \\ \text{1-1 and 1'-1'} \end{array} \right] = \left[\begin{array}{l} \text{Fluid mass} \\ \text{between sections} \\ \text{2-2 and 2'-2'} \end{array} \right]$$

Let p_1 and p_2 be the pressure intensities and V_1 and V_2 be the velocities of flow at sections 1-1 and 2-2 of the stream tube where the cross-sectional areas

of the tube are dA_1 and dA_2 respectively. Then the pressure forces ($p_1 dA_1$) ($p_2 dA_2$) act at sections 1-1 and 2-2 respectively. During the time interval dt the pressure force ($p_1 dA_1$) acting at section 1-1 causes the liquid mass to move through a distance ($V_1 dt$). Thus the work done by the pressure force ($p_1 dA_1$) on the free body of the fluid in the stream tube during the time dt is $+(p_1 dA_1) (V_1 dt)$. The positive sign indicates that both the pressure force and the distance through which it acts are of the same direction. Similarly the work done by the pressure force ($p_2 dA_2$) at section 2-2 on the free body of flowing fluid in the stream tube is $-(p_2 dA_2) (V_2 dt)$. The work done is negative because the pressure force and the distance through which it acts are of opposite directions. Therefore the net work performed by the pressure forces on the free body of fluid in the stream tube during the interval of time dt is $[(p_1 dA_1) (V_1 dt) - (p_2 dA_2) (V_2 dt)]$. Now if dm is the total mass of fluid flowing across section 1-1 during the time interval dt , or dm is actually the mass of fluid in the stream tube within the region between sections 1-1 and 1'-1', or between sections 2-2 and 2'-2', as shown in Fig. 7'3, and if ρ_1 and ρ_2 are the mass densities of flowing fluid at sections 1-1 and 2-2, then

$$(V_1 dA_1) dt = \frac{dm}{\rho_1} ; \text{ and } (V_2 dA_2) dt = \frac{dm}{\rho_2}$$

By substituting these values the net work done by the pressure forces on the free body of the fluid in the stream tube becomes $[(p_1/\rho_1) - (p_2/\rho_2)] dm$. The pressure force on the boundaries of the stream tube from the adjacent fluids is everywhere normal to the surface and therefore the work done by them is zero.

In between sections, 1-1 and 2-2 there might be some mechanical device such as a pump which performs work on the fluid and puts mechanical energy into it, or there might be a machine such as a turbine on which the fluid performs the work and hence some of the energy possessed by the flowing fluid is utilized. If h_m represents the work done on the fluid by the machine or work done by the fluid on the machine, per unit weight of the flowing fluid then the total amount of work performed on the fluid or by the fluid may be denoted by $(+h_m g dm)$ or $(-h_m g dm)$ respectively.

As shown in Fig. 7'3, with reference to an arbitrarily chosen datum, the gravitational potential energies of the fluid in the stream tube in the region bounded by sections 1-1 and 1'-1' and 2-2 and 2'-2' are $(z_1 g dm)$ and $(z_2 g dm)$ respectively, where z_1 and z_2 are the elevations in metres of these two regions above the chosen datum. The change in gravitational potential energy of the flowing fluid in the stream tube during time dt is, therefore, $(z_2 - z_1) g dm$.

The kinetic energies of the fluid in these two regions are respectively $(V_1^2/2) dm$ and $(V_2^2/2) dm$. Therefore the change in kinetic energy of the flowing fluid in the stream tube during dt is $dm (V_2^2 - V_1^2)/2$.

Thus for the steady flow of the fluid in the stream tube of Fig. 7'3, the general energy equation may be expressed in the mathematical form as follows

$$\begin{aligned}
 & \left[\begin{array}{l} \text{Work done on the} \\ \text{fluid by external} \\ \text{forces during } dt \end{array} \right] \pm \left[\begin{array}{l} \text{Mechanical} \\ \text{work performed} \\ \text{on the fluid} \end{array} \right] \\
 = & \left[\begin{array}{l} \text{Total energy of} \\ \text{fluid between} \\ \text{1'-1' and 2'-2' at } t_2 \end{array} \right] - \left[\begin{array}{l} \text{Total energy of} \\ \text{fluid between} \\ \text{1-1 and 2-2 at } t_1 \end{array} \right] \\
 \text{But} & \left[\begin{array}{l} \text{Total energy of} \\ \text{fluid between} \\ \text{1'-1' and 2'-2' at } t_2 \end{array} \right] \\
 = & \left[\begin{array}{l} \text{Total energy of} \\ \text{fluid between} \\ \text{1'-1' and 2-2 at } t_2 \end{array} \right] + \left[\begin{array}{l} \text{Total energy of} \\ \text{fluid between} \\ \text{2-2 and 2'-2' at } t_2 \end{array} \right] \\
 \text{and} & \left[\begin{array}{l} \text{Total energy of} \\ \text{fluid between} \\ \text{1-1 and 2-2 at } t_1 \end{array} \right] \\
 = & \left[\begin{array}{l} \text{Total energy of} \\ \text{fluid between} \\ \text{1-1 and 1'-1' at } t_1 \end{array} \right] + \left[\begin{array}{l} \text{Total energy of} \\ \text{fluid between} \\ \text{1'-1' and 2-2 at } t_1 \end{array} \right]
 \end{aligned}$$

For steady flow the state of the flowing fluid in the stream tube within the region bounded between sections 1' - 1' and 2 - 2 remains unchanged with respect to time. Thus

$$\left[\begin{array}{l} \text{Total energy of} \\ \text{fluid between} \\ \text{1'-1' and 2-2 at } t_2 \end{array} \right] = \left[\begin{array}{l} \text{Total energy of} \\ \text{fluid between} \\ \text{1'-1' and 2-2 at } t_1 \end{array} \right]$$

Hence the general energy equation for the steady flow of fluid is reduced to the following form :

$$\begin{aligned}
 & \left[\begin{array}{l} \text{Work done on the} \\ \text{fluid by external} \\ \text{forces during } dt \end{array} \right] \pm \left[\begin{array}{l} \text{Mechanical} \\ \text{work performed} \\ \text{on the fluid} \end{array} \right] \\
 = & \left[\begin{array}{l} \text{Total energy of} \\ \text{fluid between} \\ \text{2-2 and 2'-2' at } t_2 \end{array} \right] - \left[\begin{array}{l} \text{Total energy of} \\ \text{fluid between} \\ \text{1-1 and 1'-1' at } t_1 \end{array} \right]
 \end{aligned}$$

Further when the flow of fluid is steady, the mass flow at sections 1-1 and 2-2 during time dt being the same, the following continuity equation holds :

$$\left[\begin{array}{l} \text{Fluid mass} \\ \text{between sections} \\ \text{1-1 and 1'-1'} \end{array} \right] = \left[\begin{array}{l} \text{Fluid mass} \\ \text{between sections} \\ \text{2-2 and 2'-2'} \end{array} \right]$$

Le: p_1 and p_2 be the pressure intensities and V_1 and V_2 be the velocities of flow at sections 1-1 and 2-2 of the stream tube where the cross-sectional areas

of the tube are dA_1 and dA_2 respectively. Then the pressure forces ($p_1 dA_1$) and ($p_2 dA_2$) act at sections 1-1 and 2-2 respectively. During the time interval dt the pressure force ($p_1 dA_1$) acting at section causes the liquid mass to move through a distance ($V_1 dt$). Thus the work done by the pressure force ($p_1 dA_1$) on the free body of the fluid in the stream tube during the time dt is $+(p_1 dA_1)(V_1 dt)$. The positive sign indicates that both the pressure force and the distance through which it acts are of the same direction. Similarly the work done by the pressure force ($p_2 dA_2$) at section 2-2 on the free body of flowing fluid in the stream tube is $-(p_2 dA_2)(V_2 dt)$. The work done is negative because the pressure force ($p_2 dA_2$) and the distance ($V_2 dt$) through which it acts are of opposite direction. Therefore the net work performed by the pressure forces on the free body of the fluid in the stream tube during the interval of time dt is $[(p_1 dA_1)(V_1 dt) - (p_2 dA_2)(V_2 dt)]$. Now if dm is the total mass of fluid flowing across section 1-1 during the time interval dt , or dm is actually the mass of fluid in the stream tube within the region between sections 1-1 and 1'-1', or between sections 2-2 and 2'-2', as shown in Fig. 7'3, and if ρ_1 and ρ_2 are the mass densities of the flowing fluid at sections 1-1 and 2-2, then

$$(V_1 dA_1) dt = \frac{dm}{\rho_1} ; \text{ and } (V_2 dA_2) dt = \frac{dm}{\rho_2}$$

By substituting these values the net work done by the pressure forces on the free body of the fluid in the stream tube becomes $[(p_1/\rho_1) - (p_2/\rho_2)]dm$. The pressure force on the boundaries of the stream tube from the adjacent fluids are everywhere normal to the surface and therefore the work done by them is zero.

In between sections, 1-1 and 2-2 there might be some mechanical device such as a pump which performs work on the fluid and puts mechanical energy into it, or there might be a machine such as a turbine on which the fluid performs the work and hence some of the energy possessed by the flowing fluid is utilized. If h_m represents the work done on the fluid by the machine or the work done by the fluid on the machine, per unit weight of the flowing fluid then the total amount of work performed on the fluid or by the fluid may be denoted by $(+h_m g dm)$ or $(-h_m g dm)$ respectively.

As shown in Fig. 7'3, with reference to an arbitrarily chosen datum, the gravitational potential energies of the fluid in the stream tube in the regions bounded by sections 1-1 and 1'-1' and 2-2 and 2'-2' are $(z_1 g dm)$ and $(z_2 g dm)$ respectively, where z_1 and z_2 are the elevations in metres of these two regions above the chosen datum. The change in gravitational potential energy of the flowing fluid in the stream tube during time dt is, therefore, $(z_2 - z_1) g dm$.

The kinetic energies of the fluid in these two regions are respectively $(V_1^2/2)dm$ and $(V_2^2/2)dm$. Therefore the change in kinetic energy of the flowing fluid in the stream tube during dt is $dm (V_2^2 - V_1^2)/2$.

Thus for the steady flow of the fluid in the stream tube of Fig. 7'3, the general energy equation may be expressed in the mathematical form as follows :

$$\left(\frac{p_1}{\rho_1} - \frac{p_2}{\rho_2} \right) dm \pm h_m g dm = (z_2 - z_1) g dm + \frac{dm}{2} (V_2^2 - V_1^2) \quad \dots(7.25)$$

Dividing both sides by the common factor ($g dm$) and rearranging the various terms, the general energy equation 7.25 takes the following form :

$$\frac{p_1}{\rho_1 g} + \frac{V_1^2}{2g} + z_1 \pm h_m = \frac{p_2}{\rho_2 g} + \frac{V_2^2}{2g} + z_2 \quad \dots(7.26)$$

Equation 7.26 is valid for the flow of both compressible and incompressible fluids. However, if the flowing fluid is incompressible then $\rho_1 = \rho_2 = \rho = \text{constant}$, then equation 7.26 becomes

$$\frac{p_1}{w} + \frac{V_1^2}{2g} + z_1 \pm h_m = \frac{p_2}{w} + \frac{V_2^2}{2g} + z_2 \quad \dots(7.27)$$

since specific weight $w = \rho g$.

Now if there is no mechanical device in between the sections 1-1 and 2-2, then neither any work is done on the fluid nor any work is done by the fluid and hence $h_m = 0$. Equation 7.27, then becomes

$$\frac{p_1}{w} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{w} + \frac{V_2^2}{2g} + z_2 \quad \dots(7.28)$$

Since both the sections 1-1 and 2-2 are chosen arbitrarily, it follows that for any section

$$\frac{p_1}{w} + \frac{V^2}{2g} + z = \text{constant} \quad \dots(7.29)$$

which is Bernoulli's equation for the steady flow of an incompressible non-viscous fluid in a stream tube, as derived earlier. However, for the steady flow of an incompressible real fluid there is certain loss of useful energy caused by the viscous and turbulent friction. Thus if h_L represents the loss of energy per unit weight of fluid between the sections 1-1 and 2-2, equation 7.28 may be modified as

$$\frac{p_1}{w} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{w} + \frac{V_2^2}{2g} + z_2 + h_L \quad \dots(7.30)$$

which states that in steady flow of real fluid the total head (or total energy per N or per kg (f) of flowing fluid) at any section is equal to that at any subsequent section, plus the loss of head (or loss of energy per N or per kg (f) of flowing fluid) occurring between the two sections.

7.6 KINETIC ENERGY CORRECTION FACTOR

In the Bernoulli's equation derived above the velocity head or the kinetic energy per unit weight of the fluid has been computed on the basis of the assumption that the velocity is uniform over the entire cross-section of the stream tube. But in the case of flow of real fluids the velocity distribution across any cross-sectional area of the flow passage is **not** uniform. Therefore along any cross-section the velocities of flow will be different at different

points and the total kinetic energy possessed by flowing fluid at any section will be obtained by integrating the kinetic energies possessed by different fluid particles. If v is the velocity of flowing fluid at any point through any elementary area ' dA ' of the cross-section, then the mass of fluid flowing per unit time will be $(\rho v dA)$ and the kinetic energy of this mass of fluid will be $(\rho v dA) \frac{v^2}{2}$. The total kinetic energy possessed by the flowing fluid across the entire cross-section A is

$$\int_A \rho \frac{v^3}{2} dA = \frac{w}{2g} \int_A v^3 dA \quad \dots(7.31)$$

By knowing the velocity distribution across any cross-section the actual kinetic energy possessed by the flowing fluid can be determined by using equation 7.31.

It is however more convenient to express the kinetic energy of the flowing fluid in terms of the mean velocity of flow. But the actual kinetic energy possessed by the flowing fluid is greater than that computed by using the mean velocity. Hence a factor called *kinetic energy correction factor* represented by α (Greek 'alpha') is introduced; so that the kinetic energy computed by using the mean velocity V may be expressed as $\left(\alpha \frac{w}{2g} A V^3 \right)$ and it is equal to the actual total kinetic energy possessed by the flowing fluid. Thus equating the two, the value of the kinetic energy correction factor α may be obtained as

$$\alpha \frac{w}{2g} A V^3 = \frac{w}{2g} \int_A v^3 dA$$

$$\text{Therefore } \alpha = \frac{1}{A V^3} \int_A v^3 dA \quad \dots(7.32)$$

Mathematically, the cube of the average is less than the average of the cubes, that is $V^3 < \frac{1}{A} \int_A v^3 dA$; the numerical value of α will always be greater

than 1. The actual value of α depends on the velocity distribution at the flow section. The value of α for turbulent flow in pipes lies between 1.03 to 1.06, which is very close to 1, because in turbulent flow the velocity distribution is very close to uniform velocity distribution. However, for laminar flow in pipes the value of α is 2.

In the application of the Bernoulli's equation between any two sections if the velocity distribution is non-uniform then the kinetic energy correction factors will be required to be introduced if the kinetic energy is expressed in terms of the mean velocity at each section. Thus equation 7.30 is modified as

$$\frac{p_1}{w} + \alpha_1 \frac{V_1^2}{2g} + z_1 = \frac{p_2}{w} + \alpha_2 \frac{V_2^2}{2g} + z_2 + h_L \quad \dots(7'33)$$

in which α_1 and α_2 are the energy correction factors at sections 1 and 2 respectively. However, in most of the problems of turbulent flow the value of α is nearly equal to 1, and therefore it may be assumed as one without any appreciable error being introduced.

7.7 BERNOULLI'S EQUATION FOR A COMPRESSIBLE FLUID

As indicated earlier for a steady irrotational flow of a compressible fluid, Bernoulli's equation is expressed by equation 7'15 as

$$\int \frac{dp}{\rho} + \frac{V^2}{2} + gz = C$$

in which it is assumed that the kinetic energy correction factor α is equal to unity. The flow of a compressible fluid may take place with either isothermal or adiabatic changes of volume, and therefore the pressure term in the above equation may be integrated by using the perfect gas laws as indicated below.

(a) **Isothermal change.** For isothermal change, we have

$$\frac{p}{\rho} = K; \text{ or } \frac{p}{\rho g} = K'$$

where K and K' are constants.

By substituting the value of p in the above equation and integrating the pressure term, we get

$$K \log_e p + \frac{V^2}{2} + gz = C \quad \dots(7'34 a)$$

$$\text{or } gK' \log_e p + \frac{V^2}{2} + gz = C \quad \dots(7'34 b)$$

Applying the above equation between any two sections 1-1 and 2-2, we get

$$K \log_e p_1 + \frac{V_1^2}{2} + gz_1 = K \log_e p_2 + \frac{V_2^2}{2} + gz_2 \quad \dots(7'35 a)$$

$$\text{or } gK' \log_e p_1 + \frac{V_1^2}{2} + gz_1 = gK' \log_e p_2 + \frac{V_2^2}{2} + gz_2 \quad \dots(7'35 b)$$

Equation 7'35 may also be expressed as

$$K \log_e (p_1/p_2) = \frac{V_2^2}{2} - \frac{V_1^2}{2} + g(z_2 - z_1) \quad \dots(7'35 c)$$

$$\text{or } K' \log_e (p_1/p_2) = \frac{V_2^2}{2g} - \frac{V_1^2}{2g} + (z_2 - z_1) \quad \dots(7'35 d)$$

For a gas, the potential head terms z_1 and z_2 are usually small as compared with the pressure head terms and hence the same may be neglected. Equation 7'35 then simplifies to

$$K \log_e (p_1/p_2) = \frac{V_2^2}{2} - \frac{V_1^2}{2} \quad \dots(7'36 a)$$

$$\text{or } K' \log_e (p_1/p_2) = \frac{V_2^2}{2g} - \frac{V_1^2}{2g} \quad \dots(7'36 b)$$

Equation 7'35 holds good for a frictionless isothermal flow of compressible fluid for which there is no loss of energy as the fluid flows. However, if there is some loss of energy between the sections 1-1 and 2-2, represented by head loss h_L , then equation 7'35 may be modified as

$$K \log_e p_1 + \frac{V_1^2}{2} + gz_1 = K \log_e p_2 + \frac{V_2^2}{2} + gz_2 + gh_L \quad \dots(7'37 a)$$

$$\text{or } K' \log_e p_1 + \frac{V_1^2}{2g} + z_1 = K' \log_e p_2 + \frac{V_2^2}{2g} + z_2 + h_L \quad \dots(7'37 b)$$

(b) **Adiabatic change.** For adiabatic change, we have

$$\frac{p}{\rho^k} = C_1; \text{ or } \frac{p}{(\rho g)^k} = C_1'$$

where C_1 and C_1' are constants and k is adiabatic exponent or adiabatic constant.

Differentiating the above equation, we get

$$\frac{dp}{d\rho} = C_1 k \rho^{k-1}; \text{ or } \frac{dp}{\rho} = C_1 k \rho^{k-2} d\rho$$

$$\text{and } \frac{dp}{d\rho} = C_1' g^k k \rho^{k-1}; \text{ or } \frac{dp}{\rho} = C_1' g^k k \rho^{k-2} d\rho$$

By substituting the value of (dp/ρ) in equation 7'15 and integrating the pressure term, we get

$$C_1 k \frac{\rho^{k-1}}{(k-1)} + \frac{V^2}{2} + gz = C$$

$$\text{or } C_1' g^k k \frac{\rho^{k-1}}{(k-1)} + \frac{V^2}{2} + gz = C$$

which may be expressed as

$$\frac{k}{k-1} \frac{p}{\rho} + \frac{V^2}{2} + gz = C \quad \dots(7'38 a)$$

$$\text{or } \frac{k}{k-1} \frac{p}{w} + \frac{V^2}{2g} + z = C' \quad \dots(7'38 b)$$

Applying the above equation between any two sections 1-1 and 2-2, we get

$$\frac{k}{k-1} \frac{p_1}{\rho_1} + \frac{V_1^2}{2} + gz_1 = \frac{k}{k-1} \frac{p_2}{\rho_2} + \frac{V_2^2}{2} + gz_2 \quad \dots(7'39 a)$$

$$\text{and } \frac{k}{k-1} \frac{p_1}{w_1} + \frac{V_1^2}{2g} + z_1 = \frac{k}{k-1} \frac{p_2}{w_2} + \frac{V_2^2}{2g} + z_2 \quad \dots(7'39 b)$$

Again for a gas the potential head terms z_1 and z_2 are usually small as compared with the pressure head terms and the same may be neglected so that equation 7.39 simplifies to

$$\frac{k}{k-1} \frac{p_1}{\rho_1} + \frac{V_1^2}{2} = \frac{k}{k-1} \frac{p_2}{\rho_2} + \frac{V_2^2}{2} \quad \dots(7.40 a)$$

or
$$\frac{k}{k-1} \frac{p_1}{w_1} + \frac{V_1^2}{2g} = \frac{k}{k-1} \frac{p_2}{w_2} + \frac{V_2^2}{2g} \quad \dots(7.40 b)$$

Equation 7.39 is applicable for a frictionless adiabatic flow of compressible fluid for which there is no loss of energy as the fluid flows. However, if there is some loss of energy between the sections 1-1 and 2-2, represented by head loss h_L , then equation 7.39 may be modified as

$$\frac{k}{k-1} \frac{p_1}{\rho_1} + \frac{V_1^2}{2} + gz_1 = \frac{k}{k-1} \frac{p_2}{\rho_2} + \frac{V_2^2}{2} + gz_2 + gh_L \quad \dots(7.41 a)$$

or
$$\frac{k}{k-1} \frac{p_1}{w_1} + \frac{V_1^2}{2g} + z_1 = \frac{k}{k-1} \frac{p_2}{w_2} + \frac{V_2^2}{2g} + z_2 + h_L \quad \dots(7.41 b)$$

7.8 PRESSURE VELOCITY RELATIONSHIP

If the two sections in a flowing fluid are lying at the same height above the assumed datum then $z_1 = z_2$ and the Bernoulli's equation applied between these sections reduces to

$$\frac{p_1}{w} + \frac{V_1^2}{2g} = \frac{p_2}{w} + \frac{V_2^2}{2g} + h_L \quad \dots(7.42)$$

which gives the pressure velocity relationship. Equation 7.42 is also applicable to the problems of fluid flow in which the effect of the fluid weight may be neglected. These include the problems of gas flow and the flow of liquid in horizontal pipes. If the energy loss between the two sections can be neglected then equation 7.42 becomes

$$\frac{p_1}{w} + \frac{V_1^2}{2g} = \frac{p_2}{w} + \frac{V_2^2}{2g} \quad \dots(7.43)$$

It is indicated from equation 7.43 that if the flow passage is such that V_2 the velocity of flow at section 2 is greater than V_1 the velocity of flow at section 1, then p_1 the pressure at section 1 will be greater than p_2 the pressure at section 2 so that equation 7.43 will be satisfied. In actual practice also it is observed that if at any section of flow passage due to the considerations of continuity, the velocity of flow increases then the pressure at this section reduces and vice versa.

7.9 APPLICATIONS OF BERNOULLI'S EQUATION

Bernoulli's equation finds wide application in the solution of many problems of fluid flow. The other equation which is commonly used in the solution of the problems of fluid flow is the continuity equation. Both these equations combinedly form a very useful tool in the solution of many problems

of fluid flow. Some of the simple applications of Bernoulli's equation are discussed in the following sections.

7.10 VENTURI METER

A venturi meter is a device which is used for measuring the rate of flow of fluid through a pipe. The principle of the venturi meter was first demonstrated in 1797 by Italian physicist G.B. Venturi (1746-1822), but the principle was first applied, by C. Herschel (1842-1930) in 1887, to develop the device in its present form for measuring the discharge or the rate of flow of fluid through pipes. The basic principle on which a venturi meter works is that by reducing the cross-sectional area of the flow passage, a pressure difference is created and the measurement of the pressure difference enables the determination of the discharge through the pipe.

As shown in Fig. 7.4 a venturi meter consists of (1) an *inlet section* followed by a *convergent cone*, (2) a cylindrical *throat*, and (3) a gradually *divergent cone*. The inlet section of the venturi meter is of the same diameter as that of the pipe which is followed by a convergent cone. The convergent cone is a short pipe which tapers from the original size of the pipe to that of the throat of the venturi meter. The throat of the venturi meter is a short parallel-sided tube having its cross-sectional area smaller than that of the pipe. The divergent cone of the venturi meter is a gradually diverging pipe with its cross-sectional area increasing from that of the throat to the original size of the pipe. At the inlet section and the throat, i.e., sections 1 and 2 of the venturi meter, pressure taps are provided through pressure rings as shown in Fig. 7.4.

The convergent cone of a venturi meter has a total included angle of $21^\circ \pm 1^\circ$ and its length parallel to the axis is approximately equal to $2.7(D-d)$, where D is the diameter of the inlet section and d is the diameter of the throat.

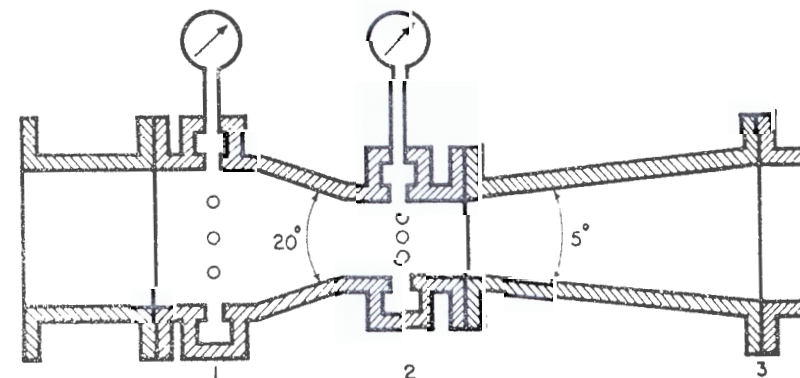


Fig. 7.4 Venturi meter

The length of the throat is equal to d . The divergent cone has a total included angle lying between 5° to 15° , (preferably about 6°). This results in the

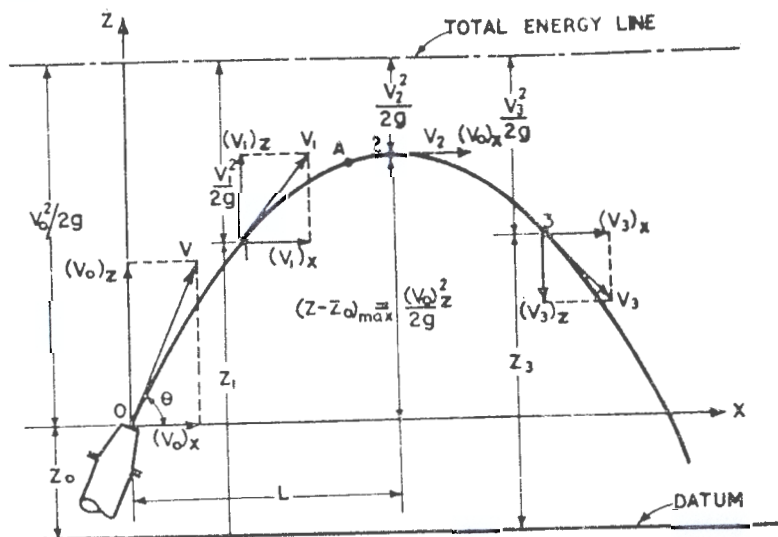


Fig. 7.14 Free liquid jet

Further if V_1 , V_2 and V_3 are the velocities at points 1, 2 and 3 along the jet, which are at heights z_1 , z_2 and z_3 respectively above the assumed datum, then according to Bernoulli's equation for no losses, we get

$$\frac{V_0^2}{2g} + z_0 = \frac{V_1^2}{2g} + z_1 = \frac{V_2^2}{2g} + z_2 = \frac{V_3^2}{2g} + z_3$$

The velocities V_1, V_2 and V_3 etc., can also be resolved along x and z directions. At the topmost point 2 of the trajectory the velocity V_2 will be acting parallel to the x direction and therefore its components in the z direction will be equal to zero. Moreover, there is no acceleration in x direction, the velocity component in the x direction is same at all the points along the jet, that is,

$$(V_0)_x = (V_1)_x = V_2 = (V_3)_x \text{ and so on.}$$

Further it is observed that from point 0 to 2, the z component of the velocity decreases, at point 2 it is equal to zero and from point 2 to 3 it increases.

The basic equations of projectile motion may be used to determine the velocity components at any point along the trajectory. Thus if V is the velocity and V_x and V_z are its components in the x and z directions respectively at any point A on the trajectory which is at a height z above the assumed datum, then we have

$$V_x = (V_0)_x \quad \dots(i)$$

$$V_j = (V_0)_j - g^l \quad \dots(ii)$$

and

Further the coordinates of the trajectory are expressed as follows :

$$x = (V_0)_x I \quad \dots(iii)$$

$$(z - z_0) = (V_0)_z t - \frac{1}{2} g t^2 \quad \dots(iv)$$

where t is the time elapsed after the liquid jet leaves the nozzle. By combining equations (ii) and (iv), we get

$$V_z^2 = (V_0)^2 - 2g(z - z_0) \quad \dots(v)$$

For point 2 since z component of velocity is zero, from equation (v) the maximum vertical elevation of the jet profile is obtained as

$$(z - z_0)_{\max} = \frac{(V_0)^2}{2g} \quad \dots(7.53)$$

By eliminating 't' between equations (iii) and (iv) and combining the resulting equation with equation 7.53 it can be shown that the horizontal distance of point 2 from point 0 is obtained as

$$L = \frac{(V_0)_x (V_0)_z}{g} = \frac{V_0^2 \sin 2\theta}{2g} \quad \dots(7.54)$$

As can be seen from equation 7.54 the distance L will be maximum for the given velocity when angle θ is equal to 45° .

In Fig. 7-14 the total energy (or head) line is also shown which is horizontal since there are no losses. Obviously the total energy line is at a distance of $\frac{V_2^2}{2g}$ [which is also equal to $\frac{(V_0)^2}{2g}$, $\frac{(V_1)^2}{2g}$ and $\frac{(V_3)^2}{2g}$] above point 2.

7.16) VORTEX MOTION

Bernoulli's equation may also be applied to the problems of vortex motion. A rotating mass of fluid is known as *vortex* and the motion of rotating mass of fluid is known as *vortex motion*. The vortex motion is of two types viz., free vortex motion and forced vortex motion.

A *free vortex motion* is that in which the fluid mass rotates without any external force being impressed on it. In this motion the whole of the moving mass of fluid rotates either by virtue of some internal action or due to rotation previously imparted to it. Hence in a free vortex motion no expenditure of energy from any external source takes place. Some of the examples of a free vortex motion are : a whirlpool in a river ; the flow of a liquid drained through an outlet provided in the bottom of a shallow vessel such as a wash basin or a bath tub draining water through an outlet at a bottom ; flow around a circular bend in a pipe system ; flow of liquid in a centrifugal pump casing after it has left the impeller ; flow of water in a turbine casing before it enters the guide vanes, etc.

A forced vortex motion is that in which the fluid mass is made to rotate by means of some external source of power, which exerts a constant torque on the fluid mass, thereby causing the whole mass of fluid to rotate at constant angular velocity ω . As such in a forced vortex motion there is always a constant external torque required to be applied to the fluid mass resulting in an

pendent of energy. A most common example of a forced vortex motion is that of a vertical cylinder containing liquid rotated about its central axis with a constant angular velocity. Some of the other examples of forced vortex motion are the flow of liquid inside the impeller of a centrifugal pump; flow of water in the runner of a turbine etc.

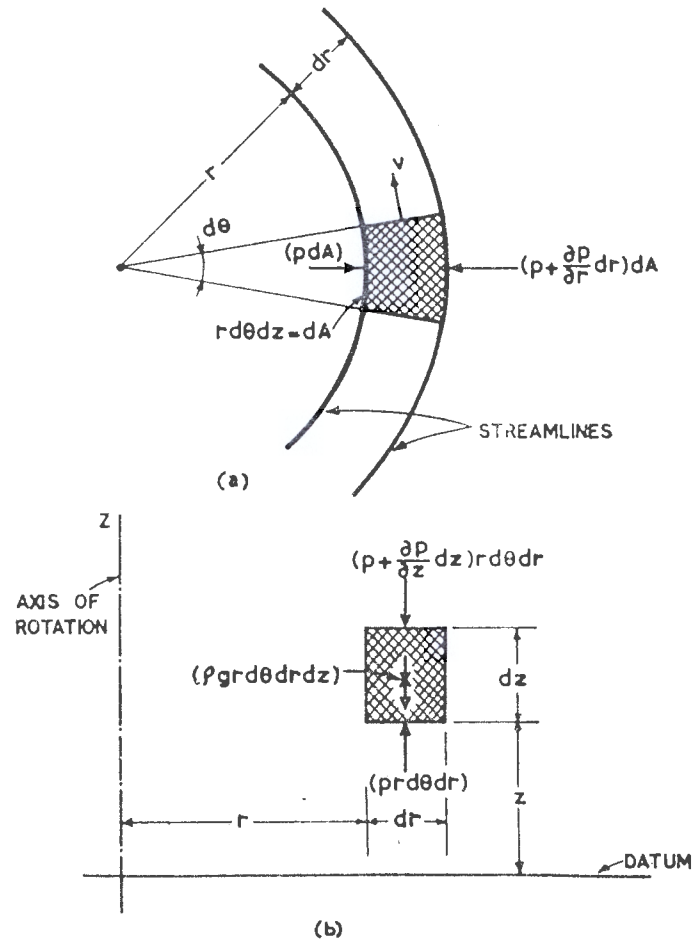


Fig. 7.15 Vortex motion : (a) plan view of a vortex motion
(b) elevation of a vortex motion

A vortex motion may also be characterised as cylindrical vortex motion and spiral vortex motion. A *cylindrical vortex motion*, is that in which the fluid mass rotates in concentric circles i.e., this motion may be described by concentric circular streamlines. A *spiral vortex motion* is a combination of cylindrical vortex motion and radial flow that is, when a cylindrical vortex motion is superimposed over the radial flow, then the resulting vortex motion

is known as spiral vortex motion. All these types of vortex motions can exist independent of each other so that any of the four types of the combination of vortex motions is possible, viz., (a) Cylindrical free vortex ; (b) Cylindrical forced vortex ; (c) Spiral free vortex ; and (d) Spiral forced vortex. In the following paragraphs different types of vortex motions have been analysed.

Fundamental differential equations representing the variation of pressure in a cylindrical free and forced vortex motion may first be developed. In a cylindrical vortex motion the fluid particles rotate along circular path at a constant speed. When the fluid particles move at a constant speed along a curved path, the pressure within the fluid mass on a horizontal plane varies in the radial direction only. This variation in the fluid pressure may be determined by considering the dynamic equilibrium of an element of fluid mass within a stream tube moving in a curved path on a horizontal plane as shown in Fig. 7.15. The fluid element of mass $(\rho dA dr)$ is located at a distance r from the centre of its curved path where its tangential velocity is v . Since the magnitude of the tangential velocity is constant, there exists no variation of pressure in the tangential direction. However, on the two radial faces there exists a difference in the pressure forces $(p dA)$ and $\left[\left(p + \frac{\partial p}{\partial r} dr \right) dA \right]$ due to the centrifugal force on the fluid element. Therefore

$$\left(p + \frac{\partial p}{\partial r} dr \right) dA - p dA = \rho dA dr \frac{v^2}{r}$$

$$\text{or} \quad \frac{\partial p}{\partial r} = \rho \frac{v^2}{r} \quad \dots (7.55)$$

If there is no acceleration (other than gravity) in the vertical direction then considering the equilibrium of fluid element in the vertical direction we have from Fig. 7.15 (b)

$$p r d\theta dr - \left(p + \frac{\partial p}{\partial z} dz \right) r d\theta dr = \rho g r d\theta dr dz$$

$$\text{or} \quad \frac{\partial p}{\partial z} = -\rho g \quad \dots (7.56)$$

Equations 7.55 and 7.56 are the fundamental relationships which may be used for studying the flow characteristics in vortex motion. It may be noted that equations 7.55 and 7.56 are same as equations 5.17 and 5.18 derived in Chapter 5.

(a) **Free vortex motion.** As stated earlier in free vortex motion no external torque is required to be exerted on the fluid mass. Therefore the rate of change of angular momentum of the flow must be zero. Thus again referring to the fluid element in Fig. 7.15, if m is the mass of the fluid element then following equation of angular momentum of flow may be obtained,

$$\frac{\partial (mvr)}{\partial t} = 0$$

which upon integrating, becomes

$$vr = \text{constant} = C \quad \dots(7.57)$$

It is thus seen from equation 7.57 that the velocity of flow in a free vortex motion varies inversely with the radial distance from the centre of vortex motion; that is when $r \rightarrow \infty$, $v \rightarrow 0$, and for $r \rightarrow 0$, $v \rightarrow \infty$. The point where the velocity v becomes infinite is called *singular point*. This condition for the velocity of flow to approach infinitely large magnitude at the centre of a free vortex motion is however not developed in actual practice as explained below.

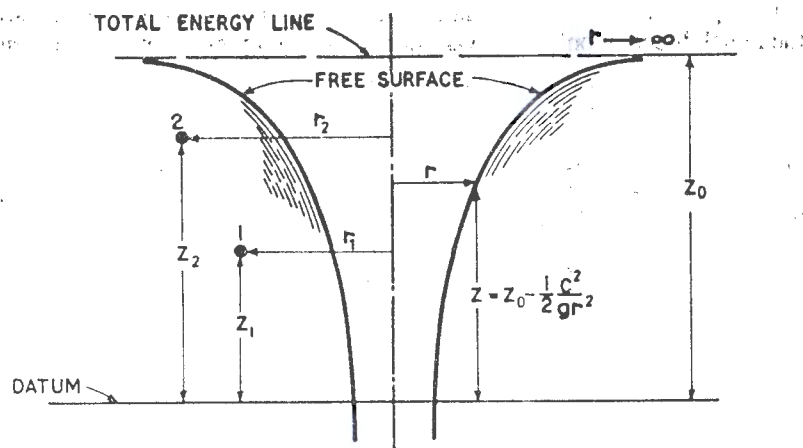


Fig. 7.16 Free vortex motion

For a free vortex motion the circulation around any streamline may be calculated as $\Gamma = (2\pi r)v = 2\pi C$, which is independent of r . Hence the circulation around all the streamlines of a free vortex motion is constant. Further it may be shown that the flow-field of a free vortex motion is everywhere irrotational except at the axis and therefore the free vortex motion is also called irrotational vortex motion. It is thus observed that for a free vortex motion although the circulation around the various streamlines is not equal to zero yet the motion is irrotational. This paradoxical situation may however be made clear as follows. If the axis which is a singular point in this case is separated from the rest of the flow field by enclosing it within an infinitesimal area surrounded by a streamline considered at an infinitesimal distance from the axis then since the circulation around the various streamlines is constant, it may be readily shown that the net circulation for the entire flow-field outside the infinitesimal area is equal to zero which thus satisfies the condition for an irrotational motion. Within the infinitesimal area close to the axis the flow is rotational. The foregoing analysis of a free vortex motion is valid only for ideal fluids. In the case of a real fluid, in the central region of a free vortex motion the viscous effects become quite predominant, due to which a central core of fluid with rotational motion is developed in which the fluid tends to rotate as a solid body with velocity proportional to the radius as in the case of a forced

vortex as indicated later. Farther away from the central region where the viscous effects are negligible irrotational motion is developed. Equation 7.57 then applies only to the region farther away from the central region of a free vortex motion. As such in the case of real fluids a combination of free and forced vortex motion frequently occurs which is known as *Rankine vortex motion*. The motion of air mass in a tornado is found to approximate this type of combined vortex motion.

The irrotationality of a free-vortex motion may also be visualized by developing a free vortex motion with a free surface on which small particles of some light-weight material capable of floating are placed. It will be seen that as these particles move with the fluid they do not rotate about their axes, which is indicated by a line marked on the particle remaining parallel to itself during the different positions of the particle.

By introducing equation 7.57 in equation 7.55, the radial pressure gradient for a free vortex motion becomes

$$\frac{dp}{dr} = \rho \frac{C^2}{r^3} \quad \dots(7.58)$$

The distribution of the fluid pressure in a free vortex motion is obtained by integrating both equations 7.56 and 7.58 between any two points 1 and 2 in the fluid (Fig. 7.16) and their results combined in the following form:

$$\frac{p_2 - p_1}{\rho} = \frac{1}{2} C^2 \left(\frac{1}{r_1^2} - \frac{1}{r_2^2} \right) + g(z_1 - z_2) \quad \dots(7.59)$$

Again from equation 7.57, $vr = v_1 r_1 = v_2 r_2 = C$, the above expression can be rearranged and written as

$$\frac{p_1}{\rho} + \frac{v_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \frac{v_2^2}{2} + g z_2$$

which indicates that Bernoulli's equation for the flow of an ideal fluid also applies to the free vortex motion. Since the points 1 and 2 have been chosen arbitrarily, this equation thus provides an additional proof of the irrotationality of a free vortex motion.

The profile of the free surface of a free vortex motion is obtained by considering $p_1 = p_2 = p_a = 0$ in equation 7.59. Thus

$$(z_2 - z_1) = \frac{1}{2} \frac{C^2}{g} \left(\frac{1}{r_1^2} - \frac{1}{r_2^2} \right) \quad \dots(7.60)$$

$$\text{or} \quad z = z_0 - \frac{1}{2} \frac{C^2}{g r^2} \quad \dots(7.61)$$

as shown in Fig. 7.16; here p_a denotes atmospheric pressure.

(b) **Forced vortex motion.** As stated earlier forced vortex motion occurs when a constant torque is applied to the fluid mass. This type of motion may be developed if a cylinder containing some liquid is rotated about its vertical axis at a constant angular velocity ω . The liquid inside the cylinder is forced to rotate with the cylinder. In this case also [which has already been dealt

with in Chapter 5 section 5.5] the streamlines are concentric circles and the velocity v of any liquid particle at a distance r from the axis of rotation may be expressed as

$$v = r\omega \quad \dots(7.62)$$

Equation 7.62 therefore indicates that in a forced vortex motion the velocity of flow is directly proportional to its radial distance from the axis of rotation.

By substituting the value of v in equation 7.55, the pressure gradient in the radial direction for a forced vortex motion becomes

$$\frac{\partial p}{\partial r} = \rho r \omega^2 \quad \dots(7.63)$$

Again the distribution of the fluid pressure in a forced vortex motion is obtained by integrating both the equations 7.56 and 7.63 between any two points 1 and 2 in the fluid and their results combined in the following form :

$$\frac{p_2 - p_1}{\rho} = \frac{1}{2} \omega^2 (r_2^2 - r_1^2) + g(z_1 - z_2) \quad \dots(7.64)$$

Equation 7.64 shows that, in a forced vortex motion, (1) on any horizontal plane (i.e., $z_1 = z_2$) the fluid pressure increases with the radial distance from the centre of vortex motion ; (2) surfaces of constant pressure (i.e., $p_1 = p_2$) are paraboloids of revolution ; and (3) the free surface (i.e., $p_1 = p_2 = p_a$) is a special surface of constant pressure and is also a paraboloid of revolution.

The forced vortex motion is essentially a rotational motion because every fluid particle in such a motion is also found to rotate about its own axis as it moves along the curved path. Mathematically also it can be shown that a forced vortex motion is rotational. Therefore in a forced vortex motion the Bernoulli's equation may although be applied to the same streamline, it cannot be applied between points on different streamlines, which is also indicated by equation 7.64.

As stated earlier a *spiral vortex motion* is a combination of a cylindrical vortex motion and radial flow. Hence before analysing spiral vortex motion is essential to discuss the radial flow, which is described in the next section.

7.17 RADIAL FLOW OR RADIAL MOTION

When a fluid flows in radial direction, so that the pressure and velocity at any point in the flow varies with respect to the radial distance of that point from the central axis, then the flow is designated as radial flow. For producing radial flow consider two horizontal circular plates of same diameter placed one above the other at a small distance B apart, as shown in fig. 7.17. The bottom plate is provided with an opening at its centre, to which is attached, a vertical pipe of the same diameter as that of the opening. Through the central pipe let some liquid be allowed to flow up continuously, which will flow radially outward between the plates, thereby developing the radial flow of liquid. Consider sections 1-1 and 2-2 in the radial flow as shown in Fig. 7.17. Section 1-1 is assumed to be located where the radial flow of liquid just commences

i.e., at this section the liquid just enters from the vertical pipe into the passage

between the plates ; and section 2-2 is just at the outer periphery of the plates where the liquid is discharged into atmospheric air. Let r_1 and r_2 be the radial distances of sections 1-1 and 2-2 respectively from the central axis ; and v_1 and v_2 be the radial velocities of flow of liquid at sections 1-1 and 2-2 respectively. Now if Q is discharge of liquid through any section then from continuity equation the following relation between the velocity of flow and the radial distance may be obtained.

$$Q = (2\pi r_1 B \times v_1) = (2\pi r_2 B \times v_2) \\ = (2\pi r_x B \times v_x)$$

$$\text{or } v_1 r_1 = v_2 r_2 = v_x r_x = \text{constant} \quad \dots(7.65)$$

where r_x is the radial distance of any section X-X shown in Fig. 7.17 and v_x is the radial velocity of flow at section X-X.

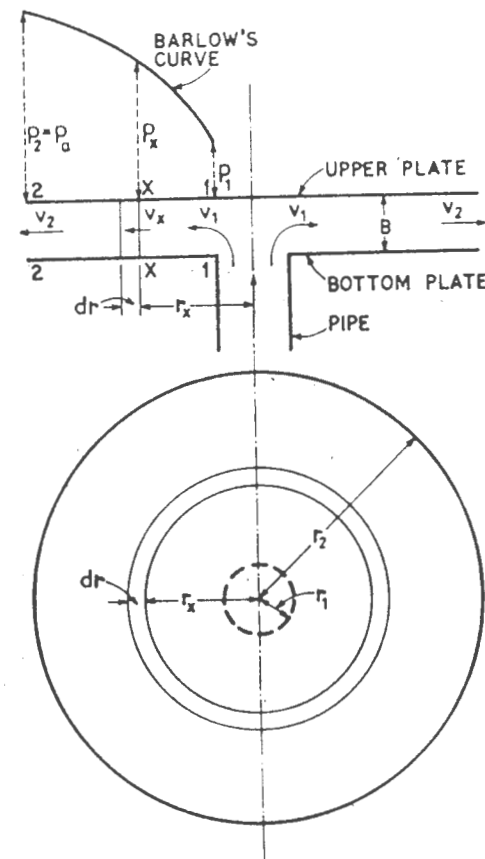


Fig. 7.17 Radial flow

Now if p_1 , p_2 and p_x are the pressure intensities at sections 1-1, 2-2 and X-X respectively then neglecting the loss of head and applying Bernoulli's equation between these sections which are located at the same elevation above the datum, the following expression may be obtained

$$\frac{p_1}{w} + \frac{v_1^2}{2g} = \frac{p_2}{w} + \frac{v_2^2}{2g} = \frac{p_x}{w} + \frac{v_x^2}{2g} = H \quad \dots(7.66)$$

where w is specific weight of the flowing liquid and H is constant, the value of which may be computed by the values of either p_1 and v_1 or p_2 and v_2 being known.

Since section 2-2 is considered to be located just at the outer periphery of the plates, the pressure $p_2 =$ atmospheric pressure p_a . Thus from equation 7.66

$$\frac{p_1}{w} = \frac{p_a}{w} + \frac{v_2^2}{2g} - \frac{v_1^2}{2g} \quad \dots(7'67)$$

Further from equation 7'65 as $r_1 < r_2$, $v_1 > v_2$ and hence it may be seen from equation 7'67 that the pressure p_1 at section 1-1 is less than the atmospheric pressure.

Again from equations 7'65 and 7'66

$$\frac{p_x}{w} = H - \frac{v_x^2}{2g} = H - \frac{1}{2g} \left(\frac{v_2^2 r_2^2}{r_x^2} \right) \quad \dots(7'68)$$

Equation 7'68 represents the pressure distribution in a radial flow, from which it may be seen that the pressure at any section in the radial flow varies with the square of the radial distance of the section from the central axis. Further the pressure increases with the increase in the radial distance following a parabolic law. The pressure distribution curve is also shown in fig. 7'17 which is known as Barlow's curve.

Total Pressure on the Upper Plate. The total pressure (or thrust) exerted on the circular plates due to radial flow of liquid may be obtained with the help of equation 7'68. For this consider a small portion of the upper plate in the form of an elementary ring at a radial distance r_x and thickness dr as shown in Fig. 7'17. The total upward pressure on the ring

$$= (2\pi r_x \times dr) \times p_x$$

$$= (2\pi r_x \times dr) \times w \left[H - \frac{v_2^2}{2g} \left(\frac{r_2^2}{r_x^2} \right) \right]$$

Thus the upward thrust on the upper plate in the portion between r_1 and r_2

$$= \int_{r_1}^{r_2} 2\pi r_x \times w \left[H - \frac{v_2^2}{2g} \left(\frac{r_2^2}{r_x^2} \right) \right] dr$$

$$= 2\pi w \left[\frac{H(r_2^2 - r_1^2)}{2} - r_2^2 \frac{v_2^2}{2g} \log_e(r_2/r_1) \right]$$

Further if p_0 is the pressure of the liquid in the central vertical pipe, then assuming this pressure to be constant over the entire cross-section of the pipe, the upward thrust on the area (πr_1^2) of the upper plate $= p_0 \times (\pi r_1^2)$.

Thus total upward pressure (or thrust) on the upper plate becomes

$$F_u = 2\pi w \left[\frac{H(r_2^2 - r_1^2)}{2} - r_2^2 \frac{v_2^2}{2g} \log_e(r_2/r_1) \right] + (p_0 \times \pi r_1^2) \quad \dots(7'69)$$

The top portion of the upper plate will be subjected to a downward force which may be either due to atmospheric pressure or it may be due to hydrostatic pressure exerted by some column of liquid lying above the plate. Thus

downward force on the top portion of the upper plate may be obtained as follows.

(i) If only the atmospheric pressure p_a acts, then the downward force

$$F_d = (p_a \times \pi r_2^2)$$

(ii) If there is a liquid column of height h and specific weight w over the entire top portion of the upper plate then the downward force

$$F_d = (wh + p_a) \pi r_2^2$$

The resultant force F_R on the upper plate may then be obtained as

$$F_R = (F_u - F_d) \text{ or } (F_d - F_u)$$

Total Pressure on the Bottom Plate. Due to radial flow of liquid the total pressure (or thrust) exerted on the top portion of the bottom plate will be acting in the downward direction, which may also be determined by adopting the same procedure as indicated in the case of the upper plate and it is obtained as

$$F_d = 2\pi w \left[\frac{H(r_2^2 - r_1^2)}{2} - r_2^2 \frac{v_2^2}{2g} \log_e(r_2/r_1) \right] \quad \dots(7'70)$$

Further the lower portion of the bottom plate will also be subjected to an upward force F_u either due to atmospheric pressure or due to hydrostatic pressure exerted by some column of liquid lying above the plate as indicated in the case of the upper plate.

The resultant force F_R on the bottom plate may also be obtained as

$$F_R = (F_d - F_u) \text{ or } (F_u - F_d)$$

7'18 SPIRAL VORTEX MOTION

A spiral vortex motion may be developed when a cylindrical vortex motion is superimposed over the radial flow. In other words a spiral vortex motion is a combination of cylindrical vortex motion and radial flow and hence the pressure difference between any two points lying on a horizontal plane in a spiral vortex motion may be obtained by combining the corresponding equations for the cylindrical vortex motion and the radial flow. Accordingly the following expressions may be obtained for a spiral vortex motion.

(i) *Spiral Free Vortex Motion*

$$\frac{p_2}{w} - \frac{p_1}{w} = \frac{C^2}{2g} \left(\frac{1}{r_1^2} - \frac{1}{r_2^2} \right) + \left(\frac{v_1^2}{2g} - \frac{v_2^2}{2g} \right) \quad \dots(7'71)$$

(ii) *Spiral Forced Vortex Motion*

$$\frac{p_2}{w} - \frac{p_1}{w} = \frac{\omega^2}{2g} (r_2^2 - r_1^2) + \left(\frac{v_1^2}{2g} - \frac{v_2^2}{2g} \right) \quad \dots(7'72)$$

Illustrative Example 7'1. A 0'25 m diameter pipe carries oil of specific gravity 0'8 at the rate of 120 litres per second and the pressure at a

point A is 19.62 kN/m^2 (gauge). If the point A is 3.5 m above the datum line, calculate the total energy at point A in metres of oil.

Solution :

Total energy in terms of oil is given by

$$\frac{p}{w} + \frac{V^2}{2g} + z$$

$$\frac{p}{w} = \frac{19.62 \times 10^3}{9.810 \times 0.8} = 2.5 \text{ m of oil}$$

By continuity

$$Q = AV$$

$$Q = 120 \times 10^{-3} = 0.12 \text{ m}^3/\text{s}$$

$$A = \frac{\pi}{4} \times (0.25)^2 = 0.049 \text{ m}^2$$

Therefore $V = \frac{Q}{A} = \frac{0.12}{0.049} = 2.45 \text{ m/s}$

and $\frac{V^2}{2g} = \frac{(2.45)^2}{2 \times 9.81} = 0.31 \text{ m of oil}$

$$z = 3.5$$

$\therefore$ Total energy = $(2.5 + 0.31 + 3.5)$
 $= 6.31 \text{ m of oil.}$

Illustrative Example 7.2. A 0.3 m pipe carries water at a velocity of 24.4 m/s . At points A and B measurements of pressure and elevation were respectively 361 kN/m^2 and 288 kN/m^2 and 30.5 m and 33.5 m . For steady flow, find the loss of head between A and B.

Solution:

Total energy in terms of metres of water is given by

$$\frac{p}{w} + \frac{V^2}{2g} + z$$

At point A

$$\frac{p}{w} = \frac{361 \times 10^3}{9.810} = 36.80 \text{ m of water}$$

$$\frac{V^2}{2g} = \frac{(24.4)^2}{2 \times 9.81} = 30.34 \text{ m of water}$$

$$z = 30.50 \text{ m}$$

$\therefore$ Total energy at A
 $= (36.80 + 30.34 + 30.50) = 97.64 \text{ m}$

At point B

$$\frac{p}{w} = \frac{288 \times 10^3}{9.810} = 29.36 \text{ m of water}$$

$$\frac{V^2}{2g} = \frac{(24.4)^2}{2 \times 9.81} = 30.34 \text{ m of water}$$

$$z = 33.5 \text{ m}$$

$\therefore$ Total energy at B
 $= (29.36 + 30.34 + 33.50) = 93.20 \text{ m}$

$\therefore$ Loss of head = $(97.64 - 93.20) = 4.44 \text{ m}$

Illustrative Example 7.3. A pipe 300 m long has a slope of 1 in 100 and tapers from 1.2 m diameter at the high end to 0.6 m diameter at the low end. Quantity of water flowing is 5400 litres per minute. If the pressure at the high end is 68.67 kPa [$0.7 \text{ kg (f) / cm}^2$], find the pressure at the low end. Neglect losses.

Solution :

Discharge $Q = \frac{5400}{60 \times 10^3} = 0.09 \text{ m}^3/\text{s}$

Area of flow section at higher end

$$= \frac{\pi}{4} (1.2)^2 = 1.131 \text{ m}^2$$

$\therefore$ Velocity at higher end

$$= \frac{0.09}{1.131} = 0.0796 \text{ m/s}$$

Area of flow section at lower end

$$= \frac{\pi}{4} (0.6)^2 = 0.2827 \text{ m}^2$$

$\therefore$ Velocity at lower end

$$= \frac{0.09}{0.2827} = 0.3184 \text{ m/s}$$

Applying Bernoulli's equation between the higher and the lower ends of the pipe we have

$$\frac{p_1}{w} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{w} + \frac{V_2^2}{2g} + z_2$$

Assume datum to be passing through the lower end of the pipe, then

$$z_2 = 0; \text{ and } z_1 = \frac{300}{100} = 3.0 \text{ m}$$

Thus by substitution

In SI Units

$$\frac{68.67 \times 10^3}{9.810} + \frac{(0.0796)^2}{2 \times 9.81} + 3.0 = \frac{p_2}{w} + \frac{(0.3184)^2}{2 \times 9.81} + 0$$

or

$$\frac{p_2}{w} = 9.995 \text{ m of water}$$

$$P_2 = 98.051 \text{ kN/m}^2 = 98.051 \text{ kPa}$$

In Metric Units

$$\frac{0.7 \times 10^4}{1000} + \frac{(0.0796)^2}{2 \times 9.81} = 3.0 = \frac{p_2}{w} + \frac{(0.3184)^2}{2 \times 9.81} + 0$$

$$\text{or } \frac{p_2}{w} = 9.995 \text{ m of water}$$

$$p_2 = 0.9995 \text{ kg(f)/cm}^2$$

Illustrative Example 7.4. A conical tube is fixed vertically with its smaller end upwards. The velocity of flow down the tube is 4.5 m/s at the upper end and 1.5 m/s at the lower end. The tube is 1.5 m long and the pressure head at the upper end is 3.1 m of the liquid. The loss in the tube expressed as a head is $\frac{0.3(V_1 - V_2)^2}{2g}$ where V_1 and V_2 are the velocities at the upper and lower ends respectively. What is the pressure head at the lower end?

Solution :

Applying Bernoulli's equation between the upper and the lower ends, we have

$$\frac{p_1}{w} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{w} + \frac{V_2^2}{2g} + z_2 + \frac{0.3(V_1 - V_2)^2}{2g}$$

$$\text{or } \frac{p_2}{w} = \frac{p_1}{w} + (z_1 - z_2) + \frac{V_1^2}{2g} - \frac{V_2^2}{2g} - \frac{0.3(V_1 - V_2)^2}{2g}$$

$$\frac{p_1}{w} = 3.1 \text{ m}, (z_1 - z_2) = 1.5 \text{ m}$$

$$V_1 = 4.5 \text{ m/s}, V_2 = 1.5 \text{ m/s}$$

Thus by substitution, we get

$$\frac{p_2}{w} = 3.1 + 1.5 + \frac{(4.5)^2}{2 \times 9.81} - \frac{(1.5)^2}{2 \times 9.81} - \frac{0.3(4.5 - 1.5)^2}{2 \times 9.81}$$

$$\text{or } \frac{p_2}{w} = 5.38 \text{ m}$$

$\therefore$ Pressure head at the lower end,

$$\frac{p_2}{w} = 5.38 \text{ m of liquid flowing in the tube.}$$

Illustrative Example 7.5. A 15 kW pump with 80% efficiency is discharging oil of specific gravity 0.85 to the overhead tank as shown in the figure. If losses in the whole system are 1.75 m of flowing fluid, find the discharge.

Solution :

Applying Bernoulli's equation between 1 and 2, we have

$$\frac{p_1}{w} + \frac{V_1^2}{2g} + Z_1 + H_m = \frac{p_2}{w} + \frac{V_2^2}{2g} + Z_2 + h_L$$

$$\frac{p_2}{w} = 0, \text{ since pressure at 2 is atmospheric;}$$

V_1 and V_2 are negligible and hence the velocity heads are neglected; and

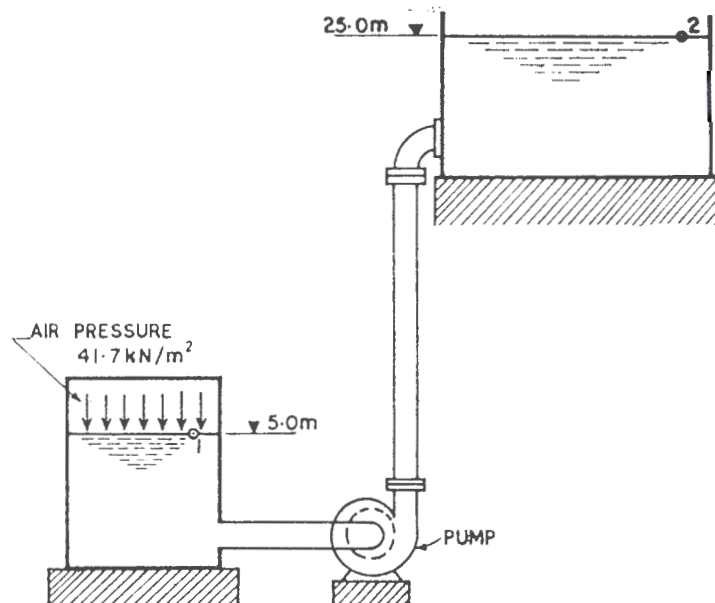
$$h_L = 1.75 \text{ m}$$

Thus by substitution, we get

$$\frac{4.17 \times 10^3}{(0.85 \times 9.810)} + 5.0 + 0 + H_m = 0 + 0 + 25 + 1.75$$

or

$$H_m = 16.75 \text{ m of oil}$$



Example 7.5

Efficiency of pump

$$\eta = \frac{wQ H_m}{P}$$

where P is power in watts.

Thus

$$0.80 = \frac{(0.85 \times 9.810) \times Q \times 16.75}{15 \times 10^3}$$

$$\therefore Q = 0.086 \text{ m}^3/\text{s}$$

Discharge = 0.086 m³/s or 86 litres/second

Illustrative Example 7.6. A portion of a pipe for conveying water is vertical and the diameter of the upper part of the pipe is 50 mm, and the

section is gradually reduced to 25 mm diameter at the lower part. A pressure gage is inserted where the diameter is 50 mm and a second gage 2 m below the first and where pipe is 25 mm diameter. When the quantity of water flowing up through the pipe is 0.205 m³ per minute, the gages show a pressure difference of 31 kN/m² (or 31 kPa). Assuming that the head loss varies as the square of the velocity determine the quantity of water passing through the pipe when the two gages show no pressure difference and the water is flowing downwards.

Solution :

The head loss varies as the square of velocity, as such if v is the velocity of flow at any section CC at a distance x from the lower end as shown in the figure then the head loss for a small length dx of the pipe is

$$dh_f = k v^2 dx$$

where k is a constant of proportionality. By integrating the above expression the total loss of head for the entire pipe between the sections 1-1 and 2-2 is obtained as

$$h_f = \int_0^2 k v^2 dx \quad \dots(i)$$

The diameter of the pipe at section CC is obtained as

$$d = 0.025 \left(\frac{x+2}{2} \right) \text{ m}$$

If Q is the discharge flowing through the pipe then

$$v = \frac{Q}{(\pi/4)d^2} = \frac{Q}{(\pi/4)(0.025)^2 \left(\frac{x+2}{2} \right)^2}$$

By substituting the value of v in equation (i), we get

$$h_f = \int_0^2 \frac{k Q^2}{(\pi/4)^2 (0.025)^4} \cdot \frac{(2)^4}{(x+2)^4} dx$$

or

$$h_f = \frac{16kQ^2}{(\pi/4)^2 (0.025)^4} \times \frac{1}{3} \left[\frac{1}{8} - \frac{1}{64} \right]$$

$$= \frac{7kQ^2}{(\pi/4)^2 (0.025)^4 (12)}$$

When water is flowing in the upward direction, then applying Bernoulli's equation between sections 1-1 and 2-2, we get

$$\frac{p_2}{w} + \frac{V_2^2}{2g} + z_2 = \frac{p_1}{w} + \frac{V_1^2}{2g} + z_1 + h_f$$

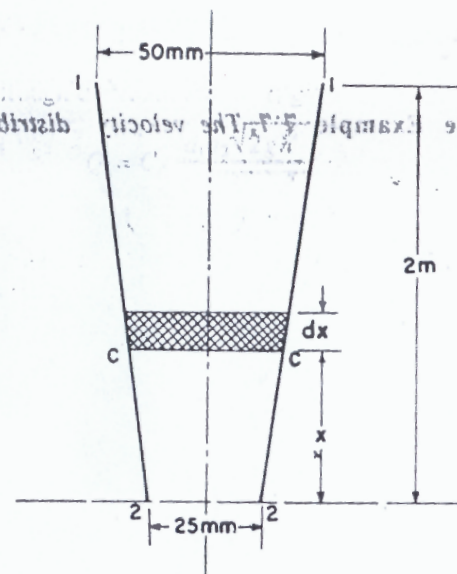
$$\left(\frac{p_2}{w} - \frac{p_1}{w} \right) = 31 \times 10^3 \quad 3.16 \text{ m}$$

$$Q = \frac{0.205}{60} = 0.00342 \text{ m}^3/\text{s}$$

$$V_2 = \frac{Q}{A_2} = \frac{0.00342}{(\pi/4)(0.025)^2} = 6.97 \text{ m/s}$$

$$V_1 = \frac{Q}{A_1} = \frac{0.00342}{(\pi/4)(0.05)^2} = 1.74 \text{ m/s}$$

$$z_2 = 0 \text{ and } z_1 = 2 \text{ m}$$



Example 7.6

Thus by substitution, we get

$$3.16 + \frac{(6.97)^2}{2 \times 9.81} - \frac{(1.74)^2}{2 \times 9.81} - 2 = \frac{7k(0.00342)^2}{(\pi/4)^2 (0.025)^4 (12)}$$

or

$$k = 0.123$$

Now when the water is flowing in the downward direction then again applying the Bernoulli's equation between the sections 1-1 and 2-2, we get

$$\frac{p_1}{w} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{w} + \frac{V_2^2}{2g} + z_2 + h_f$$

$$\left(\frac{p_1}{w} - \frac{p_2}{w} \right) = \frac{V_2^2}{2g} - \frac{V_1^2}{2g} + (z_1 - z_2) + h_f$$

If Q is the discharge flowing in this case then

$$V_2 = 4 V_1$$

Thus by substitution, we get

$$\frac{V_1^2}{2g} + 2 = \frac{16V_1^2}{2g} + 0 + \frac{7(0.123)(\pi/4)^2(0.05)^4 V_1^2}{(\pi/4)^2 (0.025)^4 \times 12}$$

or

$$V_1 = 1.02 \text{ m/s}$$

$$Q = A_1 V_1$$

$$= \left[\frac{\pi}{4} (0.05)^2 \times 1.02 \right] = 0.002 \text{ m}^3/\text{s}$$

$$\text{Discharge} = (0.002 \times 60) = 0.12 \text{ m}^3 \text{ per minute.}$$

Illustrative Example 7.7. The velocity distribution in a pipe is given by

$$v = V_{\max} \left(1 - \frac{r}{R} \right)^n$$

where R is the radius of the pipe, r is any radius at which the velocity is v and n is a constant index. Find the energy correction factor α for this case. Also determine the value of α when $n = \frac{1}{7}$

Solution:

The mean velocity of flow V is given by

$$Q = \pi R^2 V = \int_0^R 2\pi r V_{\max} \left(1 - \frac{r}{R} \right)^n dr$$

$$= \frac{2\pi R^2 V_{\max}}{(n+1)(n+2)}$$

$$V = \frac{2 V_{\max}}{(n+1)(n+2)}$$

and

$$\frac{v}{V} = \frac{(n+1)(n+2)}{2} \left(1 - \frac{r}{R} \right)^n$$

From equation 7.32, we have

$$\alpha = \frac{1}{A} \int_A \left(\frac{v}{V} \right)^3 dA$$

or

$$\alpha = \frac{1}{\pi R^2} \int_0^R \left[\frac{(n+1)(n+2)}{2} \right]^3 \left(1 - \frac{r}{R} \right)^{3n} 2\pi r dr$$

$$\alpha = \frac{(n+1)^3 (n+2)^3}{4(3n+1)(3n+2)}$$

Substituting $n = \frac{1}{7}$, we get after simplification

$$\alpha = 1.06$$

Illustrative Example 7.8. A venturi meter having a diameter of 75 mm at the throat and 150 mm diameter at the enlarged end is installed in a horizontal pipeline 150 mm in diameter carrying an oil of specific gravity 0.9. The difference of pressure head between the enlarged end and the throat recorded by a U-tube is 175 mm of mercury. Determine the discharge through the pipe. Assume the coefficient of discharge of the meter as 0.97.

Solution :

The discharge through the venturi meter is given by

$$Q = C_d \frac{a_1 a_2 \sqrt{2gh}}{\sqrt{a_1^2 - a_2^2}}$$

$$C_d = 0.97$$

$$a_1 = \frac{\pi}{4} \times \left(\frac{150}{1000} \right)^2 = 0.0177 \text{ m}^2$$

$$a_2 = \frac{\pi}{4} \times \left(\frac{75}{1000} \right)^2 = 0.0044 \text{ m}^2$$

$$h = x \left(\frac{S_m}{S} - 1 \right) = \frac{175}{1000} \left(\frac{13.6}{0.9} - 1 \right) = 2.469 \text{ m}$$

Thus by substitution, we get

$$Q = \frac{0.97 \times (0.0177 \times 0.0044) \sqrt{2 \times 9.81 \times 2.469}}{\sqrt{(0.0177)^2 - (0.0044)^2}}$$

or

$$Q = 0.03067 \text{ m}^3/\text{s} ; \text{ or } 30.67 \text{ litres/second}$$

Illustrative Example 7.9. A venturi meter has its axis vertical, the inlet and throat diameters being 150 mm and 75 mm respectively. The throat is 225 mm above inlet and $K = 0.96$. Petrol of specific gravity 0.78 flows up through the meter at a rate of $0.029 \text{ m}^3/\text{s}$. Find the pressure difference between the inlet and the throat.

Solution:

The discharge through a venturi meter is given by

$$Q = \frac{K a_1 a_2 \sqrt{2gh}}{\sqrt{a_1^2 - a_2^2}}$$

$$K = 0.96$$

$$a_1 = \frac{\pi}{4} \times \left(\frac{150}{1000} \right)^2 = 0.0177 \text{ m}^2$$

$$a_2 = \frac{\pi}{4} \times \left(\frac{75}{1000} \right)^2 = 0.0044 \text{ m}^2$$

$$Q = 0.029 \text{ m}^3/\text{s}$$

By substitution, we have

$$0.029 = \frac{0.96 \times (0.0177 \times 0.0044) \sqrt{2 \times 9.81 \times h}}{\sqrt{(0.0177)^2 - (0.0044)^2}}$$

$$\therefore h = 2.254 \text{ m of oil}$$

$$\text{Since } h = \left(\frac{p_1}{w} + z_1 \right) - \left(\frac{p_2}{w} + z_2 \right)$$

$$2.254 = \left(\frac{p_1}{w} - \frac{p_2}{w} \right) - (z_2 - z_1)$$

$$\text{or } 2.254 = \left(\frac{p_1}{w} - \frac{p_2}{w} \right) - \left(\frac{225}{1000} \right)$$

$$\text{or } \left(\frac{p_1}{w} - \frac{p_2}{w} \right) = 2.479$$

$$\begin{aligned} (p_1 - p_2) &= (2.479 \times 0.78 \times 9.810) \\ &= 18.969 \text{ N/m}^2 = 18.969 \text{ kN/m}^2 \\ &= 18.969 \text{ Pa} = 18.969 \text{ kPa} \end{aligned}$$

Illustrative Example 7.10. 215 litres of gasoline (specific gravity 0.82) flow per second upwards in an inclined venturi meter fitted to a 300 mm diameter pipe. The venturi meter is inclined at 60° to the vertical and its 150 mm diameter throat is 1.2 m from the entrance along its length. Pressure gages inserted at entrance and throat show pressures of 0.141 N/mm^2 and 0.077 N/mm^2 respectively. Calculate the discharge coefficient of venturi meter.

If instead of pressure gages the entrance and the throat of the venturi meter are connected to the two limbs of a U-tube mercury manometer, determine its reading in mm of differential mercury column.

Solution :

The discharge through the venturi meter is given by

$$Q = \frac{K a_1 a_2 \sqrt{2gh}}{\sqrt{a_1^2 - a_2^2}}$$

$$Q = 215 \times 10^{-3} = 0.215 \text{ m}^3/\text{s}$$

$$a_1 = \frac{\pi}{4} \times \left(\frac{300}{1000} \right)^2 = 0.0707 \text{ m}^2$$

$$a_2 = \frac{\pi}{4} \times \left(\frac{150}{1000} \right)^2 = 0.0177 \text{ m}^2$$

$$h = \left(\frac{p_1}{w} + z_1 \right) - \left(\frac{p_2}{w} + z_2 \right)$$

$$\frac{p_1}{w} = \frac{0.141 \times 10^6}{9.810 \times 0.82} = 17.528 \text{ m of gasoline}$$

$$\frac{p_2}{w} = \frac{0.077 \times 10^6}{9.810 \times 0.82} = 9.572 \text{ m of gasoline}$$

$$z_1 = 0; z_2 = (1.2 \sin 30^\circ) = 0.60 \text{ m}$$

$$h = (17.528 + 0) - (9.572 + 0.60) = 7.356 \text{ m}$$

Thus by substitution, we get

$$0.215 = \frac{K \times (0.0707 \times 0.0177) \sqrt{2 \times 9.81 \times 7.356}}{\sqrt{(0.0707)^2 - (0.0177)^2}}$$

$$\text{or } K = 0.979$$

If a U-tube manometer is connected then

$$h = x \left(\frac{S_m}{S} - 1 \right)$$

$$\text{or } 7.356 = x \left(\frac{13.6}{0.82} - 1 \right)$$

$$\therefore x = 0.472 \text{ m or } 472 \text{ mm}$$

Illustrative Example 7.11. A venturi meter is to be fitted in a pipe 0.25 m diameter where the pressure head is 7.6 m of flowing liquid and the maximum flow is 8.1 m^3 per minute. find the least diameter of the throat to ensure that the pressure head does not become negative. Take $K = 0.96$.

Solution :

The discharge through the venturi meter is given by

$$Q = \frac{K a_1 a_2 \sqrt{2gh}}{\sqrt{a_1^2 - a_2^2}}$$

$$Q = \frac{8.1}{60} = 0.135 \text{ m}^3/\text{s}$$

$$K = 0.96$$

$$a_1 = \frac{\pi}{4} \times (0.25)^2 = 0.049 \text{ m}^2$$

$$h = 7.6 \text{ m}$$

By substitution, we have

$$0.135 = \frac{0.96 \times (0.049) \times a_2 \sqrt{2 \times 9.81 \times 7.6}}{\sqrt{(0.049)^2 - a_2^2}}$$

or

$$a_2 = 0.0112 \text{ m}^2$$

$$\therefore d_2 = \sqrt{\frac{0.0112 \times 4}{\pi}} = 0.1195 \text{ m} = 119.5 \text{ mm}$$

Illustrative Example 7.12. The coefficient of discharge for a venturimeter used for measuring the flow of an incompressible fluid was found to be constant when the rate of flow Q exceeded a certain value. Show that under these conditions, the loss of head h_f in the convergent portion of the venturimeter can be expressed as $K_1 Q^2$, where K_1 is a constant.

Solution :

Applying Bernoulli's equation between the inlet and throat sections of the venturimeter, we get

$$\frac{p_1}{w} + \frac{V_1^2}{2g} = \frac{p_2}{w} + \frac{V_2^2}{2g} + h_f$$

If h is the venturi head then, we have

$$\frac{p_1}{w} - \frac{p_2}{w} = h$$

and hence

$$(h - h_f) = h' = \frac{V_2^2}{2g} - \frac{V_1^2}{2g}$$

Again if a_1 and a_2 are the cross-sectional areas at the inlet and throat sections of the venturimeter then, we have

$$h = \frac{Q^2}{2g} \left[\frac{1}{a_2^2} - \frac{1}{a_1^2} \right]$$

or

$$Q = \frac{a_1 a_2 \sqrt{2gh}}{\sqrt{a_1^2 - a_2^2}}$$

or

$$Q = C \sqrt{h} \quad \dots (i)$$

where $C = \frac{a_1 a_2 \sqrt{2g}}{\sqrt{a_1^2 - a_2^2}}$, is the constant of the venturimeter.

Further from equation 7.47, we have

$$Q = KC \sqrt{h} \quad \dots (ii)$$

in which K is the coefficient of discharge of the venturimeter.

Thus from equations (i) and (ii), we obtain

$$(h - h_f) = h_f = \frac{Q^2}{C^2} \left[\frac{1}{K^2} - 1 \right]$$

$$\therefore h_f = K_1 Q^2$$

where $K_1 = \frac{1}{C^2} \left[\frac{1}{K^2} - 1 \right]$ is a constant.

Illustrative Example 7.13. Water flows at the rate of $0.147 \text{ m}^3/\text{s}$ through a 150 mm diameter orifice inserted in a 300 mm diameter pipe. If the pressure gages fitted upstream and downstream of the orifice plate have shown readings of 176.58 kN/m^2 and 88.29 kN/m^2 respectively, find the coefficient of discharge C of the orifice meter.

Solution :

The discharge through an orifice meter is given by

$$Q = \frac{C a_0 a_1 \sqrt{2gh}}{\sqrt{a_1^2 - a_0^2}}$$

$$Q = 0.147 \text{ m}^3/\text{s}$$

$$h = \left(\frac{p_1}{w} - \frac{p_2}{w} \right)$$

$$= \frac{(176.58 - 88.29) \times 10^3}{9810} = 9 \text{ m of water}$$

$$a_1 = \frac{\pi}{4} \times \left(\frac{300}{1000} \right)^2 = 0.0707 \text{ m}^2$$

$$a_0 = \frac{\pi}{4} \times \left(\frac{150}{1000} \right)^2 = 0.0177 \text{ m}^2$$

Thus by substitution, we get

$$0.147 = \frac{C \times (0.0707 \times 0.0177) \sqrt{2 \times 9.81 \times 9}}{\sqrt{(0.0707)^2 - (0.0177)^2}}$$

$$\therefore C = 0.605$$

Illustrative Example 7.14. A submarine moves horizontally in a sea and has its axis below the water surface. A Prandtl pitot tube placed in front of the submarine and along its axis is connected to the two limbs of a U-tube containing mercury. The difference in mercury level is found to be 170 mm . Find the speed of submarine knowing that specific gravity of sea water is 1.025 .

Solution :

Let the submarine along with pitot tube be moving with a velocity V .

By superimposing a velocity equal in magnitude to that of the submarine but in opposite direction, i.e., $-V$, to the whole system, the submarine including pitot tube is brought to rest and sea water attains a velocity V .

Thus from equation 7.25, we have

$$V = C \sqrt{2gx \left(\frac{S_m}{S} - 1 \right)}$$

$$x = \frac{170}{1000} = 0.17 \text{ m}; S_m = 13.6; S = 1.025; \text{ and } C \text{ is taken as } 1.$$

Thus by substitution, we get

$$V = \sqrt{2 \times 9.81 \times 0.17 \left(\frac{13.6}{1.025} - 1 \right)}$$

or $V = 6.4 \text{ m/s}$

i.e., submarine is moving with a speed of 6.4 m/s

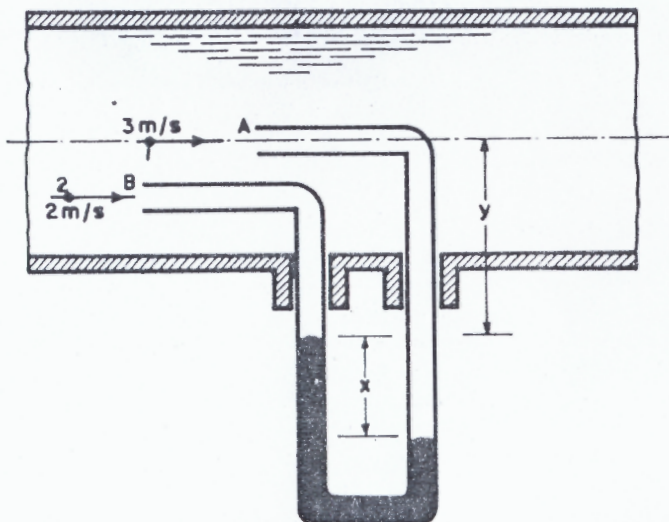
Illustrative Example 7.15. Water flows in a 300 mm pipe. Two pitot tubes are installed in the pipe, one on the centreline and the other 75 mm from the centreline. If the velocities at the two points are 3 m/s and 2 m/s respectively calculate the reading on the differential mercury manometer connected to the two tubes.

Solution :

As shown in the figure, let A and B be the stagnation points at the tips of the two pitot tubes. Considering points 1 and 2 slightly upstream of A and B and applying Bernoulli's equation, we get

$$\frac{p_1}{w} + \frac{V_1^2}{2g} = \frac{p_A}{w} \quad \dots(i)$$

and $\frac{p_2}{w} + \frac{V_2^2}{2g} = \frac{p_B}{w} \quad \dots(ii)$



Example 7.15

where p_1 and p_2 are the static pressure intensities and V_1 and V_2 are the velocities at points 1 and 2 respectively and p_A and p_B are the stagnation pressure

intensities at points A and B respectively. Also the following relation between p_1 and p_2 may be written

$$\frac{p_1}{w} + 0.075 = \frac{p_2}{w} \quad \dots(iii)$$

Thus from equations (i), (ii) and (iii), we get

$$\frac{p_B}{w} - \frac{p_A}{w} = \frac{V_2^2}{2g} - \frac{V_1^2}{2g} + 0.075 \quad \dots(iv)$$

Further if x is the difference in the levels of mercury columns in the two limbs of the manometer and y is the depth of the higher mercury column below the centreline of the pipe, then the following manometric equation may be developed

$$\frac{p_A}{w} + y + x = \frac{p_B}{w} + y - 0.075 + 13.6x$$

or $\frac{p_B}{w} - \frac{p_A}{w} = -12.6x + 0.075 \quad \dots(v)$

Solving equations (iv) and (v), we get

$$\frac{V_2^2}{2g} - \frac{V_1^2}{2g} + 0.075 = -12.6x + 0.075$$

or $\frac{V_2^2 - V_1^2}{2g} = -12.6x$

or $\frac{(2)^2 - (3)^2}{2 \times 9.81} = -12.6x$

$\therefore x = 0.0202 \text{ m or } 20.2 \text{ mm}$

Illustrative Example 7.16. A pitot-static tube placed in the centre of a 200 mm pipe line, has one orifice pointing upstream and the other perpendicular to it. If the pressure difference between the two orifices is 40 mm of water when the discharge through the pipe is 1365 litres per minute, calculate the coefficient of the pitot tube. Take the mean velocity in the pipe to be 0.83 of the central velocity.

Solution :

$$Q = \frac{1365 \times 10^{-3}}{60} = 0.02275 \text{ m}^3/\text{s}$$

The mean velocity of flow is

$$V = \frac{Q}{A} = \frac{0.02275}{\frac{\pi}{4} \left(\frac{200}{1000} \right)^2} = 0.724 \text{ m/s}$$

$$\text{Central velocity} = \frac{V}{0.83} = \frac{0.724}{0.83} = 0.872 \text{ m/s}$$

$$\text{Pressure difference } h = 40 \times 10^{-3} \text{ m}$$

Thus from equation 7.50

$$0.872 = C \sqrt{2 \times 9.81 \times (40 \times 10^{-3})}$$

$$\therefore C = 0.984$$

Illustrative Example 7.17. Water is discharged from a 50 mm diameter nozzle which is inclined at a 30° angle above the horizontal. If the jet strikes the ground at a horizontal distance of 3.6 m and a vertical distance of 0.6 m from the nozzle as shown in the figure, what is the rate of flow?

Solution :

Let V_0 be the velocity of the jet then

$$(V_0)_x = V_0 \cos 30^\circ = 0.866 V_0$$

$$(V_0)_z = V_0 \sin 30^\circ = 0.5 V_0$$

Therefore the coordinate equations for the trajectory are

$$x = 0.866 V_0 t$$

and
$$z = 0.5 V_0 t - \frac{1}{2} g t^2$$

By eliminating t and solving for V_0 from these two equations, we get

$$V_0 = \frac{x}{0.866} \left[\frac{g}{2(0.577x - z)} \right]^{1/2}$$

By substitution, we get

$$V_0 = \frac{3.6}{0.866} \left[\frac{9.81}{2[(3.6 \times 0.577) - (-0.6)]} \right]^{1/2}$$

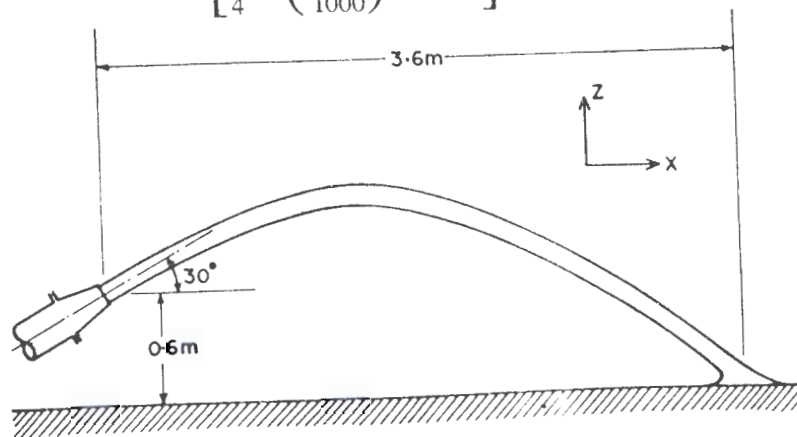
$$V_0 = 5.63 \text{ m/s}$$

or

Hence

$$Q = AV_0$$

$$= \left[\frac{\pi}{4} \times \left(\frac{50}{1000} \right)^2 \times 5.63 \right] = 0.011 \text{ m}^3/\text{s}$$



Example 7.17

Illustrative Example 7.18. A fireman intends to reach a window 25.5 m above the ground with a fire stream from a nozzle having a cylindrical tip 30

mm in diameter and discharging 1140 litres per minute. Neglecting air resistance and assuming a nozzle height of 1.5 m, determine the greatest distance from the building at which fireman can stand and still play the stream upon the window.

Solution :

$$Q = \frac{1140 \times 10^{-3}}{60} = 0.019 \text{ m}^3/\text{s}$$

$$V_0 = \frac{Q}{A} = \frac{0.019}{\frac{\pi}{4} \left(\frac{30}{1000} \right)^2} = 26.88 \text{ m/s}$$

$$x = (V_0 \cos \theta)t \quad \dots(i)$$

and
$$z = 24 = (V_0 \sin \theta)t - \frac{1}{2} g t^2 \quad \dots(ii)$$

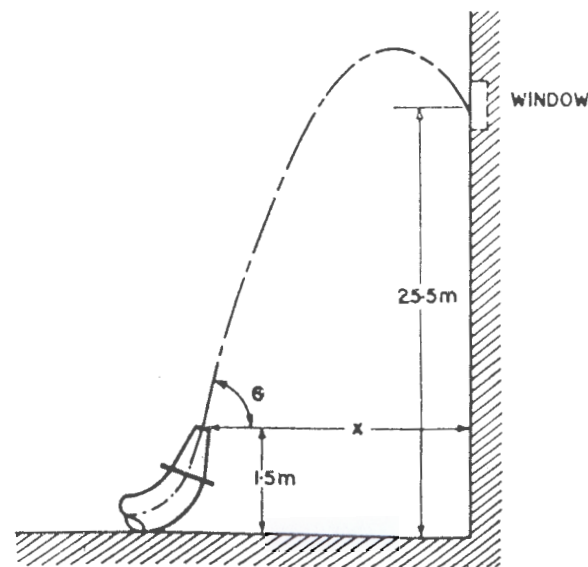
From equation (i)

$$\cos \theta = \frac{x}{V_0 t}$$

$$\therefore \sin \theta = \sqrt{1 - \left(\frac{x}{V_0 t} \right)^2}$$

Substituting this value of $\sin \theta$ in equation (ii), we get

$$24 = \left(\sqrt{V_0^2 t^2 - x^2} \right) - \frac{1}{2} g t^2$$



Example 7.18

Solving for x , we get

$$x = \sqrt{V_0^2 t^2 - (24 + \frac{1}{2} g t^2)^2}$$

and

$$\frac{dx}{dt} = \frac{2V_0^2 t - 2(24 + \frac{1}{2} g t^2) g t}{2\sqrt{V_0^2 t^2 - (24 + \frac{1}{2} g t^2)^2}}$$

For maximum x , $\frac{dx}{dt} = 0$

$$\text{Hence } V_0^2 t = 24 g t + \frac{1}{2} g^2 t^3$$

Solving for t , we get

$$t = \sqrt{\frac{2(26.88)^2 - 48g}{g^2}} = 3.18 \text{ s}$$

$$\therefore x = \sqrt{(26.88)^2 (3.18)^2 - [24 + \frac{1}{2} \times 9.81 \times (3.18)^2]^2} = 43.47 \text{ m}$$

Illustrative Example 7.19. In a free cylindrical vortex of water the tangential velocity at a radius of 0.12 m from the axis of rotation is found to be 7.2 m/s and the intensity of pressure is 245.25 kN/m² [2.5 kg(f)/cm²]. Find the intensity of pressure at a radius of 0.24 m from the axis.

Solution :

For a free vortex

$$v_1 r_1 = v_2 r_2$$

$$\text{or } 7.2 \times 0.12 = v_2 \times 0.24$$

$$\therefore v_2 = 3.6 \text{ m/s}$$

$$\text{Also } \frac{p_1}{w} + \frac{v_1^2}{2g} = \frac{p_2}{w} + \frac{v_2^2}{2g}$$

Thus by substitution

In SI units

$$\frac{245.25 \times 10^3}{9810} + \frac{(7.2)^2}{2 \times 9.81} = \frac{p_2}{w} + \frac{(3.6)^2}{2 \times 9.81}$$

$$p_2 = 264.69 \text{ kN/m}^2, \text{ or } 264.69 \text{ kPa}$$

In metric units

$$\frac{2.5 \times 10^4}{1000} + \frac{(7.2)^2}{2 \times 9.81} = \frac{p_2}{w} + \frac{(3.6)^2}{2 \times 9.81}$$

$$\therefore p_2 = 2.698 \text{ kg (f)/cm}^2$$

Illustrative Example 7.20. Water flows radially between the two flanges at the end of a 0.15 m diameter pipe as shown in the figure. Neglecting losses, if the pressure head at A is -0.3 m, find the pressure head at B and the flow in m³/s.

Solution :

By continuity equation

$$Q = \left(\frac{\pi}{4} \times 0.15^2 \right) V_A = (2\pi \times 0.6 \times 0.025) V_B \\ = (2\pi \times 1.2 \times 0.025) V_C$$

$$\text{or } V_A = \frac{16}{3} V_B ; \text{ and } V_B = 2V_C$$

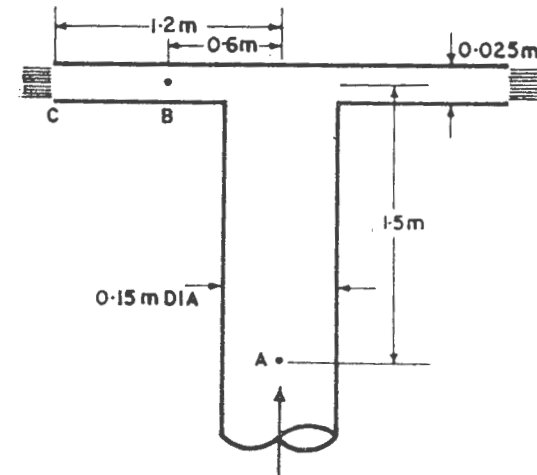
By applying Bernoulli's equation between A and C, we have

$$-0.3 + \frac{V_A^2}{2g} = \frac{V_C^2}{2g} + 1.5$$

$$\text{or } \frac{V_A^2}{2g} \left[1 - \left(\frac{3}{32} \right)^2 \right] = 1.8$$

$$\therefore V_A = 5.97 \text{ m/s}$$

$$\text{Thus } Q = \left(\frac{\pi}{4} \times 0.15^2 \times 5.97 \right) = 0.105 \text{ m}^3/\text{s}$$



Example 7.20

Again applying Bernoulli's equation between points B and C, we have

$$\frac{p_B}{w} + \frac{V_B^2}{2g} = \frac{p_C}{w} + \frac{V_C^2}{2g}$$

$$V_B = \left(\frac{3}{16} \times 5.97 \right) = 1.12 \text{ m/s}$$

$$V_C = 0.56 \text{ m/s}$$

Thus by substitution, we get

$$\frac{p_B}{w} + \frac{(1.12)^2}{2 \times 9.81} = 0 + \frac{(0.56)^2}{2 \times 9.81}$$

$$\therefore \frac{p_B}{w} = -0.048 \text{ m}$$

Illustrative Example 7.21. Assume a steady isothermal flow of air considered as a perfect gas with $R = 29.3$ metres per °C absolute. The velocity field is $u = 3x$, $v = -3y$, $w = 20$, where u, v, w are the velocity components in m/s in the coordinate directions x, y, z respectively. The coordinates are in metres. What is the ratio of pressures between the positions (10, 3, 0) and (0, -5, 0)? The temperature is 38°C.

Solution :

At point (10, 3, 0)

$$u = 30 \text{ m/s}; V = -9 \text{ m/s}; w = 20 \text{ m/s}$$

$$\begin{aligned} \therefore V_1 &= \sqrt{u^2 + v^2 + w^2} \\ &= \sqrt{(30)^2 + (-9)^2 + (20)^2} \\ &= 37.16 \text{ m} \end{aligned}$$

Similarly at point (0, -5, 0)

$$u = 0; V = 15 \text{ m/s}; w = 20 \text{ m/s}$$

$$\begin{aligned} \therefore V_2 &= \sqrt{(0)^2 + (15)^2 + (20)^2} \\ &= 25 \text{ m/s} \end{aligned}$$

From equation 7.35 for isothermal flow, we have

$$K \log_e p_1 + \frac{V_1^2}{2} + gz_1 = K \log_e p_2 + \frac{V_2^2}{2} + gz_2$$

where K is a constant given by

$$\frac{p}{\rho} = K = RT$$

Thus

$$K = 29.3 (273 + 38) = 9112.3$$

Considering $z_1 = z_2$, and substituting in the above equation, we get

$$9112.3 \log_e (p_2/p_1) = \frac{(37.16)^2}{2} - \frac{(25)^2}{2}$$

$$\text{or } \log_e (p_2/p_1) = 0.0415$$

$$\frac{p_2}{p_1} = 1.042$$

PROBLEMS

- 7.1 What are the different energies of a fluid? Explain each of them.
- 7.2 State and derive Bernoulli's theorem, mentioning clearly the assumptions underlying it.
- 7.3 Derive Bernoulli's equation from general energy equation.
- 7.4 Derive Bernoulli's equation from Euler's equation of motion.
- 7.5 What is kinetic energy correction factor?
- 7.6 Starting from first principles obtain the following expression for discharge of a liquid through a venturi meter

$$Q = K C_d h^{1/2}$$

where Q is discharge in m^3/s ; K is discharge coefficient of the venturi meter; C_d is constant of venturi meter; and h is venturi head in metres of fluid flowing through the venturi meter.

- 7.7 A vertical venturi meter of (d/D) ratio equal to 0.6 is fitted in a 0.1 m diameter pipe. The throat is 0.2 m above the inlet. The meter has a coefficient of discharge of 0.92. Determine (i) pressure difference as recorded by two gages fitted at the inlet and throat, (ii) difference on a vertical differential mercury manometer (specific gravity of mercury = 13.6) when a liquid of specific gravity 0.8 flows through the meter at the rate of 50 litres per second.

$$[(i) 130.206 \text{ kN/m}^2 \text{ or } 1.327 \text{ kg(f)/cm}^2; (ii) 1.024 \text{ m}]$$

- 7.8 Oil of specific gravity 0.90 flows in a pipe 300 mm diameter at the rate of 120 litres per second and the pressure at a point A is 24.525 kPa [0.25 kg(f)/cm^2] (gage). If the point A is 5.2 m above the datum line, calculate the total energy at point A in terms of metres of oil.

$$[8.125 \text{ m}]$$

- 7.9 In a vertical pipe conveying water, pressure gages are inserted at A and B, where the diameters are 150 mm and 75 mm respectively. The point B is 2.4 m below A, and when the rate of flow down the pipe is $0.021 \text{ m}^3/\text{s}$ the pressure at B is 117.72 kN/m^2 [0.12 kg(f)/cm^2] greater than that at A. Assuming that the losses in the pipe between A and B can be expressed as $(kV_A^2/2g)$, where V_A is the velocity at A, find the value of k . If the gages at A and B are replaced by tubes filled with water and connected to a U-tube containing mercury of specific gravity 13.6, calculate the value of the reading on the manometer.

$$[1.67; 95.2 \text{ mm}]$$

- 7.10 A 0.3 m by 0.15 m venturi meter is mounted on a vertical pipe with flow upwards. 63 litres per second of oil of specific gravity 0.80 and dynamic viscosity 1 poise flows through the pipeline. The throat section is 0.1 m above the inlet section. What is the pressure difference between the inlet and the throat? [Reynolds Number = $\frac{\rho V d}{\mu} = \frac{0.80 \times 1000 \times 3.565 \times 0.15}{1} = 4278$ for which $K = 0.96$. The required pressure difference = 5.964 kN/m^2 or 5.964 kPa or $0.0608 \text{ kg(f)/cm}^2$]

- 7.11 A venturimeter is used for measuring the flow of petrol in a pipeline inclined at 35° to horizontal. The specific gravity of the petrol is 0.81 and throat area ratio is 4. If the difference in mercury levels in the gage is 50 mm, calculate the flow in litres per hour if the pipe diameter is 0.3 m. Take discharge coefficient of the venturi meter as 0.975.

[2.52×10^5 litres/hour]

- 7.12 A venturi meter measures the flow of water in a 75 mm diameter pipe. The difference of head between the entrance and the throat of the venturi meter is measured by U-tube containing mercury, the space above the mercury on each side being filled with water. What should be the diameter of the throat of the meter in order that the difference of the levels of the mercury shall be 0.25 m when the quantity of water flowing in the pipe is 630 litres per minute? Assume the discharge coefficient as 0.97.

[40.9 mm]

- 7.13 Define kinetic energy correction factor and derive an expression for the same. Velocity distribution for laminar flow of a real fluid in a pipe is given by the equation $V = V_{max} [1 - (r^2/R^2)]$, where V_{max} is velocity at the centre of the pipe, R is pipe radius, and V is the velocity at radius r from the centre of the pipe. Show that kinetic energy correction factor for this flow is 2.

- 7.14 A venturi meter having inlet diameter 100 mm and throat diameter 25 mm is fitted in a vertical pipe, throat 0.3 m below inlet, for measuring the flow of petrol of specific gravity 0.78. Pressure gages are fitted at inlet and throat. Assuming the loss of head between inlet and throat as 30 times the velocity head at inlet, find: (a) C_d for the meter; (b) the discharge in litres per minute when the inlet gage reads 274.68 kN/m^2 [0.28 kg(f)/cm^2] more than the throat gage.

[(a) 0.946; (b) 244 litres/min]

- 7.15 A 150 mm $\times$ 75 mm venturi meter with a coefficient of discharge 0.98 is to be replaced by an orifice meter having a coefficient of discharge 0.6. If both the meters are to give the same differential mercury manometer reading for a discharge of 100 litres per second and the inlet diameter is to remain 150 mm, what should be the diameter of the orifice.

[93.5 mm]

- 7.16 A pitot tube was used to measure the quantity of water flowing in a pipe of 0.3 m diameter. The water was raised to a height of 0.25 m above the centre line of pipe in a vertical limb of the tube. If the mean velocity is 0.78 times the velocity at the centre and coefficient of pitot tube is 0.98, find the quantity of water in litres per second. Static pressure head at the centre of the pipe is 0.2 m.

[53.52 l/s]

- 7.17 A pipe carrying oil of specific gravity 0.877 changes in size from 0.15 m at section A to 0.45 m at section B. Section A is 3.6 m lower than B and the pressures are 90.252 kN/m^2 [0.92 kg(f)/cm^2] and 59.841 kN/m^2 [0.61 kg(f)/cm^2] respectively. If the discharge is $0.145 \text{ m}^3/\text{s}$, determine the head loss and the direction of flow.

[3.32 m of oil; Flow is from A to B]

- 7.18 Water is supplied to a turbine through a 0.3 m diameter pipe at the rate of $0.53 \text{ m}^3/\text{s}$. After passing through the turbine water is discharged through a divergent pipe (called draft tube) which has a diameter of 0.6 m at point B which is 1 m below a point A in the 0.3 m pipe. If the pressures at the points A and B are respectively 147.15 kN/m^2 [1.5 kg(f)/cm^2] and -34.335 kN/m^2 [-0.35 kg(f)/cm^2], determine the power delivered to the turbine by the water.

[115.35 kW (or 156.8 h.p.)]

- 7.19 Find the velocity of flow of carbon tetrachloride specific gravity 1.59 through a pipe when a differential gage attached to a pitot static tube shows a reading of 0.1 m. Take the coefficient of the pitot tube as 0.98.

[3.77 m/s]

- 7.20 Two horizontal parallel circular discs are placed 50 mm apart and each plate has diameter of 0.25 m. The lower plate has central hole of 50 mm diameter and a 50 mm diameter vertical pipe is connected centrally to it. Water flows through the pipe at the rate of 500 litres per minute. If the pressure at the outer edge of the disc is 10 m of water, determine (i) pressure inside the vertical pipe; (ii) velocity and pressure at a radius of 80 mm from the centre; (iii) the resultant pressure on the upper disc; and (iv) the resultant pressure on the bottom disc.

[(i) 89.117 kN/m^2 ; (ii) 0.332 m/s , 98.068 kN/m^2 ; (iii) 22.209 N in the downward direction; (iv) 2.502 N in the upward direction]

- 7.21 In a free cylindrical vortex of water, the tangential velocity at a radius of 0.1 m from the axis of rotation is found to be 10 m/s and the intensity of pressure is 196.2 kN/m^2 [2 kg(f)/cm^2]. Find the intensity of pressure at a radius of 0.2 m from the axis.

[233.7 kN/m^2 , or 233.7 kPa , or 2.382 kg(f)/cm^2]

- 7.22 The wind velocity in a cyclone may be assumed to be free vortex flow. If at 50 km from centre of cyclone, velocity is 16 km/h, what pressure gradient should be obtained at this point. What reduction in barometric pressure should occur over radial distance of 10 km from this point towards centre of storm. Mass density of air = 1.2 kg/m^3 .

[1 in 2110; 0.566 m]